Toughness and Existence of Fractional (g, f)-factors in Graphs *

Shuli Liu † Jiansheng Cai
School of Mathematics and Information Sciences, Weifang University, Weifang 261061, P. R. China

Abstract

Let G be a graph with vertex set $V(G)$. For any $S \subseteq V(G)$ we use $\omega(G-S)$ to denote the number of components of $G-S$. The toughness of G, $t(G)$, is defined as $t(G) = min\{|S|/\omega(G-S)|S \subseteq V(G), \omega(G-S) > 1\}$ if G is not complete; otherwise, set $t(G) = +\infty$. In this paper, we consider the relationship between the toughness and the existence of fractional (g, f)-factors. It is proved that a graph G has a fractional (g, f)-factor if $t(G) \geq (b^2-1)/a$.

Key words: toughness; fractional (g, f)-factor; graph

1 Introduction

The graphs considered here will be finite undirected graph which may have multiple edges but no loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $S \subseteq V(G)$ we use $G[S]$ and $G-S$ to denote the subgraph of G induced by S and $V(G)-S$. For a vertex $x \in V(G)$, we use $N_G(x)$ to denote the set of vertices of $V(G)$ adjacent to x, and $d_G(x)$ and $\delta(G)$ to denote the degree of x and minimum degree of G. A subset S of $V(G)$ is called a covering set (an independent set) of G if every edge of G is incident with at least (at most) one vertex of S.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for every $x \in V(G)$, and $h : E(G) \to [0,1]$ be

*This work is supported by research grant NSFC (60673047) and the Natural Science Foundation of Shandong Province(Y2008A20) .

†The corresponding author: Shuli Liu, mail address: School of Mathematics and Information sciences, Weifang University, Weifang 261061, P.R.China. E-mail: shuli007@163.com

a function. If $g(x) \le d_G^h(x) \le f(x)$ holds for any vertex $x \in V(F_h)$ where $d_G^h(x) = \sum_{e \ni x} h(e)$, we call $G[F_h]$ a fractional (g, f)-factor of G with indicator function h where $F_h = \{e \in E(G) | h(e) > 0\}$. If $g(x) = f(x)$ or $g(x) = f(x) = k$, then a fractional (g, f)-factor is called a fractional f-factor or a fractional k-factor.Other terminologies and notations not defined here can be found in [2,6].

A graph is t-tough if $|S| \ge t\omega(G - S)$ holds for any $S \subseteq V(G)$ and $\omega(G - S) > 1$, where $\omega(G - S)$ denotes the number of components of $G - S$. A complete graph is t-tough for any positive real number t. If G is not complete, there exists the largest t such that G is t-tough, this number is denoted by $t(G)$ and is called the toughness of G, namely

$$t(G) = min\{\frac{|S|}{\omega(G - S)} | S \subseteq V(G), \omega(G - S) > 1\},$$

for complete graph K_n, we define $t(K_n) = +\infty$.

The toughness of a graph was first introduced by Chvátal in [3].Since then, much work has been contributed to the relations between toughness and the existence of factors and fractional factors of a graph.

G.Liu and L.Zhang discussed the sufficient condition for the existence of fractional k-factors with $k \ge 1$ related to toughness of graph, and obtain the following result.

Theorem 1.1[4] Let $k \ge 2$ be an integer. A graph G has a fractional k-factor if $t(G) \ge k - \frac{1}{k}$.

Q.Bian also discussed the toughness condition for the existence of fractional f-factors .

Theorem 1.2[1] Let G be a graph and f is an integer-valued function on $V(G)$ satisfying $a \le f(x) \le b$ with $1 \le a \le b$ and $b \ge 2$ for all $x \in V(G)$. If $t(G) \ge \frac{b^2+b}{a} - \frac{b+1}{b}$, then G has a fractional f-factor.

In this paper we consider the relationship between the toughness and the existence of fractional (g, f)-factors,which extends the results of Liu's and Bian's.

Theorem 1.3 Let G be a graph and let g, f be two nonnegative integer-valued functions defined on $V(G)$ satisfying $a \le g(x) \le f(x) \le b$ with $1 \le a \le b$ and $b \ge 2$ for all $x \in V(G)$, where a, b are positive integers. If $t(G) \ge \frac{b^2-1}{a}$, then G has a fractional (g, f)-factor.

Obviously, we can obtain Theorem 1.1 with $a = b = k$. Since $\frac{b^2+b}{a} - \frac{b+1}{b} \ge \frac{b^2-1}{a}$, we can improve Theorem 1.2 with $g(x) = f(x)$ for all $x \in V(G)$. From the example of [4], our result is also sharp in the sense of $f(x) = g(x) = k$ for all $x \in V(G)$.

2 Proof of Theorem 1.3

To prove the result, we need the following lemmas.

Lemma 2.1[5] A graph G has a fractional (g, f)-factor if and only if for any subset S of $V(G)$.

$$g(T) - d_{G-S}(T) \leq f(S),$$

where $T = \{x \in V(G) \setminus S | d_{G-S}(x) \leq g(x)\}$.

Lemma 2.2[3] If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.

Lemma 2.3[4] Let G be a graph and let $H = G[T]$ such that $d_G(x) = k - 1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k, where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and a covering set $C = V(H) \setminus I$ of H satisfying

$$|V(H)| \leq (k - \frac{1}{k+1})|I|,$$

and

$$|C| \leq (k - 1 - \frac{1}{k+1})|I|.$$

Lemma 2.4[4] Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k - 1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \cdots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$, where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k - 2$ in G, then G has a maximal independent set I and a covering set $C = V(H) \setminus I$ such that

$$\sum_{j=1}^{k-1}(k - j)c_j \leq \sum_{j=1}^{k-1}(k - 2)(k - j)i_j,$$

where $c_j = |C \bigcap T_j|$ and $i_j = |I \bigcap T_j|$ for every $j = 1, \cdots, k - 1$.

Proof of Theorem 1.3. Suppose, by the contrary, that there exist two integer-valued functions g and f satisfying all the conditions of the theorem 1.3, but G has no fractional (g, f)-factors. From Lemma 2.1 there exists a subset S of $V(G)$ such that

$$g(T) - d_{G-S}(T) > f(S), \tag{1}$$

where $T = \{x \in V(G) \setminus S | d_{G-S}(x) \leq g(x)\}$.

Choose T such that T is minimal subject to (1). Suppose that there exists $x \in T$ such that $d_{G-S}(x) = g(x)$. Then the sets S and $T - \{x\}$ satisfy

(1), which contradicts the choice of T. Hence we have $d_{G-S}(x) \leq g(x) - 1$ for all $x \in T$. Obviously, $d_{G-S}(x) \leq b - 1$ for all $x \in T$. By Lemma 2.2, we have

$$\delta(G) \geq 2t(G) \geq 2\frac{b^2-1}{a} \geq \frac{(2b-2)(b+1)}{a} \geq \frac{b}{a}(b+1) \geq b+1.$$

Therefore $S \neq \emptyset$ by (1). Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those components isomorphic to K_b. If $|V(H)| = 0$, then

$$a|S| \leq f(S) < g(T) - d_{G-S}(T) \leq b|T_0| + bl$$

or

$$1 \leq |S| < \frac{b}{a}(|T_0| + l).$$

Hence

$$|T_0| + l > \frac{a}{b}.$$

Clearly

$$\omega(G-S) \geq |T_0| + l \geq 1.$$

If $\omega(G-s) > 1$ then $t(G) \leq \frac{|S|}{\omega(G-s)} < \frac{\frac{b}{a}(|T_0|+l)}{|T_0|+l} = \frac{b}{a}$. This contradicts that $t(G) \geq \frac{b^2-1}{a} > \frac{b}{a}$. If $\omega(G-s) = 1$ then $|T_0| + l = 1$. Hence $d_{G-S}(x) = b-1$ or $d_{G-S}(x) = 0$ for $x \in V(G) \setminus S$. Since $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G)$, we have $|S| \geq 2t(G) - b + 1 \geq t(G) > \frac{b}{a} = \frac{b}{a}(|T_0| + l)$, this is a contradiction.

Now we consider that $|V(H)| > 0$ and $\delta(H) \geq 1$. Let $H = H_1 \bigcup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b - 1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 2.3, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq (b - \frac{1}{b+1})|I_1| \tag{2}$$

and

$$|C_1| \leq (b - 1 - \frac{1}{b+1})|I_1|. \tag{3}$$

On the other hand, it is obvious that $\delta(H_2) \geq 1$ and $\Delta(H_2) \leq b - 1$. Let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq b - 1$. By the definition of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. According to Lemma 2.4, H_2 has a maximal independent set I_2 and a covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \tag{4}$$

where $c_j = |C_2 \bigcap T_j|$ and $i_j = |I_2 \bigcap T_j|$ for every $j = 1, \cdots, b-1$.

Set $W = V(G) - S - T$ and $U = S \bigcup C_1 \bigcup C_2 \bigcup (N_G(I_2) \bigcap W)$. Since $|C_2| + |N_G(I_2) \bigcap W| \leq \sum_{j=1}^{b-1} j i_j$, we obtain

$$|U| \leq |S| + |C_1| + \sum_{j=1}^{b-1} j i_j, \tag{5}$$

and

$$\omega(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \tag{6}$$

where $t_0 = |T_0|$. Let $t(G) = t$. Then when $\omega(G-U) > 1$, we have

$$|U| \geq t\omega(G-U). \tag{7}$$

In addition, the above also holds when $\omega(G-U) = 1$. From Lemma 2.2,

$$|U| \geq d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G) > t\omega(G-U)$$

holds for any $x \in T$. By (5), (6) and (7), the following inequality

$$|S| + |C_1| + \sum_{j=1}^{b-1} j i_j \geq t(t_0 + l) + t|I_1| + t\sum_{j=1}^{b-1} i_j$$

or

$$|S| + |C_1| \geq \sum_{j=1}^{b-1}(t-j)i_j + t(t_0 + l) + t|I_1|$$

holds. From (1) we have

$$b|T| - d_{G-S}(T) > a|S|.$$

Then

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1|$$
$$> a(|S| + |C_1|) \geq \sum_{j=1}^{b-1}(at - aj)i_j + at(t_0 + l) + at|I_1|.$$

Therefore

$$\sum_{j=1}^{b-1}(b-j)c_j + |V(H_1)| + a|C_1| > \sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1|. \tag{8}$$

By (2)and (3),we have

$$|V(H_1)| + a|C_1| \leq (b - \frac{1}{b+1} + ab - a - \frac{a}{b+1})|I_1|. \tag{9}$$

By (4),(8) and (9), we have

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j + (b - \frac{1}{b+1} + ab - a - \frac{a}{b+1})|I_1|$$
$$> \sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1|.$$

Thus at least one of the following two cases must hold.

case1. There is at least one j such that

$$(b-2)(b-j) > at - aj - b + j.$$

Then

$$\begin{aligned} at &< (b-2)(b-j) + aj + b - j \\ &= b(b-2) + (a-b+1)j + b. \end{aligned}$$

If $a = b$, then $at < a(a-2) + j + a \leq a^2 - 1$, which contradicts $t \geq \frac{a^2-1}{a}$. If $a < b$, then $at < b(b-2) + (a-b+1) + b = b(b-2) + a + 1 = (b^2-1) + (a-b) + (2-b) \leq b^2 - 1$, which contradicts $t \geq \frac{b^2-1}{a}$.

case 2. $b - \frac{1}{b+1} + ab - a - \frac{a}{b+1} > at$

In this case, if $a = b$, then $at < a^2 - 1$, which contradicts $t \geq \frac{a^2-1}{a}$. If $a < b$, since

$$at < \frac{(a+1)b^2 + b - 2a - 1}{b+1} = \frac{(a+2)b^2}{b+1} - \frac{b^2 + 2a - b + 1}{b+1},$$

$$a + 2 \leq b + 1,$$

$$(b^2 + 2a - b + 1) - (b + 1) = b^2 - 2b + 2a = b(b - 2) + 2a > 0,$$

then

$$at < b^2 - 1 \quad \text{or} \quad t < \frac{b^2 - 1}{a}.$$

This also contradicts the condition of Theorem 1.3.

The proof is complete.

References

[1] Q. Bian, On Toughness and Fractional f-factors, *Ars Combinatoria.* 87(2008),175-180

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications.* Macmillan, London (1976).

[3] V.Chvátal, Tough Graphs and Hamiltonian Circuits, *Discrete Math.* 5(1973), 215-228.

[4] G.Liu and L.Zhang, Toughness and The Existence of Fractional k-factors of Graphs, *Discrete Mathematics.* 308(2008),1741-1748

[5] G. Liu, and L.Zhang, Fractional (g,f)-factors of Graphs, *Acta Math. Scientia Ser B.* 21(4)(2001),541-545.

[6] Edward R. Scheinerman, Daniel H. Ullman, Fractional Graph Theory, New York, John Wiley and Sons, Inc. (1997)