A study on the chaotic numbers of graphs

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Abstract

Given a sequence $X = (x_1, x_2, \cdots, x_k)$, let $Y = (y_1, y_2, \cdots, y_k)$ be a sequence obtained by rearranging the terms of X. The total self-variation of Y relative to X is $\zeta_X(Y) = \sum_{i=1}^{k} |y_i - x_i|$. On the other hand, let $G = (V, E)$ be a connected graph and ϕ be a permutation of V. The total relative displacement of ϕ is $\delta_\phi(G) = \sum_{\{x \neq y\} \subset V} |d(x, y) - d(\phi(x), \phi(y))|$, where $d(x, y)$ means the distance between x and y in G. It's clear that the total relative displacement of ϕ is a total self-variation relative to the distance sequence of the graph.

In this paper, we determine the sequences which attain the maximum value of the total self-variation of all possible rearrangements Y relative to X. Applying this result to the distance sequence of a graph, we find a best possible upper bound for the total relative displacement of a graph.

Keywords: Total self-variation, Total relative displacement, Chaotic Numbers

1 Introduction

Let $G = (V, E)$ be a connected graph and ϕ be a permutation of V. Define the total relative displacement of the permutation ϕ to be $\delta_\phi(G) = \sum_{\{x \neq y\} \subset V} |d(x, y) - d(\phi(x), \phi(y))|$, where $d(x, y)$ means the distance between x and y, i.e., the length of a shortest path between x and y. This parameter is related to the sorting problem in computer science[2] and it measures the disorderliness of data. Chartrand, Gavlas and VanderJagt[1] considered this concept. They also studied the near-automorphisms of graphs, i.e., permutations that attain the minimum value $\pi(G)$ of the nonzero total relative displacement of the graph G. They got a lot of fundamental properties including the property $\pi(G) \geq 2$ which we will cite later. On the other hand, Fu et al.[2] studied the maximum value of the total relative displacements among all permutations of a graph G, denoted by $\pi^*(G)$, and called it the chaotic number of G. In [2], the problem of finding

$\pi^*(K_{n_1,n_2,\cdots,n_t})$ was transformed into a quadratic integer programming problem, and a characterization of the optimal solution was proposed, and then a polynomial time algorithm was given to solve the problem.

In the next section, we will develop the concept of the total self-variation of a sequence Y relative to a given sequence X. How to determine the sequences which attain the maximum value $M = \max\{\zeta_X(Y) : Y$ is obtained by rearranging the terms of $X\}$ was solved by Mitchell in [3]. For the convenience of the reader, we describe the determination in our way to make our exposition self-contained. Then applying this result to the distance sequence of a graph, we find an upper bound for the total relative displacements of all permutations of a graph. And then we construct infinitely many graphs of all orders that attain the upper bound. These constructions show that the upper bound is best possible.

2 Main Results

Given a sequence $X = (x_1, x_2, x_3, \cdots, x_k)$, let $Y = (y_1, y_2, y_3, \cdots, y_k)$ be a sequence obtained by rearranging the terms of X. The total self-variation of Y relative to X is $\zeta_X(Y) = \sum_{i=1}^{k} |y_i - x_i|$. Define $\eta^*(X) = \max\{\zeta_X(Y) :$ Y is a sequence that obtained by rearranging the terms of $X\}$.

Let's determine the sequences which attain $\eta^*(X)$.

Theorem 2.1. Let $X = (x_1, x_2, x_3, \cdots, x_k)$ be a sequence of real numbers. Sort X into a non-decreasing sequence. Suppose that the resulted sequence is $Y = (x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(k)})$ for some permutation τ of $\{1, 2, \cdots, k\}$. Let σ be the permutation that maps $\tau(1)$ to $\tau(n)$, $\tau(2)$ to $\tau(n-1), \cdots$, and $\tau(n)$ to $\tau(1)$. If $Y_0 = (x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$, then $\eta^*(X) = \zeta_X(Y_0)$.

To prove Theorem 2.1, let's consider the following concepts and properties first.

We say that a sequence $X = (x_1, x_2, x_3, \cdots, x_k)$ has a conversion (x_i, x_j) in X if $i < j$ and $x_i < x_j$. The number of conversions with x_i as the first component is denoted by $n_X(x_i)$ and the number of conversions in X is denoted by $n(X)$. It is clear that $n(X) = \sum_{i=1}^{k} n_X(x_i)$ and X is a non-increasing sequence if and only if $n(X) = 0$.

Lemma 1. Let $X = (x_1, x_2, x_3, \cdots, x_k)$ be a sequence and (x_i, x_j) be a conversion in X. If Y is a sequence obtained by exchanging x_i and x_j in X, then $n(X) > n(Y)$.

Proof. The following facts are clear:

1. $n_X(x_l) = n_Y(x_l)$ if $l < i$ or $l > j$;

2. $n_X(x_i) = n_Y(x_i) + 1 + |\{x_l : i < l < j,$ and $x_l > x_i\}|$;

3. $n_X(x_l) \geq n_Y(x_l)$ if $i < l < j$;

4. $n_X(x_j) = n_Y(x_j) - |\{x_l : i < l < j, \text{ and } x_l > x_j\}|$; and

5. $|\{x_l : i < l < j, \text{ and } x_l > x_i\}| \geq |\{x_l : i < l < j, \text{ and } x_l > x_j\}|$.

Hence $n(X) = \sum_{l=1}^{k} n_X(x_l) > \sum_{l=1}^{k} n_Y(x_l) = n(Y)$. □

Lemma 2. Let $X = (x_1, x_2, x_3, \cdots, x_k)$ be a non-decreasing sequence. If $Y_0 = (x_k, x_{k-1}, x_{k-2}, \cdots, x_1)$, then $\eta^*(X) = \zeta_X(Y_0)$.

Proof. Since for a finite sequence X the value $\eta^*(X)$ exists, it is sufficient to prove that if $Y \neq Y_0$ then we can find a sequence Z such that $\zeta_X(Z) \geq \zeta_X(Y)$ and $n(Z) < n(Y)$.

Suppose that $Y = (y_1, y_2, \cdots, y_k) \neq Y_0$. There are two numbers i and j such that $1 \leq i < j \leq k$ and $y_i < y_j$. Let Z be the sequence obtained by exchanging the two terms y_i and y_j in Y. Then $n(Z) < n(Y)$ by Lemma 1.

Case 1. If $x_i = x_j$ then it is clear that $\zeta_X(Z) = \zeta_X(Y)$.

Case 2. If $x_i < x_j$. We can divide this case into 6 subcases, i.e. (i) $x_i < x_j \leq y_i < y_j$; (ii) $x_i \leq y_i \leq x_j \leq y_j$; (iii) $x_i \leq y_i < y_j \leq x_j$; (iv) $y_i \leq x_i \leq y_j \leq x_j$; (v) $y_i < y_j \leq x_i < x_j$; and (vi) $y_i \leq x_i < x_j \leq y_j$. For each subcase, it is clear that $\zeta_X(Z) - \zeta_X(Y) = |y_j - x_i| + |y_i - x_j| - |y_i - x_i| - |y_j - x_j| \geq 0$.

Therefore, Lemma 2 is proved. □

Proof of Theorem 2.1.

Let $Y' = (x_{\tau(k)}, x_{\tau(k-1)}, \cdots, x_{\tau(1)})$. Then by Lemma 2, $\eta^*(Y) = \zeta_Y(Y')$. Since $\eta^*(X) = \eta^*(Y), \eta^*(X) = \eta^*(Y) = \zeta_Y(Y') = \zeta_X(Y_0)$. Hence Theorem 2.1 is proved. □

After determining the sequence that attains the maximum value of the total self-variation relative to a given sequence, let's apply Theorem 2.1 to get an upper bound for the total relative displacements of permutations of graphs. For convenience, we call a sequence X a distance sequence of a graph G of order t if X consists of the distances between $\binom{t}{2}$ unordered pairs of distinct vertices of G.

Corollary 2.2. Let $G = (V, E)$ be a graph of order t and X be the distance sequence of G. Then $\delta_\phi(G) \leq \pi^*(G) \leq \eta^*(X)$ for any permutation ϕ of V.

Since $\delta_\phi(G)$ is a total self-variation relative to a distance sequence X, Corollary 2.2 is clearly true. To see that Corollary 2.2 does give a best possible upper bound for $\pi^*(G)$, let's consider the following results.

Lemma 3. Let $G = K_t \backslash \{e\}$ with t vertices ($t \geq 3$) and X be the distance sequence of G, where $e \in E(K_t)$. Then $\pi(G) = \pi^*(G) = \eta^*(X) = 2$.

The proof is obvious and we omit it.

Lemma 4. Let $G = K_t \backslash \{e_1, e_2\}$ with t vertices ($t \geq 4$) and X be the distance sequence of G, where $e_1, e_2 \in E(K_t)$ are distinct. Then $\pi^*(G) = \eta^*(X) = 4$.

Proof. Consider the distance sequence $X = (\overbrace{1,1,1,\cdots,1}^{\binom{t}{2}-2},2,2)$ of G. Then by Theorem 2.1, it is clear that $\pi^*(G) \leq \eta^*(X) = 4$.

Suppose that $e_1 = \{a_1, a_2\}$ and $e_2 = \{a_3, a_4\}$. There are two cases:

Case 1: e_1 and e_2 are not disjoint.

Without loss of generality, suppose that $a_1 = a_3$. Since $t \geq 4$, there is another vertex $a \in V(G)$. Let $\phi = (a_1\ a)$ be a transposition of $V(G)$.

Then

$\delta_\phi(G) = |d(a_1,a_2) - d(a,a_2)| + |d(a_3,a_4) - d(a,a_4)| + |d(a,a_2) - d(a_1,a_2)| + |d(a,a_4) - d(a_1,a_4)| = 4$.

Case 2: e_1 and e_2 are disjoint.

Let $\phi = (a_2\ a_4)$ be a transposition of $V(G)$. Then $\delta_\phi(G) = |d(a_1,a_2) - d(a_1,a_4)| + |d(a_3,a_4) - d(a_3,a_2)| + |d(a_1,a_4) - d(a_1,a_2)| + |d(a_3,a_2) - d(a_3,a_4)| = 4$.

In both cases, $4 \leq \pi^*(G) \leq \eta^*(X_k) = 4$. Therefore, $\pi^*(G) = 4 = \eta^*(X_k)$. □

Lemma 5. Let $G = K_t \backslash \{e_1, e_2, e_3\}$ with t vertices ($t \geq 5$) and X be the distance sequence of G, where $e_1, e_2, e_3 \in E(K_t)$ are distinct. Then $\pi^*(G) = \eta^*(X) = 6$.

Proof. Consider the distance sequence $X = (\overbrace{1,1,1,\cdots,1}^{\binom{t}{2}-3},2,2,2)$ of G. Then by Theorem 2.1, it is clear that $\pi^*(G) \leq \eta^*(X) = 6$.

Suppose that $e_1 = \{a_1, a_2\}$, $e_2 = \{a_3, a_4\}$, and $e_3 = \{a_5, a_6\}$. There are five cases as follows:

Case 1: Without loss of generality, suppose that $a_1 = a_3$ and $a_4 = a_6$. Since $t \geq 5$, there is another vertex $a \in V(G)$. Let $\phi = (a_1\ a_4\ a)$ be a permutation of $V(G)$. Then $\delta_\phi(G) = |d(a_1,a_2) - d(a_4,a_2)| + |d(a_3,a_4) - d(a_4,a)| + |d(a_5,a_6) - d(a_5,a)| + |d(a,a_2) - d(a_1,a_2)| + |d(a,a_1) - d(a_3,a_4)| + |d(a_5,a_1) - d(a_5,a_6)| = 6$.

Case 2: Without loss of generality, suppose that $a_1 = a_3 = a_5$. Since $t \geq 5$, there is another vertex $a \in V(G)$. Let $\phi = (a_1\ a)$ be a transposition of $V(G)$. Then $\delta_\phi(G) = |d(a_1,a_2) - d(a,a_2)| + |d(a_3,a_4) - d(a,a_4)| + |d(a_5,a_6) - d(a,a_6)| + |d(a,a_2) - d(a_1,a_2)| + |d(a,a_4) - d(a_3,a_4)| + |d(a,a_6) - d(a_5,a_6)| = 6$.

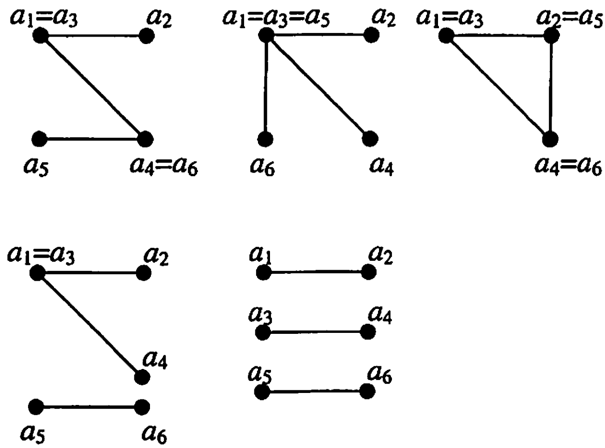

Figure 1: e_1, e_2, and e_3

Case 3: Without loss of generality, suppose that $a_1 = a_3$, $a_2 = a_5$, and $a_4 = a_6$. Since $t \geq 5$, there is another vertex a, $b \in V(G)$. Let $\phi = (a_1\ a)(a_4\ b)$ be a permutation of $V(G)$. Then $\delta_\phi(G) = |d(a_1, a_2) - d(a, a_2)| + |d(a_3, a_4) - d(a, b)| + |d(a_5, a_6) - d(a_5, b)| + |d(a, a_2) - d(a_1, a_2)| + |d(a, b) - d(a_3, a_4)| + |d(a_5, b) - d(a_5, a_6)| = 6$.

Case 4: Without loss of generality, suppose that $a_1 = a_3$. Let $\phi = (a_1\ a_5)$ be a transposition of $V(G)$. Then $\delta_\phi(G) = |d(a_1, a_2) - d(a_5, a_2)| + |d(a_3, a_4) - d(a_5, a_4)| + |d(a_5, a_6) - d(a_1, a_6)| + |d(a_2, a_5) - d(a_1, a_2)| + |d(a_5, a_4) - d(a_3, a_4)| + |d(a_1, a_6) - d(a_5, a_6)| = 6$.

Case 5: Let $\phi = (a_1\ a_3\ a_5)$ be a permutation of $V(G)$. Then $\delta_\phi(G) = |d(a_1, a_2) - d(a_5, a_2)| + |d(a_3, a_4) - d(a_1, a_4)| + |d(a_5, a_6) - d(a_3, a_6)| + |d(a_3, a_2) - d(a_1, a_2)| + |d(a_5, a_4) - d(a_3, a_4)| + |d(a_1, a_6) - d(a_5, a_6)| = 6$.

In all cases, $6 \leq \pi^*(G) \leq \eta^*(X_k) = 6$. Therefore, $\pi^*(G) = 6 = \eta^*(X_k)$. □

Theorem 2.3. Let G be a connected graph of order t and $e \in E(G)$. Then $\pi^*(G) = 2$ if and only if $G = K_t \backslash \{e\}$.

Proof. Lemma 3 gives the sufficient condition of $\pi^*(G) = 2$.

If $G = K_t \backslash E'$ and $|E'| \geq 2$, then by Lemma 4, we have $\pi^*(G) \geq \pi^*(K_t \backslash \{e_1, e_2\}) = 4$, where e_1, e_2 are distinct edges in E'. Hence we have that if $\pi^*(G) = 2$, then $G = K_t \backslash \{e\}$. □

Theorem 2.4. If G is a connected graph of order t and $e_1, e_2 \in E(G)$. Then $\pi^*(G) = 4$ if and only if $G = K_t \backslash \{e_1, e_2\}$ or $G = K_{1,3}$.

Proof. It is easy to see that if $G = K_t \backslash \{e_1, e_2\}$ or $G = K_{1,3}$, then $\pi^*(G) = 4$.
Suppose that $\pi^*(G) = 4$.

1. If $t \leq 4$, then it is easy to see that only $K_t \backslash \{e_1, e_2\}$ and $K_{1,3}$ are the graphs with $\pi^*(G) = 4$.

2. If $t \geq 5$ and $G = K_t \backslash E'$ with $|E'| \geq 3$, then by Lemma 5, we have $\pi^*(G) \geq \pi^*(K_t \backslash \{e_1, e_2, e_3\}) = 6$, where e_1, e_2, e_3 are distinct edges in E'.

Therefore, we have that if $\pi^*(G) = 4$, then $G = K_t \backslash \{e_1, e_2\}$. □

A graph is called a complete splitting graph, denoted by $S_{m,n}$, if the vertex set can be partitioned into two subsets A and B with $|A| = m$ and $|B| = n$ such that each pair of vertices in A are unadjacent, each pair of vertices in B are adjacent and each vertex in A is adjacent to each vertex in B. The maximum total relative displacement of $S_{m,m}$ can be found as follows.

Theorem 2.5. $\pi^*(S_{m,m}) = \eta^*(X) = 2\binom{m}{2}$.

Proof. Consider the distance sequence $X = (\overbrace{1,1,1,\cdots,1}^{\binom{m}{2}+m^2},\overbrace{2,2,\cdots,2}^{\binom{m}{2}})$ of $S_{m,m}$. Then $\pi^*(S_{m,m}) \leq \eta^*(X) = 2\binom{m}{2}$.

Let ϕ be a permutation which maps A into B and vice versa. Then $\delta_\phi(S_{m,m}) = 2\binom{m}{2}$.

Hence $\pi^*(S_{m,m}) = \eta^*(X) = 2\binom{m}{2}$. □

According to Theorem 3, 4, 2.5, the upper bound in Corollary 2.2 can be attained by a family of infinitely many graphs of all orders. In other words, the upper bound is best possible.

Acknowledgement

The authors want to express their thanks to the anonymous referee for reminding them of extending Lemma 3, Lemma 4 to Theorem 2.3, and Theorem 2.4.

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