Embedding handcuffed designs into a maximum packing of the complete graph with 4-cycles *

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Abstract

A *packing* of K_n with copies of C_4 (the cycle of length 4), is an ordered triple $(V, \mathcal{C}, L)$, where V is the vertex set of the complete graph K_n, $\mathcal{C}$ is a collection of edge-disjoint copies of C_4, and L is the set of edges not belonging to a block of $\mathcal{C}$. The number n is called the *order* of the packing and the set of unused edges L is called the *leave*. If $\mathcal{C}$ is as large as possible, then $(V, \mathcal{C}, L)$ is called a *maximum packing* $MPC(n, 4, 1)$. We say that an handcuffed design $H(v, k, 1)$ $(W, \mathcal{P})$ is *embedded* into an $MPC(n, 4, 1)$ $(V, \mathcal{C}, L)$ if $W \subseteq V$ and there is an injective mapping $f : \mathcal{P} \rightarrow \mathcal{C}$ such that P is a subgraph of $f(P)$ for every $P \in \mathcal{P}$. Let $\mathcal{SH}(n, 4, k)$ denote the set of the integers v such that there exists an $MPC(n, 4, 1)$ which embeds an $H(v, k, 1)$. If $n \equiv 1 \pmod 8$ then an $MPC(n, 4, 1)$ coincides with a 4-cycle system of order n and $\mathcal{SH}(n, 4, k)$ is found by Milici and Quattrocchi, *Discrete Math.*, **174** (1997).

The aim of the present paper is to determine $\mathcal{SH}(n, 4, k)$ for every integer $n \not\equiv 1 \pmod 8$, $n \geq 4$.

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1 Introduction

Let G be a subgraph of K_n, the complete undirected graph on n vertices. A *G-design* of K_n is a pair $(V, \mathcal{B})$, where V is the vertex set of K_n and $\mathcal{B}$ is an edge-disjoint decomposition of K_n into copies of the graph G. If $B \in \mathcal{B}$ we say that B is a block (or a G-block) of the G-design. The set $\mathcal{B}$ is called the block-set. A G-design of K_n is also called a G-design of order n.

A G-design is *balanced* if each vertex belongs to the same number of blocks. Obviously not every G-design is balanced.

An *handcuffed design* $H(v, k, 1)$ [5] is a balanced P_k-design of K_v, where P_k is the simple path with $k-1$ edges (k vertices) $[a_1, a_2, \dots, a_k] = \{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{k-1}, a_k\}\}$. Clearly an $H(v, 2, 1)$ $(V, \mathcal{P})$ exists for every $v \geq 2$ because V is a v-set and $\mathcal{P}$ is the set of all 2-subsets of V. Hung and Mendelsohn [4] proved that an $H(v, 2h+1, 1)$ $(h \geq 1)$ exists if and only if $v \equiv 1 \pmod{4h}$, and an $H(v, 2h, 1)$ $(h \geq 2)$ exists if and only if $v \equiv 1 \pmod{2h-1}$.

A *4-cycle system* of order n is a C_4-design of K_n, where C_4 is the 4-cycle (cycle of length 4) $(a_1, a_2, a_3, a_4) = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}\}$. It is well-known [6] that the spectrum for 4-cycle systems is precisely the set of all $n \equiv 1 \pmod 8$.

Let G_1 be a subgraph of G_2. We say that a G_1-design $(W, \mathcal{P})$ of order v is *embedded* into a G_2-design $(V, \mathcal{C})$ of order n if $W \subseteq V$ and there is an injective mapping

$$f : \mathcal{P} \to \mathcal{C}$$

such that P is a subgraph of $f(P)$ for every $P \in \mathcal{P}$.

The following embedding problem arises: *for every admissible integer n determine the set of the integers v such that there exists a G_2-design of order n which embeds some G_1-design of order v.* This embedding problem has been investigated in many cases [1, 2, 3, 7, 8, 9]. Milici and Quattrocchi [7] gave a complete answer to the embedding problem of an $H(v, k, 1)$ into a 4-cycle system of order n. The following question arises: what can we say when $n \not\equiv 1 \pmod 8$?

A *packing* of K_n with copies of C_4 is an ordered triple $(V, \mathcal{C}, L)$, where V is the vertex set of K_n, $\mathcal{C}$ is a collection of edge-disjoint copies of C_4, and L is the set of edges not belonging to a block of $\mathcal{C}$. The number n is called the *order* of the packing and the set of unused edges L is called the *leave*. If $\mathcal{C}$ is as large as possible, then $(V, \mathcal{C}, L)$ is called a *maximum packing* $MPC(n, 4, 1)$.

An $MPC(n, 4, 1)$ can be considered as the natural generalization of a 4-cycle system of order n when $n \not\equiv 1 \pmod 8$. In fact, an $MPC(n, 4, 1)$ with $n \equiv 1 \pmod 8$ coincides with a 4-cycle system of order n. In this paper we give a complete answer to the embedding problem of an $H(v, k, 1)$ into an $MPC(n, 4, 1)$.

Definition 1.1 *An $H(v,k,1)$ $(V,\mathcal{P})$ is embedded into an $MPC(n,4,1)$ $(W,\mathcal{C},L)$ if $V \subseteq W$ and there is an injective mapping*

$$f : \mathcal{P} \to \mathcal{C}$$

such that P is a subgraph of $f(P)$ for every $P \in \mathcal{P}$.

Example 1.1 An $H(3,2,1)$ on vertex set $V = \{0,1,2\}$ embedded into an $MPC(7,4,1)$ on vertex set $W = V \cup \{a_0,a_1,a_2,a_3\}$: $L = (a_0,0,a_1,1,a_2)$, $\mathcal{C} = \{(0,1,a_3,a_2),(0,2,a_0,a_3),(1,2,a_1,a_0),(a_2,a_1,a_3,2)\}$.

Example 1.2 An $H(5,2,1)$ on vertex set $V = \{0,1,\dots,4\}$ embedded into an $MPC(11,4,1)$ on vertex set $W = V \cup \{a_0,a_1,\dots,a_5\}$: $L = (3,a_4,a_5)$, $\mathcal{C} = \{(0,1,a_0,a_1),(0,2,a_1,a_2),(0,3,a_2,a_3),(0,4,a_5,a_0),(1,2,a_0,a_2),$ $(1,3,a_3,a_1),(1,4,a_0,a_3),(2,3,a_1,a_4),(2,4,a_2,a_5),(3,4,a_4,a_0),(a_4,0,a_5,1),$ $(a_2,2,a_3,a_4),(a_1,4,a_3,a_5)\}$.

Example 1.3 An $H(5,3,1)$ on vertex set $V = \{0,1,\dots,4\}$ embedded into an $MPC(10,4,1)$ on vertex set $W = V \cup \{a_0,a_1,\dots,a_4\}$: $L = \{[i,a_i] \mid i = 0,1,\dots,4\}$, $\mathcal{C} = \{(i,3+i,2+i,a_{4+i}),(a_i,a_{1+i},a_{3+i},2+i) \mid i = 0,1,\dots,4\}$ (the sums are $\pmod 5$).

Example 1.4 An $H(5,3,1)$ on vertex set $V = \{0,1,\dots,4\}$ embedded into an $MPC(11,4,1)$ on vertex set $W = V \cup \{a_0,a_1,\dots,a_5\}$: $L = (a_0,a_1,a_3)$, $\mathcal{C} = \{(0,1,4,a_0),(1,2,0,a_1),(2,3,1,a_2),(3,4,2,a_3),(4,0,3,a_4),$ $(2,a_0,a_2,a_1),(2,a_4,a_2,a_5),(4,a_1,a_4,a_3),(0,a_5,a_3,a_2),(1,a_0,a_4,a_5),$ $(3,a_1,a_5,a_0),(0,a_3,1,a_4),(3,a_2,4,a_5)\}$.

Definition 1.2 *Denote by $\mathcal{SH}(n,4,k)$ the set of the integers v, $v \geq k$, such that there exists an $H(v,k,1)$ embedded into an $MPC(n,4,1)$.*

The aim of the present paper is to determine $\mathcal{SH}(n,4,k)$ for every integer $n \geq 4$ and $k = 2,3$.

When an $MPC(n,4,1)$ coincides with a 4 cycle system of order n, the spectrum $\mathcal{SH}(n,4,k)$ is given in the following theorem.

Theorem 1.1 ([7]) *Let $n \equiv 1 \pmod 8$, $n \geq 9$. Then*

- $\mathcal{SH}(n,4,2) = \{v \mid 2 \leq v \leq \frac{n-1}{2}\}$;
- $\mathcal{SH}(n,4,3) = \{v \mid 5 \leq v \leq \frac{2n+1-\alpha(n)}{3},\ v \equiv 1 \pmod 4\}$, *where* $\alpha(n) = 12$ *if* $n \equiv 1 \pmod{24}$, $\alpha(n) = 4$ *if* $n \equiv 9 \pmod{24}$ *and* $\alpha(n) = 8$ *if* $n \equiv 17 \pmod{24}$.

Theorem 1.2 (Schöneim and Bialostocki [10]) *For every integer $n \geq 4$ there exists an $MPC(n,4,1)$ $(V,\mathcal{C},L)$ and it is:*

- $|\mathcal{C}| = \lceil \frac{n}{4} \lceil \frac{n-1}{2} \rceil \rceil$ *if* $n \not\equiv 5, 7 \pmod 8$,
- $|\mathcal{C}| = \lceil \frac{n}{4} \lceil \frac{n-1}{2} \rceil \rceil - 1$ *otherwise.*

If $n \equiv 1 \pmod 8$ *then the leave* L *does not contain any edge. If* $n \not\equiv 1 \pmod 8$ *then the non-packed edges may be chosen so that the leave* L *is isomorphic to a one-factor if* n *is even, a* 3*-cycle if* $n \equiv 3 \pmod 8$*, two* 3*-cycles having a common vertex if* $n \equiv 5 \pmod 8$ *and a* 5*-cycle if* $n \equiv 7 \pmod 8$.

2 $\mathcal{SH}(\mathbf{n}, 4, 2)$

Lemma 2.1 *For every even integer* $n \geq 4$, $\mathcal{SH}(n, 4, 2) \subseteq \{v \mid 2 \leq v \leq \frac{n}{2}\}$. *For every odd integer* $n \geq 5$, $\mathcal{SH}(n, 4, 2) \subseteq \{v \mid 2 \leq v \leq \frac{n-1}{2}\}$.

Proof. Let $(V, \mathcal{P})$ be the $H(v, 2, 1)$ embedded into an $MPC(n, 4, 1)$ $(W, \mathcal{C}, L)$. Then $\binom{n-v}{2} \geq \binom{v}{2}$. ∎

Lemma 2.2 *If* $v \in \mathcal{SH}(2v+1, 4, 2)$*, then* $v + x \in \mathcal{SH}(2v + 2x + 1, 4, 2)$ *for every* $x \equiv 0 \pmod 4$, $x \geq 4$.

Proof. Let $(V, \mathcal{P})$ be the $H(v, 2, 1)$ embedded into an $MPC(2v + 1, 4, 1)$ $(W, \mathcal{B}, L)$ and let $x = 4k$, $k \geq 1$. Put $V = \{0, 1, \ldots, v-1\}$, $W = V \cup \{v, v+1, \ldots, 2v-1, \infty\}$, $A_i = \{a_0^i, a_1^i, a_2^i, a_3^i\}$, $W_i = A_i \cup \{a_4^i, a_5^i, a_6^i, a_7^i, \infty\}$. For $i = 1, 2, \ldots, k$, let $(W_i, \mathcal{B}_i)$ be a 4-cycle system of order 9 which embeds an $H(4, 2, 1)$ on vertex set A_i (see Theorem 1.1). Let $\overline{V} = V \cup (\cup_{i=1}^{k} A_i)$ and $\overline{W} = W \cup (\cup_{i=1}^{k} W_i)$. Assign to $\mathcal{C}$ the 4-cycles of $\mathcal{B} \cup (\cup_{i=1}^{k} \mathcal{B}_i)$ and the following:

- $(\rho, a_h^i, \rho + v, a_{h+4}^i)$, $i = 1, 2, \ldots, k$, $\rho = 0, 1, \ldots, v-1$, $h = 0, 1, 2, 3$;
- if $k \geq 2$, $(a_j^{i_2}, a_h^{i_1}, a_{j+4}^{i_2}, a_{h+4}^{i_1})$, $i_1, i_2 = 1, 2, \ldots, k$, $i_1 < i_2$, and $j, h = 0, 1, 2, 3$.

Then $(\overline{W}, \mathcal{C}, L)$ is an $MPC(2v+2x+1, 4, 1)$ which embeds the $H(v+x, 2, 1)$ on vertex set $\overline{V}$. ∎

Theorem 2.1 *For every even integer* n, $n \geq 4$, $\mathcal{SH}(n, 4, 2) = \{v \mid 2 \leq v \leq \frac{n}{2}\}$. *For every odd integer* n, $n \geq 5$, $\mathcal{SH}(n, 4, 2) = \{v \mid 2 \leq v \leq \frac{n-1}{2}\}$.

Proof. By Lemma 2.1, it suffices to prove that $\{v \mid 2 \leq v \leq \frac{n}{2}\} \subseteq \mathcal{SH}(n, 4, 2)$ if n is even, and $\{v \mid 2 \leq v \leq \frac{n-1}{2}\} \subseteq \mathcal{SH}(n, 4, 2)$ if n is odd. Every $H(v, 2, 1)$ with $v \geq 3$ embeds every $H(w, 2, 1)$ with $2 \leq w \leq v - 1$.

Therefore the above inclusions are proved if $\frac{n}{2} \in \mathcal{SH}(n,4,2)$ for every even $n \geq 4$ and $\frac{n-1}{2} \in \mathcal{SH}(n,4,2)$ for every odd $n \geq 5$.

Let n be even. We prove that $\frac{n}{2} \in \mathcal{SH}(n,4,2)$ for every even $n \geq 4$ by induction. Clearly $\mathcal{SH}(4,4,2) = \{2\}$. Suppose $\frac{n}{2} \in \mathcal{SH}(n,4,2)$. Let $(V,\mathcal{P})$ be the $H(\frac{n}{2},2,1)$ embedded into an $MPC(n,4,2)$ $(W,\mathcal{C},L)$. Put $\overline{W} = W \cup \{\infty_1,\infty_2\}$, $\overline{L} = L \cup \{[\infty_1,\infty_2]\}$ and $\overline{\mathcal{C}} = \mathcal{C} \cup \{(\infty_1,x,\infty_2,g(x)) \mid x \in V\}$, where g is a bijection from V to $W \setminus V$. Then $(\overline{W},\overline{\mathcal{C}},\overline{L})$ is an $MPC(n+2,4,2)$ which embeds the $H(\frac{n+2}{2},2,1)$ on vertex set $V \cup \{\infty_1\}$.

Let n be odd. By Theorem 1.1, $\frac{n-1}{2} \in \mathcal{SH}(n,4,2)$ for every $n \equiv 1 \pmod 8$, $n \geq 9$. If $n \equiv 3,5,7 \pmod 8$, then apply Lemma 2.2 to the starting cases $2 \in \mathcal{SH}(5,4,2)$, $3 \in \mathcal{SH}(7,4,2)$ and $5 \in \mathcal{SH}(11,4,2)$. Note that $2 \in \mathcal{SH}(5,4,2)$ is straightforward. The remaining two cases follow from Examples 1.1 and 1.2 respectively. ∎

3 $\mathcal{SH}(\mathrm{n},4,3)$

Let

$$\theta(n) = \begin{cases} \frac{n-6}{2} & \text{if } n \equiv 0 \pmod 8,\ n \geq 16; \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 8,\ n \geq 10; \\ \frac{n-2}{2} & \text{if } n \equiv 4 \pmod 8,\ n \geq 12; \\ \frac{n-4}{2} & \text{if } n \equiv 6 \pmod 8,\ n \geq 14; \\ \frac{2n-11}{3} & \text{if } n \equiv 1 \pmod 6,\ n \geq 13; \\ \frac{2n-3}{3} & \text{if } n \equiv 3 \pmod 6,\ n \geq 9; \\ \frac{2n-7}{3} & \text{if } n \equiv 5 \pmod 6,\ n \geq 11. \end{cases}$$

Lemma 3.1 $\mathcal{SH}(n,4,3) \subseteq \{v \mid 5 \leq v \leq \theta(n), v \equiv 1 \pmod 4\}$.

Proof. Let $(V,\mathcal{P})$ be an $H(v,3,1)$, $v \geq 5$, embedded into an $MPC(n,4,1)$ $(W,\mathcal{C},L)$.

Let n be even. The leave L is a one-factor whose edge set does not contain any edge of K_V. Then $n \geq 2v$ and, by $v \equiv 1 \pmod 4$, the proof follows.

Let n be odd. Theorem 1.1 proves the lemma for $n \equiv 1 \pmod 8$. Suppose $n \equiv 3,5,7 \pmod 8$. Let $\mathcal{B}$ be the set of blocks of $\mathcal{C}$ having some block of $\mathcal{P}$ as a subgraph. Put $\mathcal{D} = \mathcal{C} \setminus \mathcal{B}$. Every block of $\mathcal{C}$ covers two edges of $K_{V,W\setminus V}$. Then $v(n-v) \geq \binom{v}{2}$. It follows $n \geq \frac{3v-1}{2}$. Suppose $n = \frac{3v-1}{2}$. Then $\mathcal{D} \cup L$ is a graph decomposition of $K_{W\setminus V}$. Every vertex x of $W \setminus V$ has odd degree in $K_{W\setminus V}$. Therefore the degree of x in L is odd. Moreover x has even degree in K_W, then the degree of x in L should be even. Therefore $n > \frac{3v-1}{2}$ and, by $v \equiv 1 \pmod 4$, we get the proof. ∎

Lemma 3.2 *Let (W, C, L) be an $MPC(n, 4, 1)$ with n even, $n \geq 4$. Then there exists an $MPC(n + 2, 4, 1)$ $(\overline{W}, \overline{C}, \overline{L})$ such that $W \subset \overline{W}$, $C \subset \overline{C}$ and $L \subset \overline{L}$.*

Proof. Put $\overline{W} = W \cup \{\infty_1, \infty_2\}$, $\overline{L} = L \cup \{[\infty_1, \infty_2]\}$. Let $\mathcal{B}$ be an edge-disjoint decomposition of the complete bipartite graph $K_{W,\{\infty_1,\infty_2\}}$ into subgraphs isomorphic to a 4-cycle. Put $\overline{C} = C \cup \mathcal{B}$. ∎

Lemma 3.3 *If $v \in \mathcal{SH}(2v, 4, 3)$, then $v + 4 \in \mathcal{SH}(2v + 8, 4, 3)$.*

Proof. Let $(V, \mathcal{P})$ be an $H(v, 3, 1)$ embedded into an $MPC(2v, 4, 1)$ (W, C, L). Put $v = 1 + 4k$, $k \geq 1$. Let $A_i = \{a_1^i, a_2^i, a_3^i, a_4^i\}$, $B_i = \{b_1^i, b_2^i, b_3^i, b_4^i\}$, $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$, $D = \cup_{i=1}^k \{b_3^i, b_4^i\}$, $V = A \cup \{\infty_1\}$, $W = V \cup B \cup \{\infty_2\}$. Let $(X, \mathcal{D})$ denote an $H(5, 3, 1)$ embedded into an $MPC(10, 4, 1)$ $(X \cup Y, \mathcal{B}, L_1)$ with $X = \{1, 2, 3, 4\} \cup \{\infty_1\}$, $Y = \{5, 6, 7, 8\} \cup \{\infty_2\}$ and $[\infty_1, \infty_2] \in L_1$ (see Example 1.3). Now we construct an $MPC(2v + 8, 4, 3)$ $(\overline{W}, \overline{C}, \overline{L})$. Put $\overline{W} = W \cup X \cup Y$, $\overline{L} = L \cup L_1$ and assign to $\overline{C}$ the following 4-cycles:

- the blocks of $C \cup \mathcal{B}$;
- $(1, a_1^i, 2, b_1^i), (1, a_2^i, 2, b_2^i), (3, a_3^i, 4, b_1^i), (3, a_4^i, 4, b_2^i), (a_3^i, 1, a_4^i, 5),$ $(a_3^i, 2, a_4^i, 6), (a_1^i, 3, a_2^i, 5), (a_1^i, 4, a_2^i, 6)$ for $i = 1, 2, \ldots, k$;
- the blocks of an edge-disjoint decomposition of $K_{D,\{1,2,3,4\}}$, $K_{A,\{7,8\}}$, $K_{B,\{5,6,7,8\}}$ into 4-cycles.

It is easy to see that $(\overline{W}, \overline{C}, \overline{L})$ is an $MPC(2v + 8, 4, 1)$ which embeds an $H(v + 4, 3, 1)$ on vertex set $V \cup \{1, 2, 3, 4\}$. ∎

Lemma 3.4 *Let $v \in \mathcal{SH}(n, 4, 3)$, $n \equiv 3, 5, 7 \pmod 8$. Then $v + 4 \in \mathcal{SH}(n + 8, 4, 3)$.*

Proof. Let $(V, \mathcal{P})$ be an $H(v, 3, 1)$ embedded into an $MPC(n, 4, 1)$ $(V \cup W, C, L)$. Put $v = 1 + 4k$, $k \geq 1$, $A_i = \{a_1^i, a_2^i, a_3^i, a_4^i\}$, $A = \cup_{i=1}^k A_i$, $V = A \cup \{\infty\}$ and $W = \{b_j \mid j = 1, 2, \ldots, n - 4k - 1\}$. By Theorem 1.1, there exists an $H(5, 3, 1)$ $(V_1, \mathcal{P}_1)$ embedded into a 4-cycle system of order 9 (W_1, C_1). Put $V_1 = \{\infty, 1, 2, 3, 4\}$ and $W_1 = V_1 \cup \{5, 6, 7, 8\}$. In order to construct an $MPC(n + 8, 4, 1)$ $(\overline{W}, \overline{C}, \overline{L})$, put $\overline{W} = V \cup W \cup W_1$, $\overline{L} = L$ and assign to $\overline{C}$ the following 4-cycles:

- the blocks of $C \cup C_1$;
- $(1, a_1^i, 2, b_{2i-1}), (1, a_2^i, 2, b_{2i}), (3, a_3^i, 4, b_{2i-1}), (3, a_4^i, 4, b_{2i}), (a_3^i, 1, a_4^i, 5),$ $(a_3^i, 2, a_4^i, 6), (a_1^i, 3, a_2^i, 5), (a_1^i, 4, a_2^i, 6)$ for $i = 1, 2, \ldots, k$ (as showed in the proof of Lemma 3.1, $n - v > \frac{v-1}{2}$);

- the blocks of an edge-disjoint decomposition of $K_{W,\{5,6,7,8\}}$, $K_{A,\{7,8\}}$ and $K_{\{b_{2k+1},b_{2k+2},\dots,b_{n-4k-1}\},\{1,2,3,4\}}$ into 4-cycles.

It is easy to see that $(\overline{W},\overline{C},\overline{L})$ embeds an $H(v+4,4,1)$ on vertex set $V \cup V_1$.
∎

Lemma 3.5 *Let* $v \in \mathcal{SH}(n,4,3)$, $n \equiv 3,5,7 \pmod 8$. *Then* $v+16 \in \mathcal{SH}(n+24,4,3)$.

Proof. Let $(V,\mathcal{P})$ be an $H(v,3,1)$ embedded into an $MPC(n,4,1)$ (W,C,L). Put $v = 1+4k$, $V = \{0,1,\dots,4k\}$ and $W = V \cup \{4k+1, 4k+2,\dots,n-1\}$. Let $A_i = \{a_1^i, a_2^i, a_3^i, a_4^i\}$, $A = \cup_{i=1}^4 A_i$, $B_i = \{\alpha_1^i, \alpha_2^i\}$, $B = \cup_{i=1}^4 B_i$, $\overline{V} = V \cup A$ and $\overline{W} = W \cup A \cup B$.

In order to construct the required $MPC(n+24,4,3)$, we introduce the following notation. Let $X = \{x_1,x_2,x_3,x_4\}$, $Y = \{y_1,y_2,y_3,y_4\}$, $T = \{t_1,t_2\}$, $U = \{u_1,u_2\}$. Define

- $\Gamma(X,T) = \{(0,x_1,x_4,t_1),(x_1,x_2,0,t_2),(x_2,x_3,x_1,t_1),(x_3,x_4,x_2,t_2),$ $(x_4,0,x_3,1+4k)\}$;
- $\Phi(X,Y,T,U) = \{(y_1,x_1,y_2,t_1),(y_1,x_2,y_2,t_2),(y_3,x_3,y_4,t_1),$ $(y_3,x_4,y_4,t_2),(x_3,y_1,x_4,u_1),(x_3,y_2,x_4,u_2),(x_1,y_3,x_2,u_1),$ $(x_1,y_4,x_2,u_2)\}$;
- $\Lambda_j(X,T) = \{(x_1,1+4j,x_2,1+4k+2j),(x_1,2+4j,x_2,2+4k+2j),(x_3,3+4j,x_4,3+4k+2j),(x_3,4+4j,x_4,4+4k+2j),(3+4j,x_1,4+4j,t_1),(3+4j,x_2,4+4j,t_2),(1+4j,x_3,2+4j,t_1),(1+4j,x_4,2+4j,t_2)\}$.

Assign to $\overline{C}$ the following blocks:

- the 4-cycles of C;
- the 4-cycles of $\Gamma(A_i,B_i)$ for every $i = 1,2,3,4$ (these cycles induce the paths of an $H(5,3,1)$ on $A_i \cup \{0\}$);
- the 4-cycles of $\Phi(A_i,A_j,B_i,B_j)$ for every $i,j = 1,2,3,4$, $i<j$ (these cycles induce a balanced decomposition of K_{A_i,A_j} into P_3s);
- the 4-cycles of $\cup_{i=1}^4 \Lambda_j(A_i,B_i)$ for $j = 0,1,\dots,k-1$ (these cycles induce a balanced decomposition of $K_{A_i,\{1+4j,2+4j,3+4j,4+4j\}}$ into P_3s);
- $(\alpha_1^i,\alpha_2^i,2+4k,a_3^i),(a_4^i,\alpha_2^i,\alpha_1^{i+1},2+4k)$ for $i = 1,2,3,4$ ($i+1$ is reduced (mod 4) to the range $\{1,2,3,4\}$);
- $(1+4k,\alpha_1^1,\alpha_1^2,\alpha_1^3),(1+4k,\alpha_1^2,\alpha_2^3,\alpha_2^4),(1+4k,\alpha_2^1,\alpha_2^2,\alpha_2^3),$ $(1+4k,\alpha_2^2,\alpha_1^1,\alpha_1^4),(\alpha_1^3,\alpha_1^1,\alpha_2^3,\alpha_2^1),(\alpha_1^4,\alpha_2^1,\alpha_2^4,\alpha_1^2),(\alpha_1^4,\alpha_2^2,\alpha_2^4,\alpha_1^3)$;

- for $i = 1,2,3,4$, the blocks of an edge-disjoint decomposition into 4-cycles of $K_{\{a_1^i,a_2^i\},\{1+6k,2+6k,\dots,n-1\}}$, $K_{B_i,\{3+4k,4+4k,\dots,n-1\}}$ and (if $n \geq 5+6k$) $K_{\{a_3^i,a_4^i\},\{3+6k,4+6k,\dots,n-1\}}$.

It is easy to check that $(\overline{W},\overline{C},L)$ embeds an $H(v+16,3,1)$ on vertex set $\overline{V}$. ∎

Remark 3.1 If in Lemma 3.5 we suppose that the handcuffed design $(V,\mathcal{P})$ embeds an $H(1+4\rho,3,1)$ $(V_\rho,\mathcal{P}_\rho)$ for every $\rho = 1,2,\dots,\frac{v-5}{4}$, then the produced $H(v+16,3,1)$ $(\overline{V},\overline{\mathcal{P}})$ embeds an $H(1+4\rho,3,1)$ for every $\rho = 1,2,\dots,\frac{v+11}{4}$.

Theorem 3.1 *$\mathcal{SH}(n,4,3) = \{v \mid 5 \leq v \leq \theta(n), v \equiv 1 \pmod 4\}$ for every integer $n \geq 10$.*

Proof. By Lemma 3.1, it is sufficient to prove that $\{v \mid 5 \leq v \leq \theta(n),\ v \equiv 1 \pmod 4\} \subseteq \mathcal{SH}(n,4,3)$. Let n be even. Then apply Lemmas 3.2 and 3.3 to Example 1.3.

Let n be odd. An $MPC(11,4,1)$ which embeds an $H(5,3,1)$ is given in Example 1.4.

An $MPC(13,4,1)$ (W,C,L) embedding an $H(5,3,1)$ on vertex set $V = \{0,1,\dots,4\}$: $W = V \cup \{a_0,a_1,\dots,a_7\}$, $L = \{(a_1,a_3,a_2),(a_2,a_4,a_6)\}$ and $C = \{(0,1,4,a_0),(1,2,0,a_1),(2,3,1,a_5),(3,4,2,a_3),(4,0,3,a_4),(0,a_3,1,a_4),$ $(2,a_1,3,a_2),(2,a_6,3,a_7),(1,a_2,4,a_7),(a_1,a_7,a_2,a_0),(a_0,a_6,a_7,a_3),$ $(a_5,a_1,a_6,4),(a_2,a_5,a_7,0),(a_0,a_7,a_4,2),(a_0,a_5,a_6,1),(a_5,a_3,a_6,0),$ $(a_0,a_4,a_5,3),(a_1,a_4,a_3,4)\}$.

An $MPC(15,4,1)$ (W,C,L) embedding an $H(5,3,1)$ on vertex set $V_1 = \{0,1,\dots,4\}$ and an $H(9,3,1)$ on vertex set $V = V_1 \cup \{5,6,7,8\}$: $W = V \cup \{a_0,a_1,\dots,a_5\}$, $L = \{(a_0,a_3,a_1,a_4,a_2)\}$, $C = (0,1,4,a_0),(1,2,0,a_1),$ $(2,3,1,a_4),(3,4,2,a_1),(4,0,3,a_2),(0,5,8,a_3),(5,6,0,a_5),(6,7,5,a_0),$ $(7,8,6,a_3),(8,0,7,a_0),(5,1,6,a_2),(5,2,6,a_4),(7,3,8,a_1),(7,4,8,a_2),$ $(3,5,4,a_3),(3,6,4,a_4),(1,7,2,a_2),(1,8,2,a_3),(a_5,a_0,a_4,a_3),(a_4,a_5,a_2,0),$ $(a_0,a_1,a_5,1),(a_1,a_2,a_3,5),(2,a_0,3,a_5),(4,a_1,6,a_5),(7,a_4,8,a_5)$.

To complete the proof apply Lemmas 3.4, 3.5 and Remark 3.1 to the above $MPC(n,4,1)$s for $n = 11,13,15$. ∎

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