DECOMPOSITION OF COMPLETE GRAPHS AND COMPLETE BIPARTITE GRAPHS INTO α-LABELLED TREES

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Abstract

Let G be a graph with r vertices of degree at least two. Let H be any graph. Consider r copies of H. Then $G \oplus H$ denotes the graph obtained by merging the chosen vertex of each copy of H with every vertex of degree at least two of G. Let T_0 and T^{A_1} be any two caterpillars. Define the first attachment tree $T_1 = T_0 \oplus T^{A_1}$. For $i \geq 2$, define recursively the i^{th} attachment tree $T_i = T_{i-1} \oplus T^{A_i}$, where T_{i-1} is the $(i-1)^{th}$ attachment tree. Here one of the penultimate vertices of T^{A_i}, $i \geq 1$ is chosen for merging with the vertices of degree at least two of T_{i-1}, for $i \geq 1$. In this paper, we prove that for every $i \geq 1$, the i^{th} attachment tree T_i is graceful and admits α - valuation. Thus it follows that the famous graceful tree conjecture is true for this infinite class of i^{th} attachment trees $T_i's$, for all $i \geq 1$. Due to the results of Rosa [21] and El-Zanati et al. [5] the complete graphs K_{2cm+1} and complete bipartite graphs $K_{qm,pm}$, for $c, p, m, q \geq 1$ can be decomposed into copies of i^{th} attachment tree T_i, for all $i \geq 1$, where m is the size of such i^{th} attachment tree T_i.
Key words: Graph labeling, α-valuation, decomposition of graphs, caterpillar, caterpillar attachment.

1 Introduction

In 1963 at the Smolenice Symposium Ringel [19] conjectured that K_{2m+1}, the complete graph on $2m+1$ vertices, can be decomposed into $2m+1$ isomorphic copies of a given tree with m edges. Almost during the same time, Kotzig [11] also conjectured that the complete graph on $2m+1$ vertices can be cyclically decomposed into $2m+1$ isomorphic copies of a given tree with m edges. These conjectures led to the introduction of classical graph labelings by Rosa [21] in 1967. In his classical paper, Rosa introduced β-labelings as a tool to attack Ringel and Kotzig's conjectures. This labeling was later called *graceful* by Golomb [7] and now this is the term most widely used. A function f is called a *graceful labeling* of a graph G with m edges, if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. A graceful labeling f is called an *α-valuation* of G, if there exists an integer λ such that for each edge $e = uv$, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. A graph admitting an *α-valuation* must necessarily be bipartite. In the same paper [21], Rosa proved that if a tree on m edges is graceful, then K_{2m+1} can be decomposed into isomorphic copies of this tree. This theorem led to pose the Ringel-Kotzig-Rosa conjecture that *all trees are graceful.* The Ringel-Kotzig-Rosa conjecture, popularly known as Graceful Tree Conjecture, has been the focus of many papers for about four decades. So far, no proof of the truth or falsity of the conjecture has been found. In the absence of a generic proof, one approach used in investigating the graceful tree conjecture is proving the gracefulness of special classes of trees. The graceful tree conjecture is shown to be true for all trees with at most 29 vertices [12]. It is proved that caterpillars (trees whose removal of all the end vertices produces a path) are graceful [21]. Bermond [1] conjectured that all lobsters (trees whose removal of all end vertices produces a caterpillar) are graceful. Special cases of this conjecture are shown to be graceful (refer to [14], [22]). A banana tree is a tree obtained by connecting a vertex v to one leaf of each of any number of stars, where v is not in any of the stars. Chen et al. [4] have conjectured that all banana trees are graceful. Some special classes of banana trees were shown to be graceful in [13] and [2]. Recently, Sethuraman and Jeba Jesintha [16] have shown that all banana trees are graceful. An olive tree is a collection of k paths joined in one of the end vertices, where the i^{th} path has length i. Pastel and Raynaud [15] has shown that olive trees are graceful. A symmetrical tree is a rooted tree where all the vertices at the same distance from the root have the same degree. Bermond and Sotteau have proved that all symmetrical trees are graceful. Sethuraman and Jeba Jesintha [17] have extended this result. They have shown that a rooted tree in which every level contains the vertices that have a degree either k or one (k may vary for each level) is graceful. Pavel Hrnciar and Alfons Haviar [18] introduced a new technique of transferring pendant edges incident at a vertex to some other suitable vertex in a graceful tree, thereby they showed that all tree with at most diameter five are graceful. Apart from the above families of trees a few more specific families of trees are shown to be graceful. For an exhaustive survey on this topic refer the excellent dynamic survey by Gallian [6].

A very few results deal with general method of constructing graceful trees from known graceful trees. In [20] Stanton and Zarnke and in [10] Koh et al., have given different methods for constructing graceful trees to obtain bigger graceful trees from known graceful trees. Burzio and Ferrarese [3] extended the method of Koh et al., and consequently they have shown an interesting and significant result that subdivision of every graceful tree is graceful. In this paper we introduce a new method of combining graceful trees called recursive attachment of trees.

Let G be a graph with r vertices of degree at least two. Let H be any graph. Consider r copies of H. Then $G \oplus H$ denotes the graph obtained by merging the chosen vertex of each copy of H with every vertex of degree at least two of G. Let T_0 and T^{A_1} be any two caterpillars. Define the first attachment tree $T_1 = T_0 \oplus T^{A_1}$. For $i \geq 2$, define recursively the i^{th} attachment tree $T_i = T_{i-1} \oplus T^{A_i}$, where T_{i-1} is the $(i-1)^{th}$ attachment tree. Here one of the penultimate vertices of T^{A_i}, $i \geq 1$ is chosen for merging with the vertices of degree at least two of T_{i-1}, for $i \geq 0$. In this paper, we prove that for every $i \geq 1$, the i^{th} attachment tree T_i is graceful and admits α-valuation. Consequently, due to the results of Rosa [21] (If a graph G with m edges has an α-valuation, then there exists cyclic decomposition of the edges of the complete graph K_{2cm+1} into subgraphs isomorphic to G, where c is an arbitrary natural number) and due to the result of El-Zanati and Vanden Eynden [5] (For any non-trivial connected graph G with m edges, there exists decomposition of the edges of the complete bipartite graph $K_{mq,mp}$ into subgraphs isomorphic to G, where p and q are arbitrary positive integers) complete graphs K_{2cm+1} and certain complete bipartite graphs $K_{mq,mp}$, for $c, p, q, m \geq 1$ into copies isomorphic to any i^{th} recursive attachment tree T_i, for $i \geq 1$.

2 T_1 is graceful

In this section we show that the first attachment tree $T_1 = T_0 \oplus T^A$ is graceful.

In a tree a vertex of degree at least two is called *admissible vertex*. A vertex of a caterpillar which has at most one adjacent vertex of degree greater than or equal to two is called a *penultimate vertex*. Call one of the penultimate vertices of a caterpillar as its *head* and the other penultimate vertex as its *tail*.

By an *orientation* of a graph G we mean an assignment of directions to all the edges of G. In an orientation of a tree T, a pendant edge is said to have *outward orientation* if it is given directions from the end vertex of the edge of degree greater than one to the other end vertex of degree one.

Let $\{u_1, u_2, \ldots, u_r\}$ be the set of all admissible vertices of a caterpillar T. Observe that $< u_1, u_2, u_3, \ldots, u_r >$ is the diametrical path of T, with u_r as the head of T and u_1 as the tail of T. Then the orientation of a caterpillar T is called

1. *head to tail* orientation if the edges $u_i u_{i+1}$, $1 \leq i \leq r-1$ are given directions from u_{i+1} to u_i, for $1 \leq i \leq r-1$ and the pendant edges

are given outward orientations.

2. *tail to head* orientation if the edges u_iu_{i+1}, $1 \le i \le r-1$ are given directions from u_i to u_{i+1}, for $1 \le i \le r-1$ and the pendant edges are given outward orientations.

3. *alternate orientation* if the edges u_iu_{i+1}, $1 \le i \le r-1$ are given directions from u_i to u_{i+1}, for $1 \le i \le r-1$, when i is odd, while for $2 \le i \le r-1$ and i even the edges u_iu_{i+1} are given directions from u_{i+1} to u_i and the pendant edges are given outward orientations.

Let T_0 denote any caterpillar having alternate orientation. Let $\{u_1, u_2, \ldots, u_r\}$ be the set of all admissible vertices of T_0. Let T^A be any caterpillar. Consider r copies of T^A and denote them by T_i^A, $1 \le i \le r$. We call a copy of T^A, T_i^A as odd or even depending on i is odd or even. For $1 \le i \le r$ and i odd, give tail to head orientation, while for $2 \le i \le r$, and i even, give head to tail orientation. Attach (merge) the head of each copy T_i^A at each admissible vertex u_i of T_0, for $1 \le i \le r$. Then the resulting tree is the first attachment tree $T_1 = T_0 \oplus T^A$.

Theorem 1. *The first attachment tree T_1 admits α-labeling.*

Proof. Consider the first attachment tree T_1. Let N be the number of edges of T_1.

STEP 1 (INTRODUCTION OF ARTIFICIAL EDGES)

For convenience of the graceful labeling we introduce edges between certain pairs of vertices of T_1. More precisely, join the tail of the T_i^A, say w_i, to the tail of the T_{i+1}^A, say w_{i+1}, by an arc from w_i to w_{i+1}, for $2 \le i \le r-1$ and i even (we call such arcs as artificial edges). The graph thus obtained is denoted by T_1^+. Observe that T_1^+ has no odd cycle. Thus T_1^+ is a bipartite graph.

We designate the tail of T_1^A as the *source* of T_1. If r is even, tail of T_r^A is designated as the *sink* of T_1 and if r is odd, head of T_r^A is designated as the *sink* of T_1.

STEP 2 (NAMING OF THE ADMISSIBLE VERTICES IN T_1^+)

Observe that in the construction of T_1 some of the vertices are labeled and the other vertices are unlabeled. Consider T_1^+ as the unlabeled graph by ignoring the labels of the vertices of T_1^+ if any exists. Now name the tail of T_1^A of T_1^+ as v_1 and its unique unlabeled adjacent admissible vertex (with out-degree atleast one) by v_2. In T_1^+, for $i \ge 1$ if an admissible vertex is named v_i, then name its unique unlabeled adjacent admissible vertex (with out-degree atleast one) by v_{i+1}, continue to name all the admissible vertices and finally name the sink.

STEP 3 (FINDING THE UNIQUE PATH PASSING THROUGH ALL THE ADMISSIBLE VERTICES OF T_1^+)

Observe that in T_1^+ there is a unique (directed) path starting with the source v_1 and ending with the sink v_t of T_1^+ containing all the admissible vertices $v_1, v_2, v_3, \ldots, v_t$. Denote this path $v_1 v_2 v_3 \ldots v_t$ by P.

STEP 4 (DEFINING BIPARTITION OF T_1)

Let $N_1(v)$ denote the set of pendant vertices which are adjacent from v.

For $1 \le i \le t-1$ and for each v_i, define, $S_{v_i} = \{v_i\} \cup N_1(v_{i+1})$

$$\text{Consider } A = \bigcup_{\substack{1 \le i \le t \\ i \text{ odd}}} S_{v_i} \text{ and } B = N_1(v_1) \bigcup_{\substack{1 \le i \le t \\ i \text{ even}}} S_{v_i}$$

Note that $v_t \in A$ or B depends on whether t is odd or even. Clearly (A, B) as defined above gives the bipartition of T_1^+. Observe that after removing all the artificial edges from T_1^+, the above partition (A, B) also defines a bipartition for the tree T_1.

STEP 5 (GRACEFUL LABELING OF T_1)

Observe that corresponding to each artificial edge $w_i w_{i+1}$, joining the tail of T_i^A to the tail of T_{i+1}^A, for i even, there is a unique arc e in T_1^+, joining the head of T_{i+1}^A to the head of T_i^A, with i even, then such arc e of T_1^+ is called the *counter edge* of the artificial edge $w_i w_{i+1}$. Denote the vertices of A and B by $a_1, a_2, a_3, \ldots, a_p$ and by $b_1, b_2, b_3, \ldots, b_q$ respectively, where $p+q = N+1$. Give label $N, N-1, N-2, N-3, \ldots, N-(p-1)$ respectively to the vertices $a_1, a_2, a_3, \ldots, a_p$ of A and $0, 1, 2, \ldots, q-1$, respectively to the vertices $b_1, b_2, \ldots, b_q$ of B. In the above labeling observe that in T_1^+, edge value of an artificial edge and its counter edge in T_1^+ have the same edge value. Thus removal of all the artificial edges from T_1^+ results in T_1 with distinct edge values ranging from 1 to N. Therefore the above labeling gives the required graceful labeling of T_1. It is easy to see that this graceful labeling is also an α-labeling. □

3 The tree T_i is graceful

In this section we prove that the i^{th} attachment tree T_i is graceful, for $i \ge 2$.

By induction we assume that the $(i-1)^{th}$ attachment tree T_{i-1}, $i \ge 2$ has α-valuation. For $i \ge 2$, consider the graph T_{i-1}^+ ($(i-1)^{th}$ attachment tree together with the artificial edges). Ignore the α-labeling of T_{i-1}. Consider the unique path $P : v_1 v_2 v_3 \ldots v_l$ of T_{i-1}^+ as defined the STEP 3 of proof of Theorem 1 (Note that $\{v_1, v_2, \ldots, v_l\}$ is the set of all admissible vertices of T_{i-1}^+. For each edge $v_j v_{j+1}$ of P retain the directions v_j to v_{j+1}, for $1 \le j \le l-1$ and j odd and while for $2 \le j \le l-1$ and j even, reorient the edges $v_j v_{j+1}$ in the opposite direction as $v_{j+1} v_j$. Denote this reoriented graph as $\tilde{T}_{i-1}^{+}$. Now, remove all the artificial edges of $\tilde{T}_{i-1}^{+}$. The resulting graph thus obtained is a tree $\tilde{T}_{i-1}$ with different orientation on certain edges.

Let T^A be any caterpillar. Consider l copies of the T^A and denote them by T_j^A, for $1 \leq j \leq l$. Give tail to head orientation or head to tail orientation to the copies T_j^A, for $1 \leq j \leq l$ depending on whether j is odd or even.

Attach the head of each copy of T_j^A at each admissible vertex v_i of T_{i-1}, for $1 \leq j \leq l$. The resulting tree thus obtained is T_i, the i^{th} attachment tree $T_{i-1} \oplus T^A$.

Theorem 2. *For $i \geq 2$, the i^{th} attachment tree T_i admits α-labeling.*

Proof. Let N be the number of edges of T_i.

For $i \geq 2$, consider T_i.

For convenience of the graceful labeling we introduce artificial edges between certain pairs of vertices of T_i. More precisely, for $2 \leq j \leq l-1$ and j even join the tail of the T_j^A, say w_j, to the tail of the T_{j+1}^A, say w_{j+1}, by an arc from w_j to w_{j+1} (we refer such arcs as artificial edges). The graph thus obtained is denoted by T_i^+. Observe that T_i^+ has no odd cycle. Thus T_i^+ is a bipartite graph.

Designate the tail of the first copy T_1^A as the *source* of T_i^+. If l is even, tail of T_l^A is designated as the *sink* of T_i^+ and if l is odd, head of T_l^A is designated as the *sink* of T_i^+. Ignore all the labels of the admissible vertices in T_i.

Similarly as in STEP 2 of Theorem 1 here too we NAME the tail of T_1^A of T_i^+ as v_1 and its unique unlabeled adjacent admissible vertex (with out-degree at least one) by v_2. For $j \geq 1$ in T_i^+ if an admissible vertex is named v_j then name its unique unlabeled adjacent admissible vertex (with out-degree at least one) by v_{j+1} and continue to name all the admissible vertices of T_i^+ and finally name the sink of T_i^+.

As in STEP 3 of Theorem 1 here too we observe that there is a unique path P starting with the source and ending with the sink of T_i^+ containing all the admissible vertices $v_1, v_2, \ldots, v_l$.

Define the bipartition (A, B) for T_i as defined in STEP 4 of Theorem 3. Denote the elements of A and B of (A, B) by $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ respectively, where $p+q = N+1$. Give the labels $N, N-1, N-2, \ldots, N-(p-1)$ to the vertices $a_1, a_2, \ldots, a_p$ of A and $0, 1, 2, \ldots, q-1$ to the vertices $b_1, b_2, \ldots, b_q$ of B. This labeling as seen in STEP 5 of Theorem 1, also results in a graceful labeling of T_i. It is easy to see that this graceful labeling is also an α-labeling. □

From Theorems 1 and 2 following corollary 1 is an immediate consequence of Rosa's theorem [21].

Corollary 1. *For each tree T_i, for $i \geq 1$, there exists a decomposition of the edges of the complete graph K_{2cm+1} into subgraphs isomorphic to T_i, where c is an arbitrary natural number and $m = |E(T_i)|$.*

Due to the recent result on El-Zanati and Vanden Eynden [5] we have the following corollary.

Corollary 2. *For each tree T_i, for $i \geq 1$, there exists a decomposition of the edges of the complete bipartite graph $K_{qm,pm}$ into subgraphs isomorphic to T_i, where p, q are arbitrary natural numbers with $m = |E(T_i)|$.*

4 Discussion

Observe that for $i \geq 1$, T_i is defined recursively from T_{i-1}, with T_0 as any caterpillar. As T_0 is graceful, our attachment process led T_i to be graceful. So it tempts to ask the question that if T_i is defined recursively from T_{i-1} with T_0 as any α-valuation tree, whether is it true that T_i always admits graceful labeling. We believe that the answer to this question is affirmative. So we pose the following.

Conjecture 1. *For $i \geq 1$, $T_i = T_{i-1} \oplus T^A$ is graceful where T_0 is any α-labeled tree and T^A is any caterpillar.*

Acknowledgement

The authors thankfully acknowledge the referee for his/her valuable suggestions in improving the presentation of the paper.

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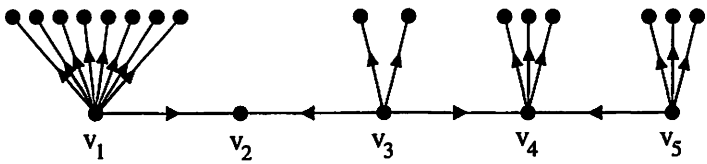
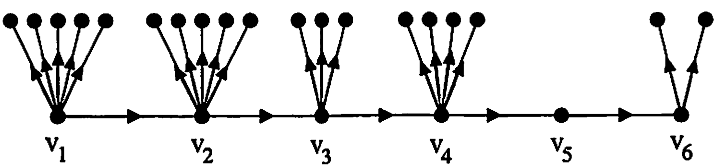
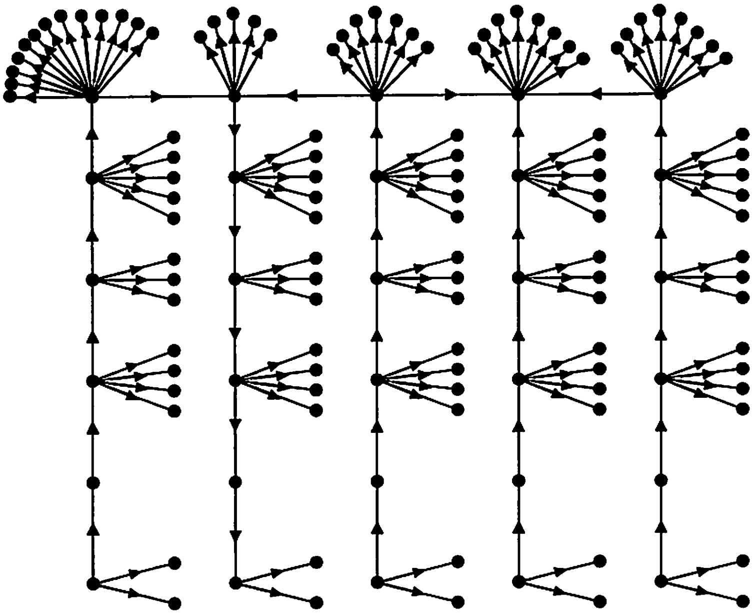
APPENDIX



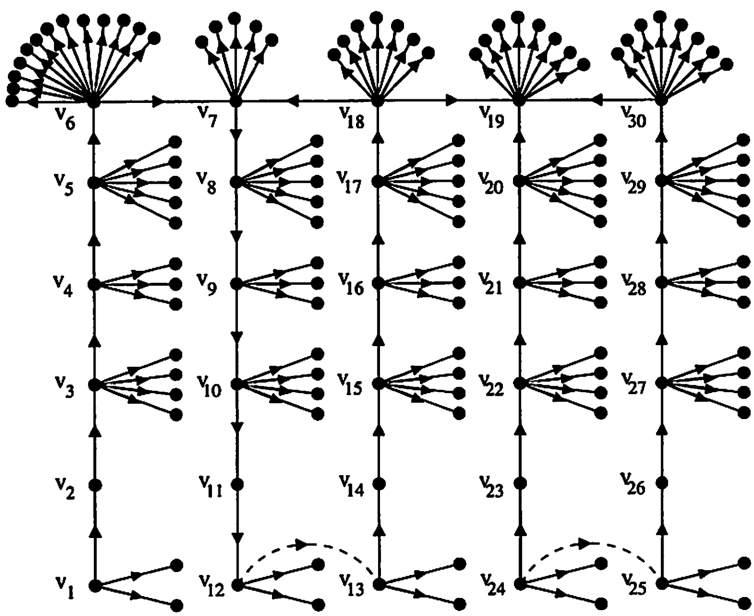
The tree T_0



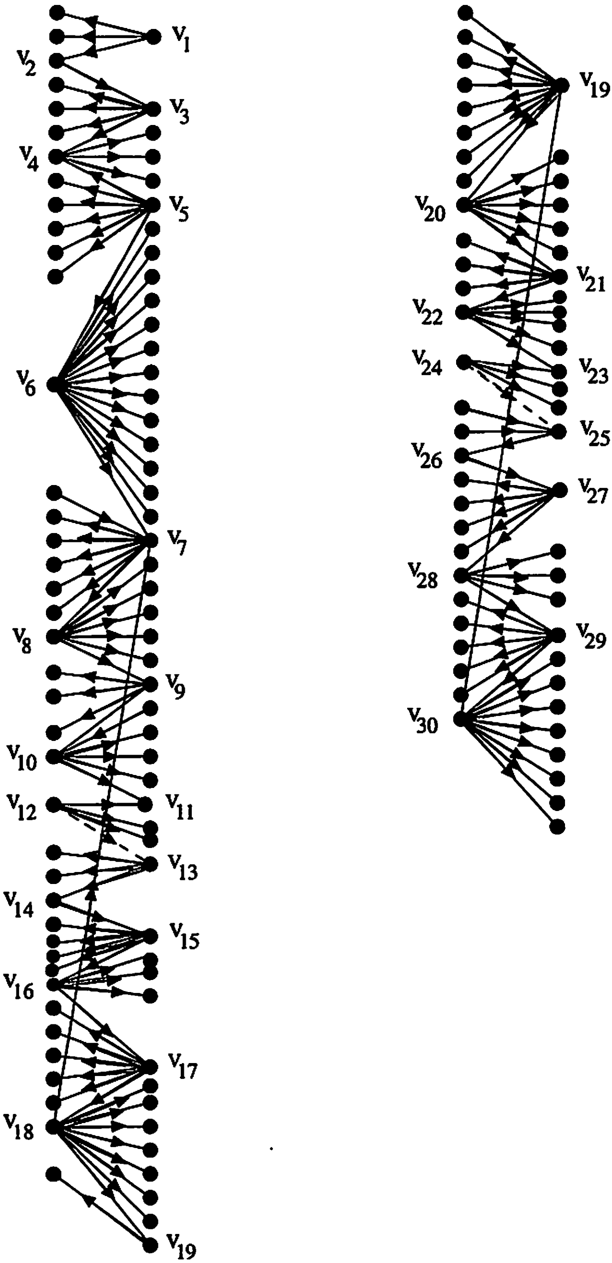
The tree T^A with the head v_1 and the tail v_6



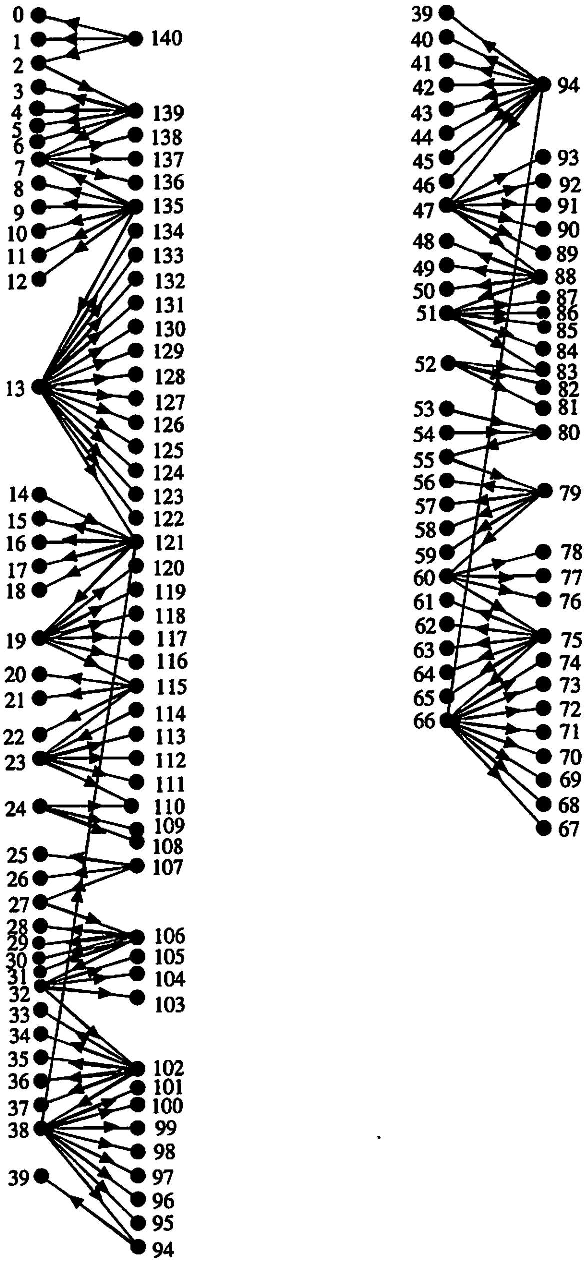
$T_1 = T_0 \oplus T^A$



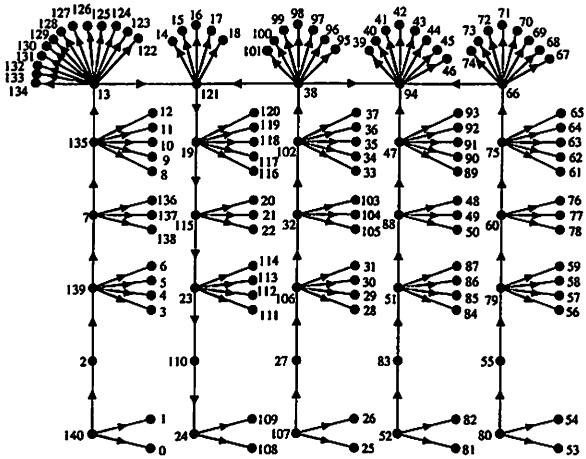
T_1^+ (together with artificial edges)



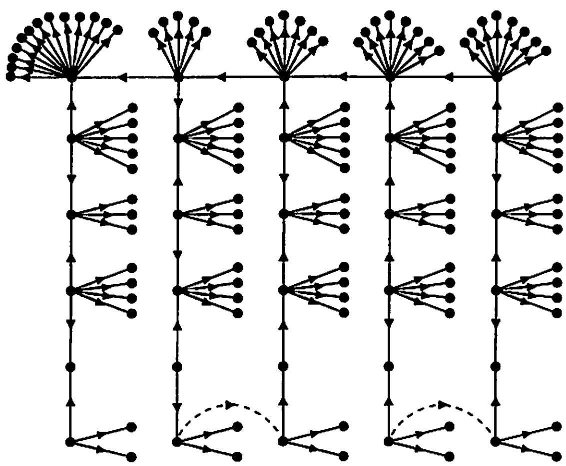
Bipartition of T_1^+



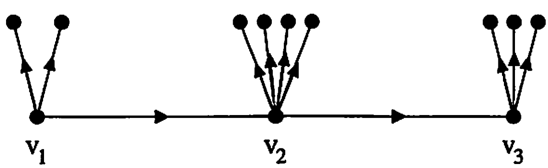
α-valuation of T_1^+



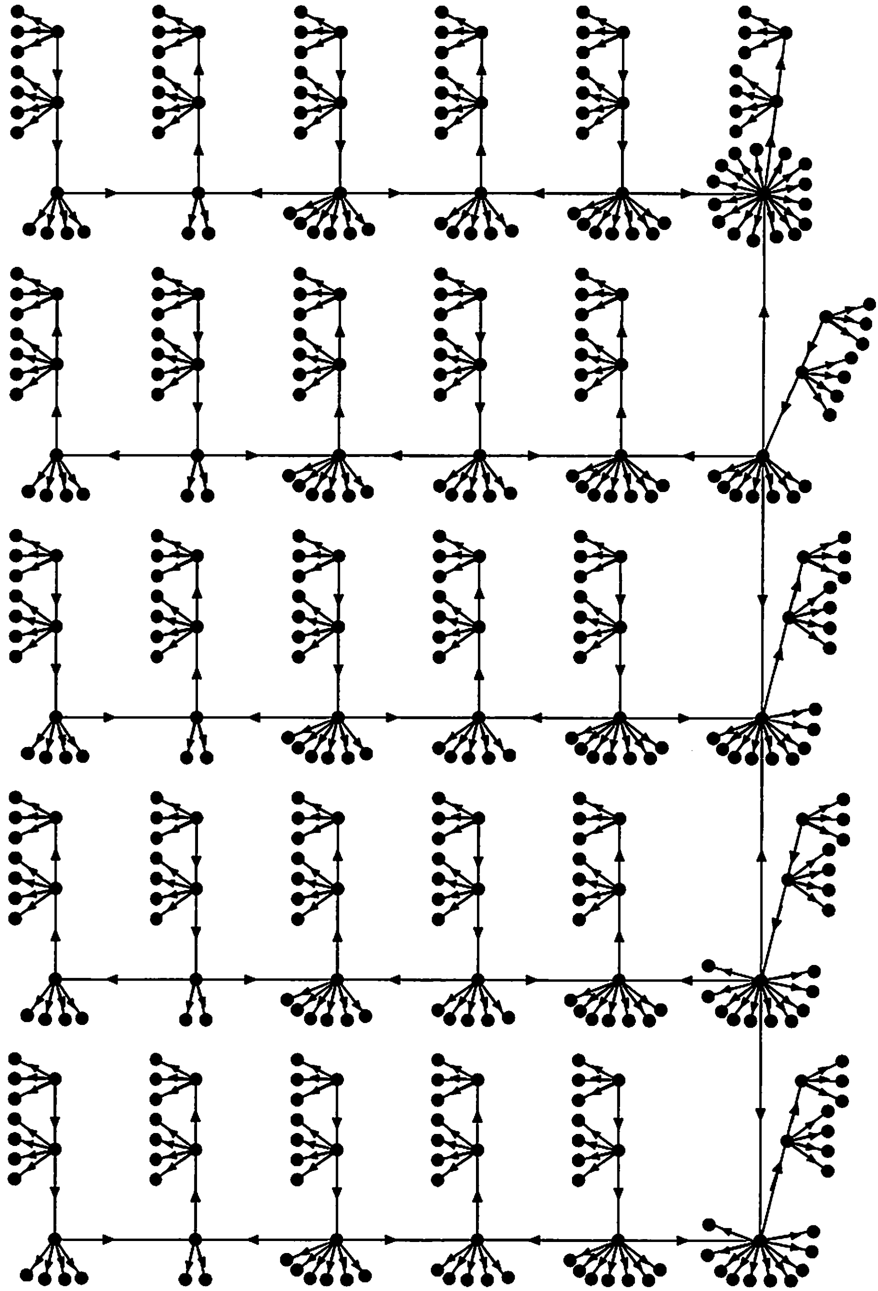
T_1 with α-valuation



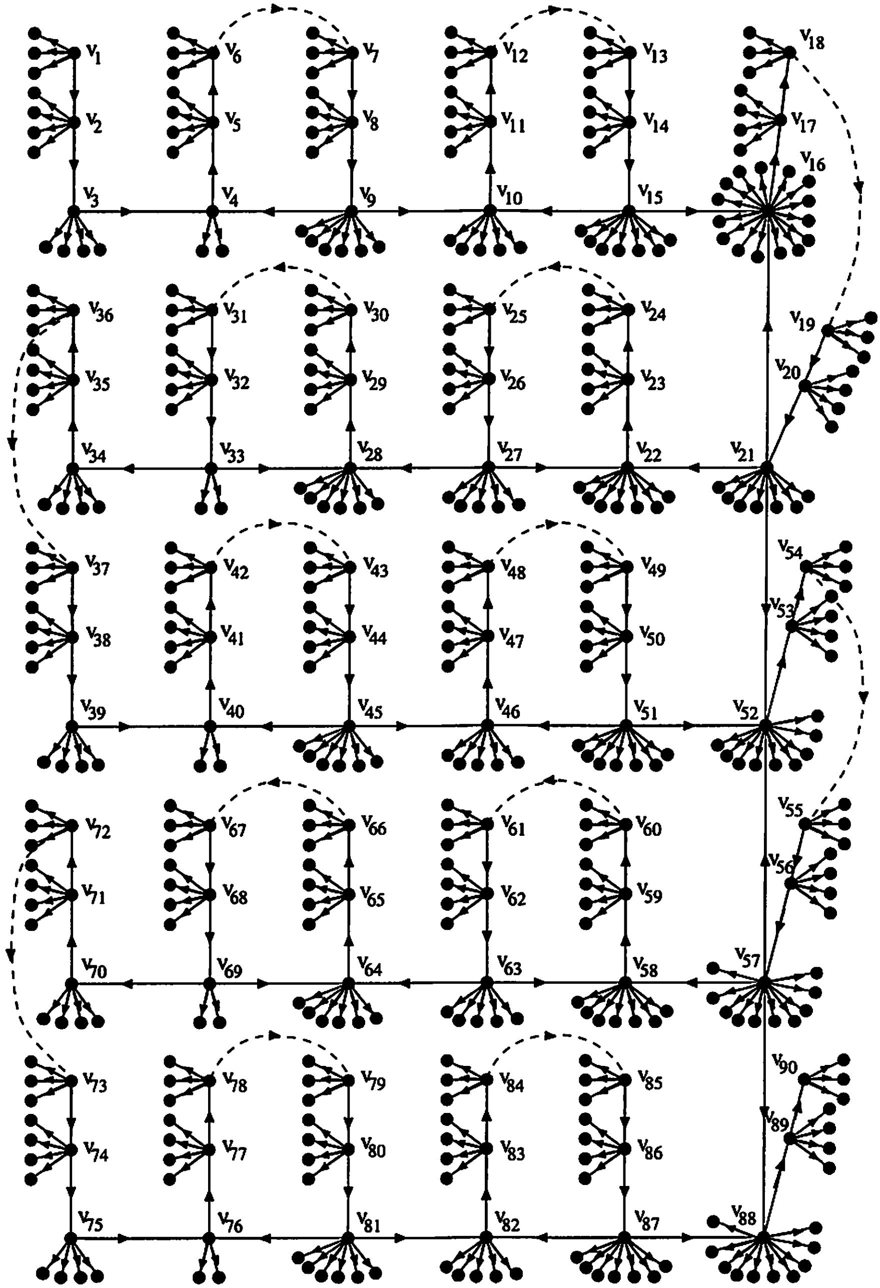
The graph $T_1 = \tilde{T}_{i-1}^{+}$



The Attachment tree T^A with the head v_1



$T_2 = T_1 \oplus T^A$, where T_1 as in Fig.7 and T_A as in Fig. 8



$T_2^+ = T_1 \oplus T^A$