A note on fragments in a locally k-critical n-connected graph *

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Abstract

We note that with only a slight modification, Su's proof on the fragments in k-critical n-connected graphs (see *J. Graph Theory* 45 (2004), 281-297) can imply the following more general result: every non-complete W-locally k-critical n-connected graph has $2k+2$ distinct fragments $F_1, F_2, \ldots, F_{2k+2}$ such that $F_1 \cap W, F_2 \cap W, \ldots, F_{2k+2} \cap W$ are pairwise disjoint.

The graphs considered here are finite, undirected, and simple (without loops or parallel edges). Let n, k be positive integers. Maurer and Slater in [4] introduced the notion of k-critical n-connected graphs. A graph is called a k-critical n-connected graph or simply an (n,k)-graph, if $\kappa(G-S) = n-|S|$ holds for any $S \subseteq V(G)$ with $|S| \leq k$, where $\kappa(G)$ denotes the connectivity of G. A generalization of this concept is the W-locally k-critical n-connected graphs or simply $W-(n,k)$-graphs (see [5], [3]): for given $W \subseteq V(G)$, $F \cap W \neq \emptyset$ for every fragment F of G, $\kappa(G-W') = n-|W'|$ holds for any $W' \subseteq W$ with $|W'| \leq k$. Clearly, for $W = V(G)$ we get back (n,k)-graphs. For (n,k)-graphs, Mader [2] conjectured that every non-complete (n,k)-graph has $2k+2$ pairwise disjoint fragments. This has been proved by Su recently in [6]. So we have.

Theorem 1 *([6]) Let G be a non-complete (n,k)-graph. Then G has at least $2k+2$ pairwise disjoint fragments.*

*This work is partially supported by NNSF (China) (Grant number: 10171022) and Guang Xi Youth Science Foundation (Grant number: 0135028).

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By using this result, several other conjectures on (n,k)-graphs are implied (see [6]). For $W-(n,k)$-graphs, the corresponding problem has been discussed in [5], [1].

Theorem 2 *([5], [1]) Let G be a non-complete $W-(n,k)$-graph. Then G has at least $2k+2$ distinct fragments and $|W| \geq 2k+2$. In particular, if $2k \geq n$, then G has $2k+2$ pairwise disjoint fragments.*

We remark that T. Jordan showed in [1] that if $2k > n$, then there exist no non-complete $W-(n,k)$-graphs. In general, we can not expect that $W-(n,k)$-graph contains at least $2k+2$ pairwise disjoint fragments as the following example shows.

Let $n \geq 4$ and K be a complete graph of order $2n$. Let $x_1,\dots,x_{n+1}$ be $n+1$ vertices of K. Let G be obtained by adding two adjacent vertices a_1, a_2 to K and joining a_1 to $x_1,\dots,x_{n-1}$, joining a_2 to $x_3,\dots,x_{n+1}$. Clearly, if $W=\{a_1,a_2,x_1,\dots,x_{n+1}\}$, then G is a $W-(n,1)$-graph. Note that G has only four fragments $F_1=\{a_1\}, F_2=\{a_2\}, F_3=K-\{x_3,\dots,x_{n+1}\}, F_4=K-\{x_1,\dots,x_{n-1}\}$. As $n\geq 4$, $F_3\cap F_4 \neq \emptyset$.

But we note that Su's Proof in [6] can imply the following more general result with only a slight modification.

Theorem 3 *Every non-complete W-locally k-critical n-connected graph has $2k+2$ distinct fragments $F_1, F_2, \dots, F_{2k+2}$ such that $F_1\cap W, F_2\cap W,\dots, F_{2k+2}\cap W$ are pairwise disjoint.*

As Su's Proof is rather involved, we will go over some key steps and illustrate the modification needed to prove the above theorem. For terms not defined here, we refer the reader to [6].

Let G be a graph. For $x \in V(G)$, $N_G(x)$ denotes the set of the vertices which are adjacent to x in G. For $F \subseteq V(G)$, let $N_G(F) = (\bigcup_{x\in F} N_G(x)) - F$ and $\overline{F} = V(G)-(F\cup N_G(F))$. F is called a fragment of G if $|N_G(F)| = \kappa(G)$ and $\overline{F} \neq \emptyset$. An inclusion minimal fragment is called an end, and a fragment with minimum cardinality is called an atom. A fragment F is said to be proper if $|F| \leq |\overline{F}|$. Define

$$\mathcal{F}(G) = \{[F,T,\overline{F}] | F \text{ is a proper fragment of } G \text{ and } T = N_G(F)\}.$$

For $V_0 \subseteq V(G)$, let $\mathcal{A}(V_0)$ denote the set of the proper ends contained in V_0, and $b(V_0)$ the maximum number of pairwise disjoint ends of G which are contained in V_0. We use $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ (or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset$) to denote $T \cap A_i \neq \emptyset$ for all $i \in \{1, \ldots, \lambda\}$ (or $T \cap B_j \neq \emptyset$ for all $j \in \{1, \ldots, \mu\}$, respectively). The following properties of the fragments and the ends are often used.

Lemma 1 *(see [6] Lemma 1.1) Let F and F' be two fragments of G, $T = N_G(F), T' = N_G(F')$.*

(1) If $F \cap F' \neq \emptyset$, then $|F \cap T'| \geq |\overline{F'} \cap T|, |F' \cap T| \geq |\overline{F} \cap T'|$.

(2) If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N_G(F \cup F') = (T \cap T') \cup (T \cap \overline{F'}) \cup (\overline{F} \cap T') = N_G(\overline{F} \cap \overline{F'})$.

(3) If F, F' are proper and $F \cap F' \neq \emptyset$, then $\overline{F} \cap \overline{F'} \neq \emptyset$, hence both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G.

Lemma 2 *(see [6] Lemma 1.2) Let F, F' be two proper fragments of G. If F is an end, then $F \cap F' = \emptyset$ or $F \subseteq F'$.*

We now sketch the proof of Theorem 3. Let $G = (V, E)$ be a non-complete $W-(n,k)$-graph. Clearly, if $b(V) \geq 2k+2$, then Theorem 3 holds. So we may assume that $b(V) \leq 2k+1$.

Let F_0 be a proper fragment of G with maximum cardinality and $[F_0, T_0, \overline{F_0}] \in \mathcal{F}(G)$. Suppose that $b(F_0) = \lambda_0$ and $A_1, \cdots, A_{\lambda_0}$ are λ_0 pairwise disjoint ends of G contained in F_0, and $b(\overline{F_0}) = \mu_0$ and $B_1, \cdots, B_{\mu_0}$ are μ_0 pairwise disjoint ends of G contained in $\overline{F_0}$. For the proper fragment F_0 with maximum cardinality, by applying Lemma 1 and 2 we have the following properties.

Assertion 1 *(see [6] Lemma 3.1)*

(1) Any end which is contained in $\overline{F_0}$ but not equal to $\overline{F_0}$ is a proper end of G.

(2) If C is a proper end of G distinct to $A_1, \cdots, A_{\lambda_0}, B_1, \cdots, B_{\mu_0}$, then $C \cap (A_1 \cup \cdots \cup A_{\lambda_0} \cup B_1 \cup \cdots \cup B_{\mu_0}) = \emptyset$.

(3) If $[F, T, \overline{F}] \in \mathcal{F}(G)$ and $T \cap [A_1, \ldots, A_{\lambda_0}] \neq \emptyset$ or $T \cap [B_1, \ldots, B_{\mu_0}] \neq \emptyset$, then $F \cap (F_0 \cup \overline{F_0}) = \emptyset$, and hence $F \cap (A_1 \cup \cdots \cup A_{\lambda_0} \cup B_1 \cup \cdots \cup B_{\mu_0}) = \emptyset$, and $F \subseteq T_0$.

By Assertion 1, we can choose ends group $A_1, \ldots, A_\lambda, B_1, \ldots, B_\mu$ of G such that the following conditions (P_2) are satisfied:

(i) The ends $A_1, \ldots, A_\lambda, B_1, \ldots, B_\mu$ are pairwise disjoint.

(ii) If $[F, T, \overline{F}] \in \mathcal{F}(G)$ and $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset$, then $F \cap (A_1 \cup \cdots \cup A_\lambda \cup B_1 \cup \cdots \cup B_\mu) = \emptyset$.

(iii) Subject to (i) and (ii) $\lambda + \mu$ maximum.

Define
$\mathcal{C} = \{C \in \mathcal{A}(F) | [F, T, \overline{F}] \in \mathcal{F}(G)$ and $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset\}$.

Since $b(V) \leq 2k + 1$, we have $\lambda \leq k$ or $\mu \leq k$. As G is a non-complete $W - (n, k)$-graph, there is a triple $[F, T, \overline{F}] \in \mathcal{F}(G)$ such that $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset$. So $\mathcal{C} \neq \emptyset$. Let $\mathcal{C} = \{C_1, \ldots, C_r\}$. For each $i \in \{1, \ldots, r\}$, take a $[F, T, \overline{F}] \in \mathcal{F}(G)$ such that the following conditions (P_3) are satisfied.

(i) $C_i \subseteq F_i$.

(ii) $T_i \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T_i \cap [B_1, \ldots, B_\mu] \neq \emptyset$.

(iii) $|F_i| = \max\{|F| | [F, T, \overline{F}] \in \mathcal{F}(G)$ and $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset$ and $C_i \subseteq F\}$.

By (P_3), there is a mapping ϕ between $\{C_1, \ldots, C_r\}$ and $F_1, \ldots, F_r$ such that $\phi(C_i) = F_i$ for $i = 1, \ldots, r$. For simplification, let $F_1, \ldots, F_t$ be the distinct fragments of $F_1, \ldots, F_r$ so that $|F_t| \geq \cdots \geq |F_1|$. As we shall not use any special property of W, by applying Lemma 1 and Lemma 2 (as in section 3 of [6]), we can prove the following statements.

Assertion 2 *(see [6] Lemma 3.6) Let $\mathcal{A}(F_t) = \{C_{t1}, \cdots, C_{ts}\}$, $s \geq 1$. Then*

(1) $\mathcal{A}(\overline{F_t}) = \emptyset$.

(2) $F_t \cap F_l = \emptyset$, $l = 1, \cdots, t-1$.

(3) $\overline{F_t} \cap F_l = \emptyset$, $l = 1, \cdots, t-1$.

(4) Suppose that $[F, T, \overline{F}] \in \mathcal{F}(G)$, $T \cap [A_1, \ldots, A_{\lambda_0}] \neq \emptyset$ or $T \cap [B_1, \ldots, B_{\mu_0}] \neq \emptyset$. If $F \cap F_t \neq \emptyset$, then $T \cap \overline{F_t} = \emptyset$.

(5) Suppose that $[F, T, \overline{F}] \in \mathcal{F}(G)$, $T \cap [A_1, \ldots, A_{\lambda_0}] \neq \emptyset$ or $T \cap [B_1, \ldots, B_{\mu_0}] \neq \emptyset$. If $F \cap F_t \neq \emptyset$, then $F \subseteq F_t$.

For $i \in \{1, \cdots, t\}$, consider the following proposition (Z_i):

(i) $\mathcal{A}(\overline{F_j}) \subseteq \mathcal{A}(F_t) \cup \cdots \cup \mathcal{A}(F_{j+1})$, $j = t, \cdots, i$.

(ii) If $j \in \{t, \cdots, i\}$, $1 \leq l < j$, $F_j \cap F_l = \emptyset$.

(iii) If $j \in \{t, \cdots, i\}$, $1 \leq l < j$, $\overline{F_j} \cap F_l = \emptyset$.

(iv) Suppose that $[F, T, \overline{F}] \in \mathcal{F}(G)$, $T \cap [A_1, \ldots, A_{\lambda_0}] \neq \emptyset$ or $T \cap [B_1, \ldots, B_{\mu_0}] \neq \emptyset$. If $j \in \{t, \cdots, i\}$ and $F \cap F_j \neq \emptyset$, then $T \cap \overline{F_j} = \emptyset$.

(v) Suppose that $[F, T, \overline{F}] \in \mathcal{F}(G)$, $T \cap [A_1, \ldots, A_{\lambda_0}] \neq \emptyset$ or $T \cap [B_1, \ldots, B_{\mu_0}] \neq \emptyset$. If $j \in \{t, \cdots, i\}$ and $F \cap F_t \neq \emptyset$, then $\mathcal{A}(F) - \mathcal{A}(F_t) \cup \cdots \cup \mathcal{A}(F_j) = \emptyset$.

By Assertion 2, (Z_t) is satisfied. By induction, we can prove that (Z_1) is satisfied. So we have.

Assertion 3 *(see [6] Theorem 3.1)*

(1) $F_1, \ldots, F_t$ are pairwise disjoint.

(2) $\overline{F_i} \cap (F_{i-1} \cup \cdots \cup F_1) = \emptyset$ for $i = 2, \ldots, t$.

(3) Suppose that $[F, T, \overline{F}] \in \mathcal{F}(G)$, and $T \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ or $T \cap [B_1, \ldots, B_\mu] \neq \emptyset$. If for some $i \in \{1, \ldots, t\}$, $F \cap F_i = \emptyset$, then $T \cap \overline{F_i} = \emptyset$.

For each $i = 1, \ldots, t$, let $\mathcal{A}(F_i) = \{C_{i1}, \ldots, C_{i\alpha(i)}\}$. Then, $\mathcal{C} = \{C_{11}, \ldots, C_{1\alpha(1)}, \ldots, C_{t1}, \ldots, C_{t\alpha(t)}\} = \{C_1, \ldots, C_r\}$.

For $R \subseteq W$, define that R covers $\{A_1, \ldots, A_\lambda\}$ (or $\{B_1, \ldots, B_\mu\}$) if $R \cap [A_1, \ldots, A_\lambda] \neq \emptyset$ (or if $R \cap [B_1, \ldots, B_\mu] \neq \emptyset$, respectively). Let $\mathcal{C}_0 \subseteq \mathcal{C}$, define that R covers $\mathcal{C}_0$ if for each $C_{il} \in \mathcal{C}_0$, $R \cap (C_{il} \cup \overline{F_i}) \neq \emptyset$, where $C_{il} \in \mathcal{A}(F_i)$. (This is different from the case of (n, k)-graphs.) For this definition of cover, we can obtain the following result.

Assertion 4 *(see [6] Lemma 3.8) If there is a vertex set $R \subseteq W$ such that R covers $\{A_1, \ldots, A_\lambda\}$ (or $\{B_1, \ldots, B_\mu\}$) and $\mathcal{C}$, then there is no $[F, T, \overline{F}] \in \mathcal{F}(G)$ such that $R \subseteq T$. Hence, $|R| \geq k + 1$ as G is a non-complete $W - (n, k)$-graph.*

In what follows, similarly to [6], define the Δ_1-bipartite graph as in [6] section 4, but for our purpose, we have to consider W. Before that, we include a definition of Δ_0-bipartite in [6].

Let $H = (X, Y)$ be a bipartite (perhaps $Y = \emptyset$). H is called a Δ_0-bipartite graph, if the following three conditions are satisfied: (i) For any $x \in X$, $d_H(x) = 0$ or $d_H(x) \geq 2$. (ii) If $S(Y) \neq \emptyset$, then $|T(Y)| > |S(Y)|$. (iii) There is a matching saturating $S(Y)$ in H. Where $S(Y) = N_H(Y)$, $T(Y) = N_H(S(Y))$.

Let $X_g \subseteq \mathcal{C}, Y_g = \{\overline{F_{i_l}}, \ldots, \overline{F_{i_1}}\} \subseteq \{\overline{F_t}, \ldots, \overline{F_1}\}$ (perhaps $Y_g = \emptyset$), such that $\mathcal{A}(F_{i_1}) \cup \cdots \cup \mathcal{A}(F_{i_l}) \subseteq X_g$. Define a bipartite graph $H_g = (X_g, Y_g)$ as follows: for $C \in X_g, \overline{F_{i_p}} \in Y_g$, $C\overline{F_{i_p}} \in E(H_g)$ if and only if $C \cap \overline{F_{i_p}} \cap W \neq \emptyset$.

Set $S_g = N_{H_g}(Y_g), T_g = N_{H_g}(S_g)$. The bipartite graph H_g is called a Δ_1-bipartite graph if the following conditions (P_4) are satisfied.

(i) $\overline{F_{i_l}} \cap W, \ldots, \overline{F_{i_1}} \cap W$ are pairwise disjoint, and for each $C \in \mathcal{A}(F_{i_1}) \cup \cdots \cup \mathcal{A}(F_{i_l})$, $C \cap (\overline{F_{i_l}} \cup \cdots \cup \overline{F_{i_1}}) \cap W = \emptyset$.

(ii) H_g is a Δ_0-bipartite graph (for the definition see section 2 in [6]).

(iii) There is a $D_g \subseteq X_g - S_g$ and $R_g \subseteq W$ such that R_g covers D_g, $|R_g| \leq |D_g|/2$, and $\mathcal{A}(F_{i_j}) - D_g \neq \emptyset$ for $j = 1, \ldots, l$ (if $Y_g \neq \emptyset$), and $D_g \cup \mathcal{A}(F_{i_1}) \cup \cdots \cup \mathcal{A}(F_{i_l}) \supseteq X_g - S_g$.

By using our definition for Δ_1-bipartite graph, along the construction in the proof of Theorem 4.1 of [6], we can show that the following counterpart of Theorem 4.1 of [6] still holds, i.e.

Assertion 5 *There is a Δ_1-bipartite graph $H_f = (X_f, Y_f)$ with $|X_f| = r$.*

Define a bipartite graph H^* as in section 5 of [6].

Let $X^* = X_f \cup \{A_1, \ldots, A_\lambda, B_1, \ldots, B_\mu\}, Y^* = Y_f$, $H^* = (X^*, Y^*)$. For $D \in X^*, \overline{F_{i_j}} \in Y^*$, $D\overline{F_{i_j}} \in E(H^*)$ if and only if $D \cap \overline{F_{i_j}} \cap W \neq \emptyset$.

Note that H_f is a subgraph of H^*. Let M^* be a maximum matching of H^*, set

$$M_1^* = \{A_i y \in M^* | A_i \in \{A_1, \cdots, A_\lambda\}, y \in Y^*\}$$
$$M_2^* = \{B_j y \in M^* | B_j \in \{B_1, \cdots, B_\mu\}, y \in Y^*\}$$
$$M_3^* = \{C_l y \in M^* | C_l \in X_f, y \in Y^*\}.$$

Take a maximum matching M^* in H^* such that $|M_3^*|$ is maximum, let M_f be a maximum matching of H_f. By applying the properties of matching in bipartite graphs, we can show.

Assertion 6 *(see [6] Lemma 5.1)* $|M_3^*| = |M_f|$

By applying some arguments of the M^*-alternating paths in bipartite graph H^*, we can show that H^* contains an independent set I of many vertices. By applying Assertion 4, we can show that $|I| \geq 2k + 2$. Hence,

Assertion 7 *(see [6] Theorem 5.1) H^* contains an independent set of size at least $2k+2$.*

By our definition, the independent set in H^* corresponds to the set of fragments in G, in which any two fragments share no common vertices in W. So, Assertion 7 implies that G contains $2k+2$ distinct fragments $F_1, \ldots, F_{2k+2}$ such that $F_1 \cap W, \ldots, F_{2k+2} \cap W$ are pairwise disjoint. Hence, Theorem 3 is true.

$W-(n,k)$-graphs are closely related to an important graph class: k-con-critically n-connected graphs introduced by Mader in [3]. A graph G is called k-con-critically n-connected if $\kappa(G-V') = n - |V'|$ for any V' with $|V'| \leq k$ and the induced subgraph by V' connected. As noted by Mader in [3], if G is k-con-critically n-connected, then $G-\{z\}$ is a $W-(n-1,k-1)$-graph with $W = N_G(z)$. So Theorem 3 may be used as a tool to study k-con-critically n-connected graph.

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