Stable Well-Covered Graphs

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Abstract

A graph G is said to be *well-covered* if every maximal independent set of G is of the same size. It has been shown that characterizing well-covered graphs is a co-NP-complete problem. In an effort to characterize some of these graphs, different subclasses of well-covered graphs have been studied. In this paper, we will introduce the subclass of *stable well-covered graphs*, which are well-covered graphs that remain well-covered with the addition of any edge. Some properties of stable well-covered graphs are given. In addition, the relationships between stable well-covered graphs and some other subclasses of well-covered graphs, including the suprising equivalence between stable well-covered graphs and other known subclasses, are proved.

1 Introduction

In this paper, assume any graph $G = (V, E)$ is a finite simple graph with vertex set V and edge set E. The notation $u \sim v$ denotes that vertices u and v are adjacent, while $u \nsim v$ denotes that they are not. Let $d(v)$ denote the degree of a vertex v, $\delta(G)$ the minimum $d(v)$ in G, and $\Delta(G)$ the maximum $d(v)$ in G. The *independence number* of G, denoted $\alpha(G)$, is defined as the maximum cardinality of all independent sets of G. We say that a graph is *well-covered* if every maximal independent set of G has the same cardinality.

Determining whether or not a graph is well-covered has been shown to be co-NP-complete [1] [8]. Thus many subclasses of well-covered graphs have been studied in an effort to characterize well-covered graphs with certain properties. In 1991, Pinter [6] [7] introduced *strongly well-covered* graphs, those well-covered graphs that remain well-covered with the deletion of any

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edge. Our interest lies in an analogous question: what can we say about well-covered graphs that remain well-covered with the addition of any edge?

2 Properties of Stable Well-Covered Graphs

A graph G is *stable well-covered* if G is well-covered and the graph formed by the addition of any edge $e = uv$, where $u, v \in V(G)$ and $e \notin E(G)$, is still well-covered [2]. Before we prove some properties of stable well-covered graphs we first prove a useful and more general property of well-covered graphs in the following lemma.

Lemma 1: Let e be any edge such that $e = uv$, where $u, v \in V(G)$ and $e \notin E(G)$. If G has at most one isolated vertex and is a well-covered graph, then $\alpha(G + e) = \alpha(G)$.

Proof: Suppose that G has at most one isolated vertex, is a well-covered graph and $e = uv$ where $u, v \in V(G)$ and $e \notin E(G)$. Clearly $\alpha(G + e) \leq \alpha(G)$. By assumption, at most one of u and v is an isolate. Without loss of generality, assume u is not an isolate. Let w be a neighbor of u in G. Starting with w, greedily choose a maximal independent set I in G. Then $|I| = \alpha(G)$ since G is well-covered. The set I is also maximal independent in $G + e$ since $u \notin I$. Thus since $\alpha(G + e)$ cannot be larger than $\alpha(G)$, $\alpha(G + e) = \alpha(G)$. ■

Remark 2: If G is a stable well-covered graph, then either G is K_1 or G has no isolated vertices.

Theorem 3: If G is a stable well-covered graph of order n and $G \neq K_n$, then $\Delta(G) \leq n - 3$.

Proof: Let G be a graph that fulfills the hypotheses of the theorem. Since G is not complete, $\alpha(G) \geq 2$ and so since G is well-covered, $\Delta(G) \leq n - 2$. Suppose, by way of contradiction, there exists a vertex u such that $d(u) = n - 2$. Let w be the vertex of G to which u is not adjacent. Then the graph $G + e = H$, where $e = uw$, has a maximal independent set of size one, namely $\{u\}$. By Remark 2 and Lemma 1, since $\alpha(G) \geq 2$, $\alpha(G + e) \geq 2$. Thus H is not well-covered and so G is not stable well-covered, a contradiction. Thus $\Delta(G) \leq n - 3$. ■

Let a neighbor w of v be called a *private* neighbor of v with respect to a set $S \subseteq V$ such that $v \in S$ and $w \notin S$, if no other vertex of S is adjacent to w.

Theorem 4: A non-trivial well-covered graph G is stable well-covered if and only if every vertex in any maximum independent set in G has a private neighbor.

Proof:

($\Rightarrow$) Suppose that G is a stable well-covered graph. If G is a complete graph, then $\alpha(G) = 1$ and so vacuously every neighbor of a vertex in a maximum independent set is private. Thus we may assume G is not complete.

By way of contradiction, let I be a maximum independent set of G and $u \in I$ such that u does not have a private neighbor with respect to I (i.e. the vertices of $I - \{u\}$ dominate $N(u)$). Note that $|I| \geq 2$ since G is not complete. Let w be another vertex of I. Then $G + e$ where $e = uw$ has a maximal independent set of size $\alpha(G) - 1$, namely $I - \{u\}$. Hence since $\alpha(G + e) = \alpha(G)$ by Remark 2 and Lemma 1, $G + e$ is not well-covered, a contradiction. Thus if G is stable well-covered then every vertex in any maximum independent set in G has a private neighbor.

($\Leftarrow$) Suppose that G is well-covered and every vertex in any maximum independent set in G has a private neighbor with respect to the independent set. By way of contradiction, suppose that there exists an edge $e = uv$ where $u, v \in V$, $e \notin E$, such that $H = G + e$ is not well-covered. Then there exists a maximal independent set I of H containing u or v, without loss of generality suppose v, such that $|I| \leq \alpha(G) - 1 = \alpha(G + e) - 1$ by Lemma 1. Since I is independent in G and G is well-covered, we must be able to extend I to a maximum independent set in G. Since u is the only possible vertex we could add, $|I| = \alpha(G) - 1$ and $I \cup \{u\} = J$ is a maximum independent set of G. By assumption, u has a private neighbor, w, in G with respect to J. But then $I \cup \{w\}$ is an independent set of H and so I is **not** maximal in H, a contradiction. Therefore if every vertex in any maximum independent set in G has a private neighbor, then G is stable well-covered. ■

Theorem 5: If G is a connected, stable well-covered graph and $G \neq K_1, K_2$, then $\delta(G) \geq 2$.

Proof: Let G be a graph that fulfills the hypotheses of the theorem. By way of contradiction, suppose $\delta(G) < 2$. By assumption, G is connected so $\delta(G) = 1$. Let $u \in V(G)$ such that $d(u) = 1$. Let w be the vertex of G to which u is adjacent, and let $x \neq u$ be a neighbor of w. There must be such an x since $G \neq K_2$ and G is connected. Let I be a maximal independent set containing u and x. Since G is well-covered, I is also maximum. But u has no private neighbor with respect to I, and so by Theorem 4, G is not stable well-covered; a contradiction. Thus $\delta(G) \geq 2$. ■

Consider the disconnected graph $H = K_2 \cup K_t$, where $t \geq 2$, with order $n = t + 2$. Then $\alpha(H) = 2$, $\delta(H) = 1$ and $\Delta(H) = n - 3$. Note that H is stable well-covered since each component has at least two vertices and so the addition of any edge will not eliminate the possibility of taking one vertex from each of the original components in any maximal independent set. Thus H highlights the importance of requiring the graphs in Theorem 5 to be connected, since H is stable well-covered but has minimum degree of just one. In addition, H illustrates that our bound in Theorem 3 is best possible.

3 Compared to Strongly Well-Covered Graphs

In 1991, Pinter [6][7] introduced strongly well-covered graphs. A well-covered graph, G, is *strongly well-covered* if $G - e$ is well-covered for all $e \in E$. In an attempt to characterize these graphs, Pinter proved the following theorem, which will be crucial to our comparison of these graphs with the stable well-covered graphs.

Theorem 6[6]: If G ($G \neq K_1$ or K_2) is strongly well-covered, then $G - v$ is not well-covered for all $v \in V$.

The following lemma, a general result for well-covered graphs shown by Pinter [5], is very useful both in the proof below and in section 4.

Lemma 7[5]: If G is a well-covered graph without isolated vertices, then $\alpha(G - v) = \alpha(G)$ for any $v \in V(G)$.

We now consider what can be said about $G - v$ in stable well-covered graphs.

Theorem 8: If G is stable well-covered, then $G - v$ is well-covered for all $v \in V$.

Proof: Let G be a stable well-covered graph. If G is a complete graph, then $G - v$ is also a complete graph for all $v \in V(G)$, and so $G - v$ is well-covered. Thus we may assume that G is not complete and so $\alpha(G) > 1$.

By way of contradiction, suppose that $G - v$ is not well-covered for some $v \in V$. Starting with a neighbor of v, greedily choose a maximal independent set, I_1, in G. Since G is well-covered, $|I_1| = \alpha(G)$. Since $v \notin I_1$, I_1 is also maximal in $G - v$. By Lemma 7, $\alpha(G - v) = \alpha(G)$ and so since $G - v$ is not well-covered, there exists a maximal independent set of $G - v$, call it I_2, such that $|I_2| \leq \alpha(G) - 1$. Since I_2 is independent in G, there must be no vertex of I_2 that is adjacent to v (i.e. $I_2 \cup \{v\}$ is independent in G); otherwise I_2 would be maximal in G contradicting the fact that G is well-covered. Let u be a vertex of I_2. Then $e = uv$ is not an edge of G, and I_1 and I_2 are both maximal independent sets in $G + e$. But $|I_1| > |I_2|$ and so $G + e$ is not well-covered, contradicting the fact that G is stable well-covered. Hence if G is stable well-covered, then $G - v$ is well-covered for all $v \in V$. ■

It now follows that, other than the graphs K_1 and K_2, the subclass of graphs that are strongly well-covered is disjoint from the subclass of graphs that are stable well-covered.

Corollary 9: If $G \neq K_1, K_2$ and G is stable well-covered, then G is not strongly well-covered.

Proof: If G is stable well-covered, then by Theorem 8, $G-v$ is well-covered for all $v \in V$. But then by Theorem 6, G is not strongly well-covered. ■

4 Compared to W2 & 1-Well-Covered Graphs

Staples introduced two classes of graphs called W_n and n-well-covered [9][10] in 1975. A graph G is said to be in the class W_n for positive integer n if G has at least n vertices, and every n disjoint independent sets in G are contained in n disjoint maximum independent sets. A graph G is said to be *n-well-covered* if for all sets $S \subseteq V(G)$ such that $|S| = n$ and $\alpha(G - S) = \alpha(G)$, $G - S$ is also well-covered. Staples proved that these classes were closely related.

Theorem 10[9]: For any $n \geq 1$, a graph G is $(n-1)$-well-covered if and only if $G \in W_n$.

In particular, we are interested in the class of W_2 graphs which is equivalent to the class of 1-well-covered graphs. Using the definition of 1-well-covered together with Lemma 7, these two classes are the set of graphs G for which $G - v$ is well-covered for all vertices $v \in V(G)$. Thus the natural question to ask given Theorem 8 is whether stable well-covered graphs and W_2 graphs are equivalent classes of well-covered graphs.

Theorem 11: A connected graph G is stable well-covered iff G is in W_2.

Proof:

($\Rightarrow$) Follows from Theorem 8.

($\Leftarrow$) Suppose G is in W_2, but G is not stable well-covered. Then there exist $u, v \in V(G)$ such that $e = uv \notin E(G)$ and $G + e$ is not well-covered. Thus there exists a maximal independent set of $G + e$, call it I, such that $|I| < \alpha(G)$. Since G is well-covered, either u or v (without loss of generality suppose u) is in I. But then I is maximal in $G-v$, since $v \notin I$. By Lemma 7, $\alpha(G-v) = \alpha(G)$ so $|I| < \alpha(G-v)$, which is a contradiction since G is in W_2. Therefore if G is in W_2, then G is stable well-covered.

Hence, G is stable well-covered iff G is in W_2. ∎

Thus the class of stable well-covered graphs is equivalent to the class of W_2 and 1-well-covered graphs. Now that we know this, it is clear that Theorem 3 and Theorem 5 are actually special cases of more general results in [10] for the maximum and minimum degree in W_n graphs. Note that if you know that the independence number of the graph is at least three, the Staples result gives a stronger bound on the maximum degree. We leave the proofs of our special cases here to illustrate alternate approaches, specifically how Theorem 4 may be utilized to prove theorems about stable well-covered/W_2 graphs. Knowing that stable well-covered, W_2 and 1-well-covered graphs are equivalent subclasses, gives us multiple ways to approach proving properties of this family of graphs.

For further results comparing stable well-covered graphs with the well-dominated and $\alpha = \gamma$ subclasses of well-covered graphs, see [4].

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