Competition Graphs of Acyclic Digraphs Satisfying Condition $C^*(p)$

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Abstract

Given a digraph D, its competition graph $C(D)$ has the same vertex set as D and an edge between two vertices x and y if there is a vertex u so that (x, u) and (y, u) are arcs of D. Motivated by a problem of communications, Kim and Roberts [2002] studied the competition graphs of the special digraphs known as semiorders and the graphs arising as competition graphs of acyclic digraphs satisfying conditions so called $C(p)$ or $C^*(p)$. While they could completely characterized the competition graph of an acyclic digraph satisfying $C(p)$, they obtained only partial results on $C^*(p)$ and left the general case open. In this paper, we answer their open question.

Keywords: Competition Graph, Competition Number, $C(p)$, $C^*(p)$

1 Introduction

Throughout this paper, we only consider simple graphs and simple digraphs. Suppose $D = (V, A)$ is a digraph (for all undefined graph-theoretical terms, see [1]). Its *competition graph* $G = C(D)$ has the same vertex set and has an edge xy if for some vertex $u \in V$, the arcs (x, u) and (y, u) are in D. If G is any graph, then adding sufficiently many isolated vertices produces a competition graph of an acyclic digraph ([10]). The smallest k so that $G \cup I_k$ is a competition graph of an acyclic digraph is called the *competition number* of G and is denoted $k(G)$. Clearly, then, $k(G) \geq 1$ whenever G is connected and has more than one vertex. The notion of competition graph arose from a problem in ecology and has since found application in problems of coding, channel assignment in communications, scheduling,

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and the modeling of complex systems arising in the study of energy and economic systems. (See [9] and [11] for details.) The long literature of competition graphs is summarized in several survey papers, [4], [7], [11]. There have been a number of papers about competition graphs of specific classes of digraphs. For instance, competition graphs of strongly connected digraphs have been studied in [2], of Hamiltonian digraphs in [2] and [3], of interval digraphs in [6], and for various classes of symmetric digraphs in [8] and [9]. In the same context, motivated by a problem of communications, Kim and Roberts [5] characterized the competition graphs of the special digraphs known as semiorders. They defined conditions on digraphs called $C(p)$ and $C^*(p)$ and studied the graphs arising as competition graphs of acyclic digraphs satisfying conditions $C(p)$ or $C^*(p)$.

To define conditions $C(p)$ and $C^*(p)$ on a digraph $D = (V, A)$, we need a relation on V: Given a, $b \in V$, we say a *supervises* b if $(b, u) \in A$ for $u \in V$ implies $(a, u) \in A$.
If $p \geq 2$ is an integer, we say that D satisfies *condition* $C(p)$ if whenever S is a set of p vertices of D, there is a vertex x in S so that x is supervised by every vertex in $S \setminus \{x\}$. A variant $C^*(p)$ of condition $C(p)$ is defined as follows: If $p \geq 2$ is an integer, we say that D satisfies *condition* $C^*(p)$ if whenever S is a set of p vertices of D, then there is a vertex x in S so that x supervises every vertex in $S \setminus \{x\}$. Kim and Roberts [5] characterized the competition graph of an acyclic digraph satisfying condition $C(p)$ completely. However, for the condition $C^*(p)$, they gave only partial results: They characterized the competition graphs of digraphs satisfying condition $C^*(p)$ for $p = 2, \ldots, 5$. The rest of this paper is devoted to characterizing the competition graphs of acyclic digraphs satisfying condition $C^*(p)$ for general p.

We begin by presenting some simple but useful properties on $C^*(p)$ given by Kim and Roberts [5].

In the rest of this paper, it is assumed that any digraph has no loops and therefore the competition graph of a digraph is a simple graph.

If there is a vertex x in the set S that supervises every vertex in $S \setminus \{x\}$, we call x a *head* of S and denote any such vertex by $h(S)$ by a somewhat ambiguous notation. (If there is more than one head, the context will tell us which is denoted by $h(S)$.)

Proposition 1.1 ([5]) *If $p < q$, then $C^*(p)$ implies $C^*(q)$.*

For a vertex set X of G and a vertex v of G, we mean by $v \overset{*}{\sim} X$ that v is adjacent to every non-isolated vertex in $X \setminus \{v\}$.

Proposition 1.2 ([5]) *If $G = C(D)$ for some digraph D satisfying the condition $C^*(p)$ and $S \subset V(G)$ with $|S| = p$, then:*

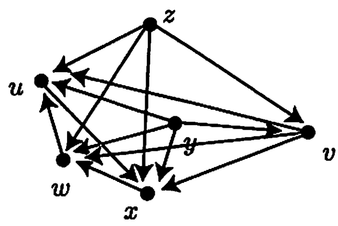

Figure 1: A digraph D satisfying $C^*(4)$. If either y or z is in a 4-element set, then it is a head. The set $\{u, v, w, x\}$ is the only 4-element set not containing y or z and v is its head. However, D does not satisfy $C^*(3)$. To see why, take 3-element set $\{u, w, x\}$. Since u, w, x form directed cycle $u \rightarrow x \rightarrow w \rightarrow u$, none of them can be a head.

(1) $h(S) \overset{*}{\sim} S$.

(2) If S has any vertices not isolated in G, then $h(S)$ cannot be isolated in G.

Lemma 1.3 ([5]) *Let D be a digraph satisfying condition $C^*(p)$, $G = C(D)$, and q be the number of isolated vertices in G. Then:*

(1) The size of an independent set T of vertices none of which is isolated in G is at most $\max\{1, p - q - 1\}$.

(2) If G has an independent set T of exactly $p - q - 1 > 1$ vertices that are not isolated in G, then every vertex outside of T not isolated in G is adjacent to every vertex of T and every pair of vertices not isolated in G other than vertices of T are adjacent.

2 Main Results

Throughout the rest of the paper, for a graph G with n vertices, G^C means the complement of a graph G, i.e., $G^C = K_n - E(G)$. In addition, for a vertex set W of a graph G (resp. digraph D), $\overline{W}$ means $V(G) \setminus W$ (resp. $V(D) \setminus W$) and $G[W]$ means the subgraph of G induced by W.

Given a graph G, we will use the following notations:

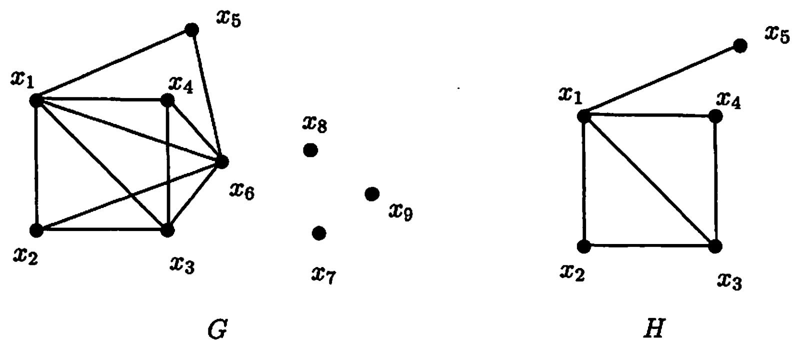

Figure 2: A branch set $U = \{x_1, x_2, x_3, x_4, x_5\}$ and the subgraph H of G induced by U

$$\begin{aligned} V_G^* &= \{v \in V(G) \mid \deg_G(v) \geq 1\}; \\ W_G &= \{v \in V(G) \mid 1 \leq \deg_G(v) \leq |V_G^*| - 2\}; \\ I_G &= \{v \in V(G) \mid \deg_G(v) = 0\}. \end{aligned}$$

Note that $V_G^* \setminus W_G = \{v \in V(G) \mid v \overset{*}{\sim} V_G^*\}$ and that $V(G)$ is the disjoint union of $(V_G^* \setminus W_G)$, W_G, and I_G.

Lemma 1.3 can be generalized as follows:

Lemma 2.1 *Let D be a digraph satisfying condition $C^*(p)$, $G = C(D)$, $|V_G^*| = r$, and $|I_G| = q$. Suppose that $q \leq p - 1$. Then the size of W_G is at most $p - q - 1$.*

Proof. By contradiction. Suppose that $|W_G| > p-q-1$. Let $S = W_G \cup I_G$. Then $|S| \geq p$. Let $|S| = t$. Since D satisfies $C^*(t)$ by Proposition 1.1, S has a head $h(S)$ that must be contained in W_G. Since any vertex in W_G is not isolated in G, $h(S) \overset{*}{\sim} W_G$ by Proposition 1.2 (1). Furthermore, since every vertex not in S has degree $r - 1$, it is adjacent to all vertices in W_G and therefore $h(S) \overset{*}{\sim} \overline{S}$. Thus $h(S) \overset{*}{\sim} V_G^*$ and so it has degree $r - 1$, which is a contradiction. Hence $|W_G| \leq p - q - 1$ and the lemma follows. □

We will characterize the competition graph G of an acyclic digraph satisfying condition $C^*(p)$ in terms of a set U of non-isolated vertices of G such that for any $v \in V_G^* - U$, $v \overset{*}{\sim} V_G^*$. We call such a set a *branch set* of G. (See Figure 2 for illustration.)

The following follows from Lemma 2.1:

Lemma 2.2 *Suppose that $p \geq 2$ and G is the competition graph of a digraph D satisfying the condition $C^*(p)$ with $|V_G^*| = r$ and $|I_G| = q$ where $r \geq p-q > 1$. Then there is a branch set of G with size $p-q-1$. Moreover, if $|W_G| = p-q-1$, then W_G is the unique branch set that has $p-q-1$ elements.*

Proof. Since $r \geq p-q > 1$, it is true that there are at least $p-q$ vertices of degree at least one. By Lemma 2.1, $|W_G| \leq p-q-1$. If $|W_G| = p-q-1$, then W_G is a branch set of size $p-q-1$ and we are done. Moreover if U is a branch set distinct from W_G, then there is an element $y \in W_G \setminus U$. By the definition of W_G, y is a non-isolated vertex of degree at most $r-2$, which contradicts the assumption that U is a branch set. Thus W_G is a unique branch set of size $p-q-1$. Now suppose that $|W_G| < p-q-1$. Then $V_G^* - W_G$ has at least $p-q-|W_G|$ vertices. For a subset J of $V_G^* - W_G$ with size $p-q-|W_G|-1$, it can easily be checked that $W_G \cup J$ is a branch set with size $p-q-1$. □

By definition, it is obvious that the set W_G itself is a branch set and that every branch set includes W_G.

The *join* $G \vee H$ of two disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in E(G)\}$.

Lemma 2.3 *Suppose that G is a graph with $|V_G^*| = r$ and $|I_G| = q$. Let U be a branch set of G and $H = G[U]$. Then $G = (K_{r-l} \vee H) \cup I_G$ where $|U| = l$.*

Proof. Since every vertex not in $U \cup I_G$ is adjacent to every vertex in $G - I_G$ except itself, it is true that $G = (K_{r-l} \vee H) \cup I_G$. □

Lemma 2.3 tells us that the structure of the competition graph of a digraph satisfying $C^*(p)$ is determined by that of the subgraph induced by a branch set. In the following, we will give necessary conditions for a graph being the competition graph of a digraph satisfying $C^*(p)$ in terms of its branch sets.

If there is an arc from a vertex in a vertex set S to a vertex in a vertex set T in a digraph, then we say that *there are arcs from S to T* for short. Given a vertex set S of a digraph D, we denote the out-neighborhood of S by

$$N_D^+(S) = \{v \in \overline{S} \mid \text{ there is an arc from a vertex in } S \text{ to } v\}.$$

The following lemma shows that the competition graph of an acyclic digraph D satisfying condition $C^*(p)$ has a branch set of size $p-q-1$ with at most q out-neighbors in D:

Lemma 2.4 *Suppose that $p \geq 2$ and G is the competition graph of a digraph D satisfying the condition $C^*(p)$ with $|V_G^*| = r$ and $|I_G| = q$ where*

$r \geq p-q>1$. Then there exists a branch set U of G with size $p-q-1$ satisfying $|N_D^+(U)| \leq q$.

Proof. Since $r \geq p-q>1$, there is a branch set of size $p-q-1$ by Lemma 2.2. Let U be a branch set of size of $p-q-1$ such that $|N_D^+(U)|$ is as small as possible. Suppose that $|N_D^+(U)| > q$. We will reach a contradiction. Since $|N_D^+(U)| > q$, there is a vertex $w \in \overline{U \cup I_G}$ such that (u,w) is an arc in D for some $u \in U$. Let $T = U \cup I_G \cup \{w\}$. Then $|T| = p$ since $w \notin U \cup I_G$. Since D satisfies the property $C^*(p)$, T has a head $h(T)$. Since w has in-neighbor $u \in T$, $h(T)$ cannot be w. For otherwise there would be a loop incident to w in D by the definition of a head. Since w is a non-isolated vertex not belonging to branch set U, $w \overset{*}{\sim} V_G^\star$ and so, by Proposition 1.2 (1), $h(T) \overset{*}{\sim} V_G^\star$. Let $U' = U \cup \{w\} \setminus h(T)$. Since $h(T) \overset{*}{\sim} V_G^\star$, U' is still a branch set of size $p-q-1$.

We claim by contradiction that there is no arc from a vertex in U' to head $h(T)$. Assume that there is an arc from a vertex v in U' to head $h(T)$. Then $v \in U' \subset T$. By the definition of head, there is an arc $(h(T), h(T))$ in D, which is a contradiction. Thus there is no arc from a vertex in U' to head $h(T)$. Hence $h(T) \notin N_D^+(U')$. In addition, since $N_D^+(w) \subset N_D^+(h(T)) \setminus \{w\}$ and $w \in N_D^+(U)$,

$$N_D^+(U') \subset N_D^+(U) \setminus \{w\}.$$

This contradicts the choice of U and the proof is complete. □

Lemma 2.5 *Suppose that $p \geq 2$ and G is the competition graph of an acyclic digraph D satisfying the condition $C^*(p)$ with $|V_G^\star| = r$ and $|I_G| = q$ where $r \geq p-q>1$. Let U be a branch set of G with size $p-q-1$, $H = G[U]$, and $I_H \neq \emptyset$. Then the following hold:*

(1) $N_D^+(U) \subset I_G$;

(2) If $N_D^+(V_H^\star) \cap I_H = \{x_1, x_2, \ldots, x_s\}$ ($s \geq 1$), then there exist distinct vertices $z_1, z_2, \ldots, z_s$ in $U \cup I_G \setminus I_H$ such that there is a directed (x_i, z_i)-path P_i in D such that every internal vertex of P_i belongs to I_H and z_i has an in-neighbor in I_H for each $i = 1, \ldots, s$.

Proof. We show (1) by contradiction. Suppose that $N_D^+(U) \not\subset I_G$. Then there exists $v \in N_D^+(U) \setminus I_G$. Then $v \notin U \cup I_G$ and, by the definition of U, $v \overset{*}{\sim} U$. Let $T = U \cup I_G \cup \{v\}$. Since $|T| = p$, T has a head $h(T)$. Since v has an in-neighbor in T, $h(T) \neq v$. Since $v \overset{*}{\sim} U$, it is true that $h(T) \overset{*}{\sim} U$ by Proposition 1.2 (1). This implies that H has no isolated vertices, which contradicts the hypothesis that H has an isolated vertex. Thus, $N_D^+(U) \subset I_G$.

Now we prove that (2) holds. Since $I_H \subset U$, every vertex in I_H has degree at least 1 in G and so each has an out-neighbor in D. Since $N_D^+(U) \subset I_G$ by (1), each vertex in I_H has an out-neighbor in $U \cup I_G$. Take $x_i \in N_D^+(V_H^*) \cap I_H$ for $i = 1, \dots, s$ and denote it by x_{i_1}. Since $N_D^+(V_H^*) \cap I_H \subset I_H$, x_{i_1} has an out-neighbor x_{i_2} in $U \cup I_G$. Since D has no loop, $x_{i_1} \neq x_{i_2}$. If x_{i_2} is in $V_H^* \cup I_G$, then we rename it z_i and we are done. If x_{i_2} is in I_H, then x_{i_2} has an out-neighbor x_{i_3} in $U \cup I_G$. Since D is acyclic, x_{i_3} is distinct from x_{i_1} and x_{i_2}. We repeat this process for x_{i_3} and so on. Since I_H is finite and all of x_{i_j} are distinct, the process will eventually end up obtaining a vertex x_{i_t} in $V_H^* \cup I_G$. We have found z_i by renaming x_{i_t} z_i.

Suppose that $z_i = z_j$ for some $i \neq j$, $1 \leq i, j \leq s$. By definition, there exist directed (x_i, z_i)-path P_i and (x_j, z_j)-path P_j such that all the vertices other than z_i (resp. z_j) on P_i (resp. P_j) are in I_H. Let v be the first vertex common to P_i and P_j. Then the vertex immediately preceding v on P_i is adjacent to the one immediately preceding v on P_j in $C(D)$. However, since both of them belong to I_H, they are isolated in H and so not adjacent in H. Since H is an induced subgraph, they are not adjacent in $C(D)$. Thus we reach a contradiction. Hence all of z_i are distinct and this completes the proof of (2). □

Now we are ready to present a necessary condition for a graph being the competition graph of an acyclic digraph D satisfying the condition $C^*(p)$:

Theorem 2.6 *Suppose that $p \geq 2$ and G is the competition graph of an acyclic digraph D satisfying the condition $C^*(p)$ with $|V_G^*| = r$ and $|I_G| = q$ where $r \geq p - q > 1$. Then there is a branch set U of G with size $p - q - 1$ such that $k(H - I_H) \leq q$ for $H = G[U]$.*

Proof. By Lemma 2.4, there is a branch set U of size $p - q - 1$ satisfying $|N_D^+(U)| \leq q$. Suppose that

$$q < k(H - I_H).$$

If $N_D^+(V_H^*) = I_G$, then the competition graph of the subdigraph of D induced by $V_H^* \cup I_G$ is $V_H^* \cup I_G$. Then $k(H - I_H) \leq q$, which contradicts the assumption that $q < k(H - I_H)$. Thus

$$N_D^+(V_H^*) \setminus I_G \neq \emptyset. \tag{1}$$

If $I_H = \emptyset$, then $|N_D^+(U)| > q$ since $q < k(H)$. This contradicts the choice of U. Thus $I_H \neq \emptyset$. Then, by Lemma 2.5 (1), $N_D^+(U) \subset I_G$. Hence $N_D^+(V_H^*) \setminus I_G \subset I_H$. This and (1) imply that $N_D^+(V_H^*) \cap I_H \neq \emptyset$. Let $N_D^+(V_H^*) \cap I_H = \{x_1, x_2, ..., x_s\}$. Then by Lemma 2.5 (2), there exist vertices z_1, z_2, …, z_s in $V_H^* \cup I_G$ such that there is a directed (x_i, z_i)-path in D and z_i has an in-neighbor in I_H for each $i = 1, \dots, s$.

Let D^* be the subdigraph of D induced by the vertex set $U \cup I_G$. Since $N_D^+(U \cup I_G) = \emptyset$, we have $C(D^*) = H \cup I_G$. Now we construct a digraph D^{**} as follows: Let

$$V(D^{**}) = V(D^*).$$

Then we let

$$\begin{aligned} A(D^{**}) =& A(D^*) - \{(v, x_i) \,|\, (v, x_i) \in A(D^*), i = 1, 2, ..., s\} \\ & - \{(v, z_i) \,|\, (v, z_i) \in A(D^*), i = 1, 2, ..., s\} \\ & \cup \{(v, z_i) \,|\, (v, x_i) \in A(D^*), i = 1, 2, ..., s\}. \end{aligned}$$

Then the acyclic labelling for D is still valid for D^{**} since newly added arcs still go from higher indices to lower indices. We shall claim that $C(D^{**} - I_H) = (H - I_H) \cup I_G$ in the following. Suppose that there are vertices y_1 and y_2 in H such that arcs (y_1, z_i) and (y_2, z_i) for some $i \in \{1, 2, \ldots, s\}$ are in D^*. Since there is an in-neighbor x of z_i that belongs to I_H, it is true that x, y_1, y_2 form a clique in $C(D^*)$, which contradicts the fact that x is isolated in H. Thus deleting the arcs in $\{(v, z_i) \,|\, (v, z_i) \in A(D^*)\}$ from D^* does not delete any edge in H in the competition graph of the resulting digraph. The arcs in $\{(v, x_i) \,|\, (v, x_i) \in A(D^*)\}$ are replaced by the ones in $\{(v, z_i) \,|\, (v, x_i) \in A(D^*)\}$ in D^{**}. Thus $C(D^{**} - J_H) = (H - J_H) \cup I_G$ where $J_H = N^+(V_H^*) \cap I_H$. In addition, since any vertex in $I_H \setminus J_H$ has no incoming arcs from a vertex in U by the definition of J_H,

$$\begin{aligned} C(D^{**} - I_H) &= C((D^{**} - J_H) - I_H \setminus J_H) = ((H - J_H) - I_H \setminus J_H) \cup I_G \\ &= (H - I_H) \cup I_G. \end{aligned}$$

This implies that $k(H - I_H) \leq q$, which contradicts the assumption that $k(H - I_H) > q$. □

The following theorem characterizes the competition graphs of acyclic digraphs satisfying the condition $C^*(p)$. We denote by I_q the set of q isolated vertices.

Theorem 2.7 *Suppose that $p \geq 2$ and G is a graph with $|V_G^*| = r$ and $|I_G| = q$. Then G is the competition graph of an acyclic digraph satisfying the condition $C^*(p)$ if and only if G is one of the following graphs:*

(1) I_q where $q > 0$;

(2) $K_r \cup I_q$ where $r > 1$ and $q > 0$;

(3) $H \cup I_q$ where H is a graph without isolated vertices, $|V(H)| = r < p - q$, and $0 < k(H) \leq q$;

(4) $(K_{r-p+q+1} \vee H) \cup I_q$ where H is a graph with $p - q - 1$ vertices, $r \geq p - q$, and $0 < k(H - I_H) \leq q$.

Proof. To show the 'only if' part, suppose that G is the competition graph of D satisfying condition $C^*(p)$. By the acyclicity of D, G has at least one isolated vertex. If G does not have edges, then $G = I_G = I_q$. Now we suppose that there are at least two non-isolated vertices (it is impossible for a graph to have exactly one non-isolated vertex), that is, $r \geq 2$. If $|V(D)| \leq p$, then D satisfies the condition $C^*(p)$ (vacuously). Since G has r non-isolated vertices and q isolated vertices, it is true that $G = H \cup I_q$ for a graph H without isolated vertices and $|V(H)| = r$. Since $C(D) = G = H \cup I_q$, $k(H) \leq q$, by the definition of competition number. Since $|V(D)| = r + q$, it is true that $r + q < p$ and so $r < p - q$. Thus G is of Type (3) if $|V(D)| \leq p$. Now suppose that $|V(D)| \geq p$. Then, since $|V(D)| = r + q$, it is true that $r \geq p - q$. If $p - q - 1 \leq 1$, then G is of Type (2) since the maximum size of independent set of vertices none of which is isolated in G is 1 by Lemma 1.3. Thus it remains to consider the case $p - q - 1 > 1$. By Theorem 2.6, there is a branch set U of size $p - q - 1$ such that $k(H - I_H) \leq q$ for $H = G[U]$. Then, by Lemma 2.3, $G = (K_{r-p+q+1} \vee H) \cup I_q$.

Now we show the converse. We will construct an acyclic digraph D satisfying the condition $C^*(p)$ for each type of graph.

If G is of Type (1), then D with vertex set $V(G)$ and the empty arc set vacuously satisfies $C^*(p)$, and $C(D) = G$.

If $G = K_r \cup I_q$ for $r > 1$ and $q > 0$, then define an acyclic digraph D as follows: $V(D) = V(G)$ and $A(D) = \{(x, y) | x \in K_r, y \in I_q\}$. We can easily check that $G = C(D)$ and D satisfies the condition $C^*(p)$.

Suppose that G is of Type (3). Since $k(H) \leq q$, there exist acyclic digraphs D with $C(D) = (H) \cup I_q$. Since $r < p - q$, it is true that $|V(D)| < p$ and so this digraph D satisfies the condition $C^*(p)$ vacuously.

Finally suppose that G is of Type (4). Since $k(H - I_H) \leq q$, there is an acyclic digraph D' such that $C(D') = (H - I_H) \cup I_q$. Without loss of generality, we may assume that D' is minimal among such digraphs. Let

$$\Gamma = \{(x, y) \mid x \in \overline{V(H) \cup I_q}, y \in V(H) \cup I_q\}.$$

We first suppose that $I_H \neq \emptyset$. Let $I_H = \{i_1, i_2, \ldots, i_l\}$ and a be a vertex in I_q. We define a digraph D as follows: Let

$$V(D) = V(G).$$

Then we let

$$\begin{aligned} A(D) =&\Gamma \cup A(D') - \{(v, a) \mid (v, a) \in A(D')\} \cup \{(v, i_l) \mid (v, a) \in A(D')\} \\ &\cup \{(i_{j+1}, i_j) \mid j = 1, 2, \ldots, l - 1\} \cup \{(i_1, a)\}. \end{aligned}$$

(See Figure 3 for an illustration.)

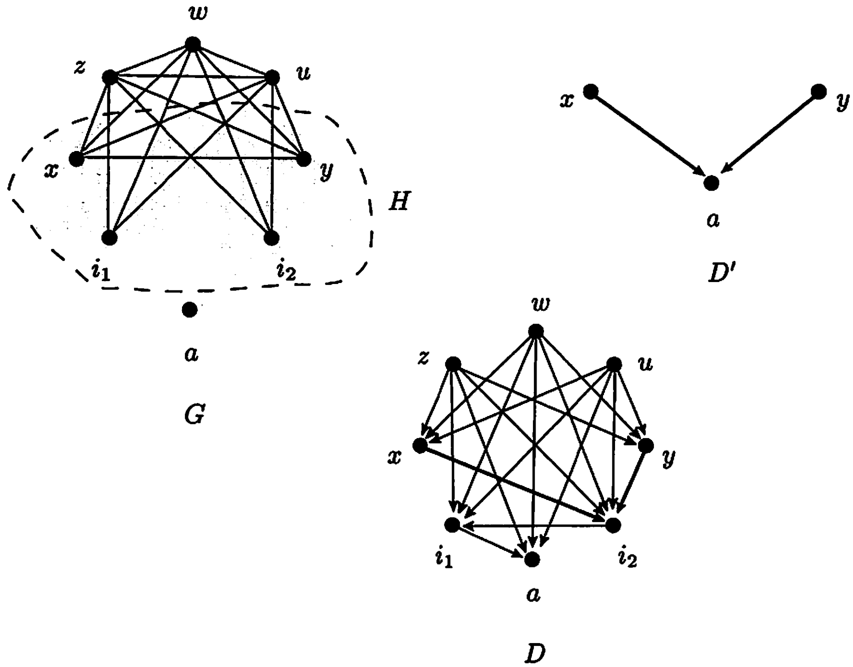

Figure 3: Given a subgraph L of K_7 with 4 vertices x, y, i_1, i_2 and $k(H - I_H) = 1$, an acyclic digraph D satisfying the condition $C^*(6)$ whose competition graph is $(K_3 \vee H) \cup \{a\}$ constructed as indicated in the proof of Theorem 2.7.

From the fact that no arcs from a vertex in D' to a vertex in D' are added and the way in which the arcs are added, it can easily be checked that D is still acyclic. Now take a subset T of $V(D)$ with $|T| = p$. Then it contains a vertex v in $V(H) \cup I_q$ since $|V(H) \cup I_q| = p - 1$. It is clear from the construction of D that v is a head of T. Thus D satisfies the condition $C^*(p)$.

It is rather tedious but not difficult to check that $E(C(D)) = E(G)$. □

The results on $C^*(p)$ ($p = 1, \ldots, 5$) in [5] follow from the above theorem in a much simpler way than in [5] as shown in the following.

Corollary 2.8 ([5]) *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C^*(5)$ if and only if $G = I_q$ or $G = K_r \cup I_q$ for $r > 1, q > 0$ or $G = K_r - e \cup I_2$ for $r > 2$, or $G = K_r - P_3 \cup I_1$ for $r > 3$ or $G = K_r - K_3 \cup I_1$ for $r > 3$.*

Proof. Suppose that G is the competition graph of an acyclic digraph satisfying condition $C^*(5)$. Let r and q be the numbers of vertices of degree at least one and isolated vertices, respectively. Then G is of one of the four types in Theorem 2.7. If G is of Type (1), then $G = I_q$ for $q > 0$. If G is of Type (2), then $G = K_r \cup I_q$ for $r > 1$, $q > 0$.

If G is of Type (3), then $r < 5 - q$. Since $q \geq 1$, it is true that $r < 4$, and so $r = 2$ or 3. If $r = 2$, then $G = I_q$ if $L = K_2$ and $G = K_2 \cup I_q$ if L is an empty graph for $q > 0$. If $r = 3$, then $G = K_3 \cup I_1$ if L is an empty graph and $G = K_3 - e \cup I_1$ if L has only one edge.

If G is of Type (4), then $G = (K_{r-5+q+1} \vee H) \cup I_q$ where $5 - q - 1 > 0$. Thus $q = 1$, 2, or 3. Then $|V(H)| = 5 - q - 1 = 3$, 2, or 1. Now if $q = 3$, then H is a trivial graph; if $q = 2$, then $H = I_2$ or K_2; if $q = 1$, then $H = I_3$, $P_2 \cup I_1$, P_3, or K_3. Hence $G = K_r \cup I_3$ for $r \geq 2$ or $G = K_r \cup I_2$ for $r \geq 3$ or $G = K_r - e \cup I_2$ for $r \geq 3$ or $G = K_r - K_3 \cup I_1$ for $r \geq 4$ or $G = K_r - P_3 \cup I_1$ for $r \geq 4$ or $G = K_r - e \cup I_1$ for $r \geq 4$ or $G = K_r \cup I_1$ for $r \geq 4$.

Since G is of one of the types given in Theorem 2.7, the converse holds. □

3 Closing Remarks

In this paper, we completely characterize the competition graph of an acyclic digraph satisfying the condition $C^*(p)$. This answers an open question given by Kim and Roberts [5].

It seems to be interesting to characterize the competition graph of a digraph satisfying the condition $C^*(p)$. The lemmas 2.1-2.4 are still valid for digraphs satisfying the condition $C^*(p)$ without the acyclicity being guaranteed. In addition, characterization of the competition graph of a digraph satisfying the condition $C(p)$ remains open.

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