ON $(K_4, K_4 - e)$-DESIGNS

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ABSTRACT. In [5], the first author posed the problem of determining the spectrum of $(K_4, K_4 - e)$-designs. In this article, we solve this problem, and also determine the spectrum of $(K_4, K_4 - e)$-designs with exactly one K_4 (or, equivalently, the spectrum of $(K_4 - e)$-designs with a hole of size 4). We also improve the bound for embedding a partial $S(2,4,v)$ into a $(K_4, K_4 - e)$-design given in [5].

1. INTRODUCTION

A K_4-design [a $(K_4 - e)$-design, and a $(K_4, K_4 - e)$-design, respectively] of order n is a decomposition of the complete graph K_n into copies of K_4, the complete graph on 4 vertices [into copies of $K_4 - e$, the complete graph on 4 vertices from which one edge was deleted, and into copies of K_4 or $K_4 - e$, respectively]. Of course, a K_4-design of order n is the same as a Steiner system $S(2,4,n)$. In [5], the first author obtained a polynomial size embedding of a partial Steiner system $S(2,4,v)$ into a $(K_4, K_4 - e)$-design, i.e. a decomposition of the complete graph into copies of K_4 or $K_4 - e$. He also posed the problem of determining the spectrum for $(K_4, K_4 - e)$-designs, i.e. the set of all orders for which there exists a $(K_4, K_4 - e)$-design. For the sake of brevity, from now on a $(K_4, K_4 - e)$-design will be termed a K^*-design, and the spectrum will be denoted by $\mathcal{K}^*$. Thus $\mathcal{K}^* = \{n :$ there exists a K^*-design of order $n\}$.

A K^*-design (or a $(K_4 - e)$-design) of order n with a hole of size m, sometimes termed an incomplete (n,m)-design, is a partition of edges of $K_n \setminus K_m$ into copies of K_4 or $K_4 - e$ (into copies of $K_4 - e$, respectively).

In this article, we prove the following.

Theorem A. $\mathcal{K}^* = \{n : n \geq 4, n \notin \{5,7,8,9\}\}$.

We also determine the spectrum for K^*-designs with a unique K_4-block (or, alternatively, the spectrum for $(K_4 - e)$-designs with a hole of size 4); this spectrum will be denoted by $\mathcal{K}_1{}^*$, and a K^*-design with a unique K_4-block will be called a K_1^*-design.

Theorem B. $\mathcal{K}_1{}^* = \{n\text{: } n \equiv 2 \text{ or } 4 \ (mod\ 5),\ n \geq 4,\ n \neq 7, 9\}$.

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In a K^*-design, a K_4-block will be indicated by figure brackets, e.g. $\{a,b,c,d\}$ while a (K_4-e)-block will be indicated by square brackets, e.g. $[a,b,c,d]$ with $\{c,d\}$ the omitted edge. It follows that $[a,b,c,d]$, $[a,b,d,c]$, $[b,a,c,d]$, $[b,a,d,c]$ all denote the same (K_4-e)-block.

We recall a classical result of Hanani [3] and a well-known result from [1].

Theorem 2.1. *An $S(2,4,n)$ (i.e., a K_4-design of order n) exists if and only if $n \equiv 1$ or 4 (mod 12).*

Theorem 2.2. *A (K_4-e)-design of order n exists if and only if $n \equiv 0$ or 1 (mod 5), $v \neq 5$.*

Next we record the following obvious lemma.

Lemma 2.3. *The following are equivalent:*
(i) There exists a K^*-design of order n with exactly one K_4-block;
(ii) There exists a (K_4-e)-design of order n with a hole of size 4;
(iii) There exists a (K_4-e)-design with a hole of size 2.

Lemma 2.4. *There exists no K^*-design of order n for $n = 7, 8, 9$.*

Proof. (i) $n = 7$: since K_7 has 21 edges, any K^*-design of order 7 necessarily contains one K_4-block and three (K_4-e)-blocks. If $\{a,b,c,d\}$ is the K_4-block then each of a,b,c,d must occur in a (K_4-e)-block as a vertex of degree 3 (since K_7 is regular of degree 6), and no two of a,b,c,d may appear as adjacent vertices in the same (K_4-e)-block, as they are already adjacent in the unique K_4-block. But there are only three (K_4-e)-blocks, a contradiction.

(ii) $n = 8$: the only solution in nonnegative integers of the equation $6x + 5y = 28$ is $x = 3, y = 2$, thus a K^*-design of order 8 would necessarily consist of three K_4-blocks and two (K_4-e)-blocks. But it is impossible to pack three K_4-blocks in a K_8.

(iii) $n = 9$: in a K^*-design of order 9 either there are six (K_4-e)-blocks and one K_4-block, or there are six K_4-blocks. The latter is impossible since there exists no S(2,4,9). So assume the former, and let $A,B,C,D,1,2,3,4,5$ be the vertices of K_9, and let $\{A,B,C,D\}$ be the K_4-block. Then there are two more (K_4-e)-blocks through each of A,B,C,D; say, w.l.o.g., $[A,1,2,3]$ and $[4,5,A,B]$ are the two (K_4-e)-blocks containing A. Now B has to occur in a (K_4-e)-block which requires a path with two edges on $\{1,2,3\}$ disjoint from the path $(2,1,3)$ - a contradiction. □

Lemma 2.5. $12 \in \mathcal{K}_1{}^*$, *i.e., there exists a K^*-design of order 12 with a unique K_4-block.*

Proof. The blocks of the design are $\{1,5,9,11\}$, and $[1,2,3,4]$, $[1,6,8,12]$, $[2,5,8,10]$, $[2,11,6,7]$, $[3,4,8,11]$, $[3,6,9,10]$, $[3,12,5,7]$, $[5,6,4,7]$, $[7,10,1,4]$, $[8,9,7,10]$, $[9,12,2,4]$, $[11,12,8,10]$. □

Lemma 2.6. $14 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_{13} \cup \{\infty\}$. The blocks of the design are $\{0,3,5,12\}$ (the unique K_4-block), and $[4,6,0,1]$, $[5,7,1,2]$, $[2,8,3,6]$, $[4,9,3,7]$, $[5,10,4,8]$, $[6,9,5,11]$, $[7,12,6,10]$, $[8,11,0,7]$, $[1,12,8,9]$, $[9,10,0,2]$, $[1,11,3,10]$, $[2,4,11,12]$, $[1,2,0,\infty]$, $[\infty,7,0,3]$, $[6,10,3,\infty]$, $[\infty,11,5,12]$, $[\infty,8,4,9]$. □

Lemma 2.7. $17 \in \mathcal{K}_1{}^*$.

Proof. With $V = \{0, 1, \ldots, 16\}$, the blocks are $[0,5,4,6], [7,8,0,4]$, $[9,10,0,5], [0,13,11,12], [0,16,14,15], [4,6,1,2], [5,7,1,2], [1,8,9,10]$, $[1,14,11,12], [1,13,15,16], [2,11,8,9], [2,16,10,12], [2,14,13,15]$, $[3,4,13,14], [3,5,11,16], [6,8,3,16], [9,12,3,6], [3,10,7,15]$, $[4,9,15,16], [4,10,11,12], 12,15,5,8], [5,8,13,14], [6,15,7,11]$, $[6,10,13,14], [7,9,13,14], [7,11,12,16]$, and $\{0,1,2,3\}$. □

Lemma 2.8. $19 \in \mathcal{K}_1{}^*$.

Proof. With $V = \{0, 1, \ldots, 18\}$, the blocks are $[0,5,4,6], [7,8,0,1]$, $[9,10,0,1], [11,12,0,1], [0,15,13,14], [0,18,16,17], [1,13,4,5], [1,16,6,14]$, $[1,15,17,18], [4,6,2,3], [5,7,2,3], [8,9,2,3], [2,10,11,12], [2,18,13,14]$, $[2,16,15,17], [3,14,10,11], [3,12,15,18], [3,13,16,17], [9,12,4,5]$, $[4,14,7,8], [4,10,15,17], [4,11,16,18], [5,15,8,11], [14,17,5,9]$, $[5,10,16,18], [6,15,7,9], [11,17,6,7], [6,8,10,18], [6,14,12,13], [7,9,16,18]$, $[7,13,10,12], [11,13,8,9], [8,12,16,17]$, and $\{0,1,2,3\}$. □

Lemma 2.9. $22 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_{18} \cup \{A, B, C, D\}$. The blocks are $[i, i+1, i+3, i+7]$, $i \in Z_{18}$, and $[A,8,0,17], [A,1,9,10], [A,2,11,12], [A,13,3,4], [A,14,5,6]$, $[A,7,15,16], [B,10,0,2], [B,11,1,3], [B,4,12,14], [B,5,13,15], [B,16,6,8]$, $[B,17,7,9], [C,14,0,10], [C,5,1,9], [C,2,6,16], [C,17,3,13], [C,8,4,12]$, $[C,11,7,15], [D,5,0,10], [D,14,1,9], [D,2,7,15], [D,8,3,13], [D,17,4,12]$, $[D,11,6,16], [0,9,4,13], [3,12,7,16], [6,15,1,10]$, and $\{A, B, C, D\}$. □

Lemma 2.10. $24 \in \mathcal{K}_1{}^*$.

Proof. With $V = \{0, 1, \ldots, 23\}$, the blocks are $[0,5,4,6], [7,8,0,4]$, $[9,10,0,4], [11,12,0,4], [13,14,0,5], [15,16,0,5], [17,18,0,5], [19,20,0,5]$, $[0,23,21,22], [4,6,1,2], [5,7,1,2], [8,9,1,2], [10,11,1,2], [12,13,1,3]$, $[14,15,1,3], [16,17,1,3], [1,21,18,19], [1,20,22,23], [2,15,12,13]$, $[2,14,16,17], [2,23,18,19], [2,21,20,22], [4,18,3,13], [3,5,8,9]$, $[3,19,6,7], [3,10,20,21], [3,11,22,23], [4,19,14,15], [4,16,20,21]$, $[4,17,22,23], [5,10,22,23], [5,21,11,12], [6,10,7,8], [9,11,6,7]$, $[12,20,6,7], [6,22,13,14], [6,21,15,17], [6,16,18,23], [7,13,16,21]$, $[7,18,14,22], [7,15,17,23], [13,23,8,9], [14,21,8,9], [8,17,11,12]$, $[8,22,15,16], [8,18,19,20], [9,12,16,18], [19,22,9,12], [9,20,15,17]$, $[15,18,10,11], 16,19,10,11], [12,14,10,23], [13,17,10,19], [11,20,13,14]$, and $\{0,1,2,3\}$. □

Lemma 2.11. $27 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_5 \times \{1,2,3,4,5\} \cup \{\infty_1, \infty_2\}$. The following 14 base blocks modulo 5 determine 70 $(K_4 - e)$-blocks of a $(K_4 - e)$-design with a hole of size 2:
$[0_2, 4_4, \infty_1, \infty_2], [\infty_1, 1_3, 0_1, 1_5], [\infty_2, 2_3, 0_1, 3_5], [1_1, 0_2, 0_1, 3_1], [1_2, 0_3, 0_2, 3_2]$, $[1_3, 0_4, 0_3, 3_3], [1_4, 0_5, 0_4, 3_4], [0_0, 0_5, 2_5, 4_5], [0_1, 1_2, 2_4, 4_4], [2_1, 3_4, 0_3, 2_3]$, $[0_4, 3_5, 0_1, 2_1], [0_2, 3_3, 0_5, 2_5], [0_4, 1_5, 0_2, 3_2], [3_2, 4_3, 0_1, 2_5]$.
Adjoining now the edge $\{\infty_1, \infty_2\}$ to any of the blocks of the first orbit produces the unique K_4-block which completes the proof. □

Lemma 2.12. $29 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_5 \times \{1,2,3,4\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following 16 base blocks yield the 80 $(K_4 - e)$-blocks of the desired $K_1{}^*$-design:
$[\infty_1, 1_3, 0_1, 4_2], [\infty_2, 4_3, 0_1, 0_2], [\infty_3, 0_3, 2_4, 2_5], [\infty_4, 0_3, 4_4, 4_5]$,
$[0_4, 3_5, \infty_1, \infty_2], [0_1, 1_2, \infty_3, \infty_4], [0_4, 0_5, 0_1, 0_2], [1_4, 3_5, 0_1, 2_2]$,
$[2_4, 1_5, 0_1, 4_2], [3_4, 4_5, 0_1, 1_2], [0_1, 3_2, 4_4, 2_5], [1_1, 3_3, 0_1, 3_1]$,
$[1_1, 1_2, 0_2, 3_2], [0_2, 1_3, 0_3, 3_3], [0_3, 1_4, 0_4, 3_4], [0_3, 1_5, 0_5, 3_5]$. □

Lemma 2.13. $37 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_{11} \times \{1,2,3\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following 12 base blocks yield the 132 $(K_4 - e)$-blocks of a $K_1{}^*$-design:
$[\infty_1, 6_2, 0_1, 0_3], [\infty_2, 8_2, 0_1, 1_3], [\infty_3, 0_1, 10_2, 10_3], [\infty_4, 6_3, 1_1, 0_2]$,
$[4_1, 7_2, 0_1, 9_1], [1_1, 3_1, 0_1, 5_2], [4_2, 7_3, 0_2, 9_2], [0_2, 1_2, 3_2, 2_3], [6_1, 4_3, 0_3, 9_3]$,
$[0_3, 1_3, 4_1, 3_3], [1_2, 1_3, 0_1, 1_1], [0_1, 5_2, 2_3, 4_3]$. □

Lemma 2.14. $39 \in \mathcal{K}_1{}^*$.

Proof. We will construct a $K_1{}^*$-design of order 39 with a sub-$K_1{}^*$-design of order 12 embedded in it. Let $V = Z_{27} \cup X$, where $X = \{x_i : i = 1,2, \ldots, 12\}$. It is an easy exercise to see that any of the circulants $C(27;1,13)$, $C(27;4,8)$, $C(27;5,11)$ and $C(27;3,6)$ can be decomposed into three 2-path factors (a 2-path factor in a graph G is a factor whose each component is a path with 2 edges). Let $F_i, i = 1,2,\ldots,12$, be these 2-path factors. For each x_i and a 2-path $(a,b,c) \in F_i$, form the $(K_4 - e)$-block $[x_i, b, a, c]$; this yields 108 $(K_4 - e)$-blocks. Adjoin to this the 27 $(K_4 - e)$-blocks obtained by developing modulo 27 the base block $[0,2,9,12]$, as well as the 12 $(K_4 - e)$-blocks of a $K_1{}^*$-design of order 12 (cf. Lemma 2.5) on the set X, for a total of 147 $(K_4 - e)$-blocks of a desired $K_1{}^*$-design of order 39. □

Lemma 2.15. $57 \in \mathcal{K}_1{}^*$.

Proof. We will construct a $K_1{}^*$-design of order 57 with a sub-$K_1{}^*$-design of order 24. Let $V = Z_{33} \cup X$ where $X = \{x_i : i = 1,2,\ldots,24\}$. The complete graph K_{33} on Z_{33} can be decomposed into 24 2-path factors (a solution to the handcuffed prisoners problem, see [4]).Let these 2-path factors be $F_1, \ldots, F_{24}$. Associate x_i, for $i = 1,2,\ldots,24$, with F_i the same way as in the proof of Lemma 2.14; this produces 24.11 $(K_4 - e)$-blocks. Adjoin to this the 54 $(K_4 - e)$-blocks of a $K_1{}^*$-design of order 24 on X. □

Lemma 2.16. $59 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_{11} \times \{1,2,3,4,5\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following 31 base blocks yield the 341 $(K_4 - e)$-blocks of the desired $K_1{}^*$-design of order 59:
$[\infty_1, 5_3, 0_1, 6_2], [\infty_2, 9_3, 0_1, 6_2], [0_4, 3_5, \infty_1, \infty_2], [\infty_3, 0_3, 6_4, 6_5]$,
$[\infty_4, 0_3, 10_4, 8_5], [0_1, 8_2, \infty_3, \infty_4], [0_1, 0_2, 6_4, 0_5], [10_4, 7_5, 6_1, 0_2]$,
$[0_2, 1_5, 2_1, 0_3], [7_2, 10_5, 5_1, 8_3], [2_3, 0_4, 0_1, 0_2], [10_3, 10_4, 0_1, 1_2], [0_2, 4_3, 7_4, 8_5]$,
$[0_2, 6_3, 3_4, 5_5], [5_4, 9_5, 0_2, 4_3], [2_4, 4_5, 0_2, 6_3], [0_1, 5_1, 2_5, 3_5], [0_2, 5_2, 1_1, 4_1]$,
$[0_3, 5_3, 10_1, 8_1], [0_4, 5_4, 9_1, 2_1], [0_5, 5_5, 10_4, 0_4], [1_1, 6_2, 0_1, 3_1], [1_2, 8_3, 0_2, 3_2]$,
$[1_3, 5_4, 0_3, 3_3], [1_4, 10_5, 0_4, 3_4], [1_1, 7_5, 0_1, 3_1], [3_1, 7_3, 0_1, 7_1], [3_2, 4_4, 0_2, 7_2]$,
$[3_3, 3_5, 0_3, 7_3], [3_1, 8_4, 0_1, 7_1], [3_2, 2_5, 0_2, 7_2]$. □

Lemma 2.17. $67 \in \mathcal{K}_1{}^*$.

Proof. Let $V = Z_{21} \times \{1,2,3\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following 21 base blocks yield the 441 $(K_4 - e)$-blocks of a $K_1{}^*$-design of order 67:
$[\infty_1, 2_2, 0_1, 1_3], [\infty_2, 4_2, 0_1, 14_3], [\infty_3, 0_1, 9_2, 14_3], [\infty_4, 0_3, 0_1, 0_2]$,
$[6_1, 19_2, 0_1, 13_1], [8_1, 15_2, 0_1, 18_1], [0_1, 3_1, 1_1, 20_2], [4_1, 9_1, 0_1, 20_2]$,
$[6_2, 8_3, 0_2, 13_2], [8_2, 1_3, 0_2, 18_2], [0_2, 1_2, 3_2, 7_3], [0_2, 4_2, 9_2, 13_3]$,
$[12_1, 6_3, 0_3, 13_3], [11_1, 8_3, 0_3, 18_3], [0_3, 1_3, 10_1, 3_3], [0_3, 4_3, 8_1, 9_3]$,
$[1_2, 20_3, 0_1, 1_1], [14_2, 5_3, 0_1, 2_1], [0_1, 3_2, 6_3, 8_3], [0_1, 5_2, 2_3, 16_3]$,
$[0_1, 4_3, 8_2, 10_2]$. □

Note that in Lemmas 2.12. 2.13, 2.16, and 2.17, $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ is the unique K_4-block.

Lemma 2.18. $69 \in \mathcal{K}_1{}^*$.

Proof. We construct a $K_1{}^*$-design of order 69 with a sub-$K_1{}^*$-design of order 12. Let $V = Z_{57} \cup X$ where $X = \{x_i : i = 1, 2, \ldots, 12\}$. It is easily verified that each of the circulants $C(57; 1, 2)$, $C(57; 18, 21)$, $C(57; 9, 24)$, $C(57; 14, 28)$ can be decomposed into three 2-path factors, for a total of 12 2-path factors; let $F_i : i = 1, \ldots, 12$ be these 12 2-path factors. Associate $x_i, i = 1, \ldots, 12$ with F_i the same way as in the proof of Lemma 2.14; one obtains $19.12 = 228$ $(K_4 - e)$-blocks. Adjoin to this the 228 $(K_4 - e)$-blocks obtained by developing modulo 57 the four base blocks $[0, 4, 10, 16], [5, 20, 0, 12], [0, 30, 17, 19], [23, 26, 0, 48]$, and the 12 $(K_4 - e)$-blocks of a $K_1{}^*$-design of order 12 on X. □

3. The main construction and K^*-designs with a unique block of size 4

A *commutative quasigroup with holes*, $(V, \circ, \mathcal{H})$, is a (finite) set V, $|V| = v$, $\mathcal{H} = \{H_1, H_2, \ldots, H_s\}$ is the set of *holes*, $|H_i| = h_i$, $H_i \cap H_j = \emptyset$ for $i \neq j$, $\Sigma_{i=1}^{s} h_i = v$, and $\circ$ is a binary operation on V defined for all $x, y \in V$ such that $x \in H_i$, $y \in H_j$ *and* $i \neq j$, and satisfying $x \circ y = y \circ x$. If $i = j$ then $x \circ y$ is undefined. In other words, $x \circ y$ is defined if and only if x and y belong to different holes.

It is well known that

(i) a commutative quasigroup of order v with all holes of size 2 exists if and only if $v \geq 6$ and $v \equiv 0$ $(mod\ 2)$ [7];

(ii) a commutative quasigroup of order $v \equiv 3$ $(mod\ 6)$ with all holes of size 3 exists if and only if $v \geq 9$;

(iii) a commutative quasigroup of order $v \equiv 1$ $(mod\ 6)$ with one hole of size 7 and all other holes of size 3 exists if and only if $v \geq 19$; and

(iv) a commutative quasigroup of order $v \equiv 5$ $(mod\ 6)$ with one hole of size 5 and all other holes of size 3 exists if and only if $v \geq 17$ [6]

Now we are able to formulate our main construction which when specified will enable us to prove our main results.

Main construction. *Let $(X, \circ, \mathcal{H})$ be a commutative quasigroup of order n with holes $\mathcal{H} = \{H_1, \ldots, H_s\}$, $|H_i| = h_i$, let T be a set, $T \cap X = \emptyset$, $|T| = t \leq 9$. Assume that for some $i \in \{1, 2, \ldots, s\}$, there exists a K^*-design of order $5h_i + t$, and for each $j \in \{1, 2, \ldots, s\}$, $j \neq i$, there exists a K^*-design of order $5h_j + t$ with a hole of size t. Then there exists a K^*-design of order $5n + t$.*

Proof. Let $(H_i \times Z_5 \cup T, \mathcal{C}_i)$ be a K^*-design of order $5h_i + t$, and for each $j \neq i$, let $(H_j \times Z_5, \mathcal{C}_j)$ be a K^*-design of order $5h_j + t$ with a hole of size t on T, and let $\mathcal{C} = \bigcup_{k=1}^{s} \mathcal{C}_k$. For any $x \in H_k, y \in H_l, k \neq l$ (i.e. for any x, y from different holes), let $\mathcal{B}_{xy} = \{[(x,i),(y,i),(x \circ y, i+1),(x \circ y, i+2)] : i \in Z_5\}$, and let $\mathcal{B} = \bigcup \mathcal{B}_{xy}$. Then $(X \times Z_5 \cup T, \mathcal{B} \cup \mathcal{C})$ is a K^*-design of order $5|X| + |T|$.

The first application of the Main Construction is the following.

Lemma 3.1. *Let $n \equiv 2$ (mod 10), $n \geq 12$. Then $n \in \mathcal{K}_1{}^*$.*

Proof. Let $n = 10k + 2$. The statement is certainly true for $k = 1, 2$ since a $K_1{}^*$-design of order 12 and 22 is given in Lemma 2.5 and Lemma 2.9, respectively. Thus we may assume $k \geq 3$, i.e., $n \geq 32$. Let $(X, \circ, \mathcal{H})$ be a commutative quasigroup of order $2k$ with k holes H_i, each of size 2. Set $V = (X \times Z_5) \cup \{\infty_1, \infty_2\}$, $V_i = (H_i \times Z_5) \cup \{\infty_1, \infty_2\}$, $i = 1, 2, \ldots, k$. Let $(V_1, \mathcal{B}_1)$ be a $K_1{}^*$-design of order 12 such that the pair $\{\infty_1, \infty_2\}$ is contained in the unique K_4-block. Let $(V_i, \mathcal{B}_i), i = 2, 3, \ldots, k$, be a $(K_4 - e)$-design of order 12 with a hole of size 2 on $\{\infty_1, \infty_2\}$. It is straightforward to see that the Main Construction yields a $K_1{}^*$-design of order n. □

We can proceed in a very similar manner when $n \equiv 4$ (*mod* 10).

Lemma 3.2. *Let $n \equiv 4$ (mod 10). Then $n \in \mathcal{K}_1{}^*$.*

Proof. Let $n = 10k + 4$. The statement is trivial for $k = 0$, and for $k = 1, 2$, the corresponding $K_1{}^*$-design is given in Lemma 2.6 and Lemma 2.10, respectively. So assume $k \geq 3$, i.e. $n \geq 34$. As before, let $(X, \circ, \mathcal{H})$ be a commutative quasigroup of order $2k$ with k holes, each of size 2. Set now $V = (X \times Z_5) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, and use the $K_1{}^*$-design of order 14 given in Lemma 2.6, making sure that $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ is the unique K_4-block (or the hole, respectively). □

Lemma 3.3. *Let $n \equiv 7$ (mod 10), $n > 7$. Then $n \in \mathcal{K}_1{}^*$.*

Proof. Let $n = 10k + 7$. A $K_1{}^*$-design of order 7 does not exist by Lemma 2.4, while $K_1{}^*$-design of order $n \in \{17, 27, 37, 57, 67\}$ is given in Lemmas 2.7, 2.11, 2.13, 2.15 and 2.17. Let $(X, \circ, \mathcal{H})$ be a commutative quasigroup of odd order $2k + 1$ of type (ii), (iii), or (iv) (cf. beginning of this Section), according as $2k + 1 \equiv 3, 1$ or 5 (*mod* 6), with all holes of size 3, or the first hole H_1 of size 7 and all remaining holes of size 3, or the first hole H_1 of size 5 and all remaining holes of size 3, respectively. Let $V = X \times Z_5 \cup \{\infty_1, \infty_2\}$. On $H_1 \times Z_5 \cup \{\infty_1, \infty_2\}$, put a $K_1{}^*$-design of order 17, 37, or 27, respectively, according as our commutative quasigroup with holes is of type (i), (iii), or (iv), respectively, making sure that the unique K_4-block contains the pair $\{\infty_1, \infty_2\}$. These designs exist by Lemmas 2.7, 2.11 and 2.13. In each case, put on the remaining holes a $K_1{}^*$-design of order 17 with a hole of size 2 (simply delete the edge $\{\infty_1, \infty_2\}$ from the unique K_4-block). Apply the Main Construction to complete the proof. □

Lemma 3.4. *Let $n \equiv 9$ (mod 10), $n > 9$. Then $n \in \mathcal{K}_1{}^*$.*

Proof. By Lemma 2.4, there is no $K_1{}^*$-design of order 9. A $K_1{}^*$-design of order 19 and 29 is given in Lemmas 2.8 and 2.12; $K_1{}^*$-designs of orders 39, 59 and 69 which are not covered by the Main Construction, are given in Lemmas 2.14, 2.16 and 2.18. Proceed as in the proof of Lemma 3.3, but take instead $V = X \times Z_5 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, with $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ the unique K_4-block. □

Proof of Theorem B. Combine Lemma 2.4 with Lemmas 3.1, 3.2, 3.3 and 3.4. □

4. K^*-DESIGNS OF ORDER $n \equiv 3$ (*mod* 5)

In view of the previous section, in order to complete the proof of Theorem A, all one needs is to show the existence of K^*-designs of order $n \equiv 3 \ (mod\ 5), n \geq 13$.

Lemma 4.1. *There exists a K^*-design of order* 13 *with a hole of size* 3.

Proof. Take the Steiner system $S(2,4,v)$ (i.e. a K^*-design where all blocks are K_4-blocks), say, $\{0,1,4,6\}$ (*mod* 13), and delete the edges between any three noncollinear points, say $0,1,2$. □

Lemma 4.2. *There exists a K^*-design of order* 18.

Proof. The complete tripartite graph $K_{4,4,4}$ can be decomposed into six 2-path factors. Indeed, if $Z_4 \times \{1,2,3\}$ is the vertex set of $K_{4,4,4}$ then, for example, the six 2-path factors

$F_1 : (0_1, 0_2, 0_3), (1_1, 1_2, 1_3), (2_1, 2_2, 2_3), (3_1, 3_2, 3_3)$
$F_2 : (0_3, 1_1, 2_2), (1_3, 2_1, 3_2), (2_3, 3_1, 0_2), (3_3, 0_1, 1_2)$
$F_3 : (0_1, 1_3, 2_2), (1_1, 2_3, 3_2), (2_1, 3_3, 0_2), (3_1, 0_3, 1_2)$
$F_4 : (2_3, 0_1, 2_2), (3_3, 1_1, 3_2), (0_3, 2_1, 0_2), (1_3, 3_1, 1_2)$
$F_5 : (0_1, 0_3, 2_2), (1_1, 1_3, 3_2), (2_1, 2_3, 0_2), (3_1, 3_3, 1_2)$
$F_6 : (0_1, 3_2, 0_3), (1_1, 0_2, 1_3), (2_1, 1_2, 2_3), (3_1, 2_2, 3_3)$

form a decomposition into six 2-path factors.

Let $V = \{a_i : i = 1,2,3,4,5,6\} \cup (Z_4 \times \{1,2,3\})$. Form the set of 24 $(K_4 - e)$-blocks $\{[a_i, y, x, z] : (x,y,z) \in F_i, i \in \{1,2,3,4,5,6\}\}$. Adjoin to this set the three $(K_4 - e)$-blocks of any $(K_4 - e)$-design on the set $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and the three K_4-blocks $Z_4 \times \{i\}$, $i = 1,2,3$, to obtain a K^*-design of order 18. □

Lemma 4.3. *There exists a K^*-design of order* 18 *with a hole of size* 8.

Proof. Let $V = Z_{10} \cup \{x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2\}$ where $x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2$ are elements of the hole. Then the 25 $(K_4 - e)$-blocks $[1,8,x_1,x_2]$, $[0,7,x_1,x_2]$, $[6,9,x_1,x_2]$, $[2,4,x_1,x_2]$, $[3,5,x_1,x_2]$, $[0,8,y_1,y_2]$, $[7,9,y_1,y_2]$, $[4,6,y_1,y_2]$, $[2,5,y_1,y_2]$, $[1,3,y_1,y_2]$, $[1,4,z_1,z_2]$, $[5,7,z_1,z_2]$, $[6,8,z_1,z_2]$, $[0,3,z_1,z_2]$, $[2,9,z_1,z_2]$, $[4,7,w_1,w_2]$, $[5,8,w_1,w_2]$, $[3,6,w_1,w_2]$, $[0,2,w_1,w_2]$, $[1,9,w_1,w_2]$, $[1,6,2,7]$, $[2,7,3,8]$, $[3,8,4,9]$, $[4,9,5,10]$, $[5,10,1,6]$ are the blocks of the design. □

Lemma 4.4. *Let $n \equiv 3$ (mod 5), $n > 13$. Then $n \in K^*$.*

Proof. We employ again the Main Construction, just as in the proofs of Section 3, using as ingredients a K^*-design of order 13 together with the K^*-design of order 13 with a hole of size 3 (given in Lemma 4.1) when $n \equiv 3$ (*mod* 10), or a K^*-design of order 18 together with a K^*-design of order 18 with a hole of size 8 (given in Lemmas 4.2 and 4.3) when $n \equiv 8$ (*mod* 10). □

Proof of Theorem A Combine Lemma 4.4 with Theorem B and Theorem 2.2. □

5. Improving the embedding bound

It was shown in [5] that a partial Steiner system $S(2,4,n)$ can be embedded in a K^*-design of order $\leq 8n + \sqrt{n} + 84$. Problem (2) in the same paper [5] asks for a reduction of this bound. This is attained below. We are able to prove the following.

Theorem 5.1. *A partial K^*-design of order n can be embedded in a K^*-design of order $\leq 6n + 6$.*

Proof. We may assume w.l.o.g that $n \geq 6$ (a partial K^*-design of order 4 or 5 may have at most one block). Let $(Y, \mathcal{P})$ be a partial K^*-design of order n. Let $m = n$ if n is odd, and let $m = n + 1$ if n is even. Let $(X, \mathcal{R})$ be a skew Room square of order m (cf. [2]) where $Y \subseteq X$. Put $S = X \times Z_6$, and define a collection of blocks K as follows:

(1) For each $x \in X$, let B_x be the blocks of a $(K_4 - e)$-design of order 6 on $\{x\} \times Z_6$; place the collection of blocks $\mathcal{B} = \bigcup_{x \in X} B_x$ in K.

(2) For each $x, y \in X, x \neq y$, place the set of six K_4-blocks $\{\{(x,i),(y,i),(r,i+1),(c,i+4)\} : i \in Z_6\}$ in K where $\{x,y\}$ belongs to the cell (r,c) of $\mathcal{R}$. Then (X, K) is a K^*-design of order $6m \leq 6n + 6$.

Now, for each block $b \in \mathcal{P}$ and for each edge $\{x,y\} \in b$, remove the edge $\{(x,0),(y,0)\}$ from the block $\{(x,0),(y,0),(r,1),(c,4)\}$ in K; at most one edge from $X \times \{0\}$ is removed from any block. What is left of the block $\{(x,0),(y,0),(r,1),(c,4)\}$ is the $(K_4 - e)$-block $[(r,1),(c,4),(x,0),(y,0)]$. Assemble now the removed edges into a copy of $\mathcal{P}$. The resulting collection of blocks $K(\mathcal{P})$ is a K^*-design containing a copy of $\mathcal{P}$ embedded in level $X \times \{0\}$. □

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