$H-$equipackable paths and cycles for $H = P_4$ and $H = M_3$ *†

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Abstract

A graph G is called $H-$*equipackable* if every maximal H-packing in G is also a maximum $H-$packing in G. All M_2-equipackable graphs and P_3-equipackable graphs have been characterized. In this paper, P_4-equipackable paths, P_4-equipackable cycles, M_3-equipac kable paths and M_3-equipackable cycles are characterized.

Keywords: Packing, equipackable, path, matching.

1 Introduction

The problem that we study stems from research of $H-$decomposable graphs, randomly packable graphs and equipackable graphs. A graph G has order $|V(G)|$ and size $|E(G)|$. The path and cycle on k vertices are denoted by P_k and C_k, respectively. A matching in the graph G is a set of independent edges in G. A matching with $t(t \geq 1)$ edges is denoted by $M_t(t \geq 1)$. Let H be a subgraph of G. By $G - H$, here we denote the graph left after we delete from G the edges of H and any resulting isolated vertices. A collection of edge disjoint copies of H, say $H_1, H_2, \cdots, H_k$, where each $H_i(i = 1, 2, \cdots, k)$ is a subgraph of G, is called an $H-$*packing* in G. A graph G is called $H-$ *packable* if there exists an $H-$packing of G. An $H-$packing in G with k copies $H_1, H_2, \cdots, H_k$ of H is called *maximal* if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to H. An $H-$packing in G with k copies $H_1, H_2, \cdots, H_k$ of H is called *maximum* if no more than k edge disjoint copies of H can be packed into G. A graph G is called $H-$ *decomposable* if there exists an $H-$packing of G which uses all edges in G and G is called *randomly* $H-$ *decomposable* if every maximal $H-$packing in G uses all edges in G(See [4]). There have been

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many results on H−decomposable graphs and randomly H−decomposable graphs(See [5] and [1], where randomly H−decomposable is called randomly H−packable).

As a relaxation of random H−decomposability, B. L. Hartnell and P. D. Vestergaard [2] gave the definition of H−equipackable: a graph G is called H − *equipackable* if every maximal H−packing in G is also a maximum H−packing in G. And they characterized P_3−equipackable graphs of girth five or more. Later, P. D. Vestergaard [5] gave the characterization of P_3−equipackable graphs with all valences at least two. Recently, B. Randerath and P. D. Vestergaard characterized all P_3−equipackable graphs. In 2006, Zhang and Fan ([6]) characterized all M_2−equipackable graphs. In this paper, we investigate P_4−equipackable paths, P_4−equipackable cycles, M_3−equipackable paths and M_3−equipackable cycles.

We first give a lemma which is easy but very crucial to our work:

Lemma 1. *Let G be an F−packable graph and H be an F−packable subgraph of G which satisfy: (1) H is not F−equipackable; (2) $G - H$ is F−decomposable. Then G is not F−equipackable.*

Proof. Since H is F−packable but not F−equipackable, by the definitions of packable and equipackable, H has at least one maximal F−packing which is not maximum. And $G-H$ is F−decomposable, $G-H$ has an F−packing which uses all edges of $G-H$. The union of the two F−packing mentioned above forms a maximal F−packing which is not maximum. So G is not F−equipackable. □

2 Main results

2.1 P_4−equipackable paths

Theorem 2. *A path P_n is P_4−equipackable if and only if $n = 4, 5, 6, 9$.*

Proof. We can easily verify that P_4, P_5, P_6, P_9 are all P_4−equipackable.

Conversely, let P_n be a P_4−equipackable path, then we have five cases:

Case 1: When $n \leq 3$, since P_n contains no copy of P_4, P_n can't be P_4−equipackable.

Case 2: When $4 \leq n \leq 6$, it's easy to know the number of P_4 in the maximum P_4-packing of P_n is 1. And P_n is P_4−packable, so each maximal P_4−packing is also a maximum P_4−packing. Then P_n must be P_4−equipackable.

Case 3: When $n = 7$ or $n = 8$, the number of P_4 in the maximal P_4-packing of P_n is 1 or 2. By the definition, P_n is not P_4−equipackable.

Case 4: When $n = 9$, it's easy to verify the number of P_4 in the maximal P_4−packing of P_n only can be 2. By the definition, P_n is P_4−equipackable.

Case 5: When $n \geq 10$, there are three subcases:

Subcase 1: when $n - 7 \equiv 0(mod3)$, $P_n - P_7$ has $3k(k \in Z, k \geq 1)$ edges, so $P_n - P_7$ is P_4−decomposable. From case 3, P_7 is not P_4−equipackable. By Lemma 1, P_n is not P_4−equipackable.

Subcase 2: when $n - 7 \equiv 1(mod3)$, $P_n - P_8$ is P_4−decomposable. Similarly, P_n is not P_4−equipackable.

Subcase 3: when $n - 7 \equiv 2(mod3)$, we can easily verify that P_{12} is not P_4−equipackable: the number of P_4 in the maximal P_4−packing of P_{12} is 2 or 3. Obviously, $P_n - P_{12}$ is P_4−decomposable, so P_n is not P_4−equipackable.

From above, P_n is P_4−equipackable if and only if $n = 4, 5, 6, 9$. □

2.2 P_4−equipackable cycles

Theorem 3. *A cycle C_n is P_4−equipackable if and only if $n = 4, 5, 6, 7, 8, 11$.*

Proof. We can easily verify that $C_4, C_5, C_6, C_7, C_8, C_{11}$ are all P_4−equipack able.

Conversely, let C_n be a P_4−equipackable cycle, then we have four cases:

Case 1: When $n \leq 3$, since C_n contains no P_4, C_n can't be P_4−equipack able.

Case 2: When $n = 4$ or $n = 5$, it's easy to know the number of P_4 in the maximum P_4-packing of C_n is 1. And C_n is P_4−packable, so each maximal P_4−packing is also a maximum P_4−packing. Then C_n must be P_4−equipackable.

Case 3: When $6 \leq n \leq 8$, it's easy to verify the number of P_4 in the maximal P_4-packing of C_n only can be 2. By the definition, C_n is P_4−equipackable.

Case 4: When $n \geq 9$, there are three subcases:

Subcase 1: When $n - 6 \equiv 0(mod3)$, $C_n - P_7$ is P_4−decomposable since $C_n - P_7$ has $3k(k \in Z, k \geq 1)$ edges. By Theorem 2, P_7 is not P_4−equipackable. By Lemma 1, C_n is not P_4−equipackable.

Subcase 2: When $n - 6 \equiv 1(mod3)$, $C_n - P_8$ is P_4−decomposable. By Theorem 2, P_8 is not P_4−equipackable. By Lemma 1, C_n is not P_4−equipackable.

Subcase 3: When $n - 6 \equiv 2(mod3)$, there are two possibilities:

(1) When $n = 11$, we can easily verify that C_{11} is P_4−equipackable: the number of P_4 in the maximal P_4−packing of C_{11} only can be 3.

(2) When $n \neq 11$, $C_n - P_{12}$ is P_4−decomposable. By Lemma 1, C_n is not P_4−equipackable.

From above, C_n is P_4−equipackable if and only if $n = 4, 5, 6, 7, 8, 11$. □

2.3 M_3−equipackable paths

Theorem 4. *A path P_n is M_3−equipackable if and only if $n = 3k(k \in Z, k \geq 2)$.*

Proof. We can easily verify that $P_{3k}(k \in Z, k \geq 2)$ are all M_3−equipackable.

Conversely, let P_n be an M_3−equipackable path, then we have three cases:

Case 1: When $n \leq 5$, P_n can't be M_3−equipackable since P_n contains no M_3.

Case 2: When $6 \leq n \leq 11$, it's easy to verify when $n = 6, 9$, each maximal M_3−packing of P_n is also maximum, P_n is M_3−equipackable; when $n = 7, 8, 10, 11$, the number of M_3 in the maximal M_3-packing of P_n is not unique, so P_n is not M_3−equipackable.

Case 3: When $n \geq 12$, there are three subcases:

Subcase 1: When $n \equiv 0(mod3)$, P_n is M_3−equipackable. We can easily give a maximal M_3−packing of P_n with $[\frac{n-1}{3}]$ copies of M_3, and P_n has only $(n-1)$ edges, so the number of M_3 in the maximum M_3−packing of P_n is also $[\frac{n-1}{3}]$. In the following, we prove by contradiction that the number of every maximal M_3−packing of P_n is $[\frac{n-1}{3}]$.

Assume that there exists a maximal M_3−packing $H = \{H_1, H_2, \cdots, H_k\}$ which uses less than $[\frac{n-1}{3}]$ copies of M_3, then the number of edges remained is more than 5. Five edges in a path must contain a copy of M_3, that is, $P_n - H$ still contains M_3 which contradicts to the fact that $H = \{H_1, H_2, \cdots, H_k\}$ is a maximal M_3−packing. So P_n is M_3−equipackable.

Subcase 2: When $n \equiv 1(mod3)$, $P_n - P_7$ has $3k(k \in Z, k \geq 2)$ edges, and it has a maximal M_3−packing which uses all its edges, so it is M_3−decomposable. Since P_7 is not M_3−equipackable, by Lemma 1, P_n is not M_3−equipackable.

Subcase 3: When $n \equiv 2(mod3)$, similarly, $P_n - P_8$ is M_3−decomposable. By Lemma 1, P_n is not M_3−equipackable.

So P_n is M_3−equipackable if and only if $n = 3k(k \in Z, k \geq 2)$. □

2.4 M_3−equipackable cycles

Theorem 5. *A cycle C_n is M_3−equipackable if and only if $n = 6, 7, 3k + 2(k \in Z, k \geq 2)$.*

Proof. We can easily verify that $C_6, C_7, C_{3k+2}(k \in Z, k \geq 2)$ are all M_3−equipackable.

Conversely, let C_n be an M_3−equipackable cycle, we have three cases:

Case 1: When $n \leq 5$, C_n can't be M_3−equipackable since C_n contains no M_3.

Case 2: When $6 \le n \le 11$, it's easy to verify when $n = 6, 7, 8, 11$, each maximal M_3−packing of P_n is also maximum, C_n is M_3−equipackable; when $n = 9, 10$, the number of M_3 in the maximal M_3-packing of C_n is 2 or 3, so C_n is not M_3−equipackable.

Case 3: When $n \ge 12$, there are three subcases:

Subcase 1: When $n \equiv 0(mod3)$, $C_n - P_7$ has $3k(k \in Z, k \ge 2)$ edges, and it has a maximal M_3−packing which uses all its edges, so it is M_3−decompos able. Since P_7 is not M_3−equipackable, by Lemma 1, C_n is not M_3−equipack able.

Subcase 2: When $n \equiv 1(mod3)$, similarly, $C_n - P_8$ is M_3−decomposable. Since P_8 is not M_3−equipackable, by Lemma 1, C_n is not M_3−equipackable.

Subcase 3: When $n \equiv 2(mod3)$, C_n is M_3−equipackable:

We can easily give a maximal M_3-packing of C_n with $[\frac{n}{3}]$ copies of M_3, and C_n has n edges, so the number of M_3 in the maximum M_3-packing of C_n is also $[\frac{n}{3}]$. In the following, we still prove by contradiction that the number of every maximal M_3−packing of P_n is $[\frac{n}{3}]$.

Assume that there exists a maximal M_3−packing $H = \{H_1, H_2, \cdots, H_k\}$ which uses less than $[\frac{n}{3}]$ copies of M_3, then the number of edges remained is more than 5. Five edges in a cycle must contain a copy of M_3, that is, $C_n - H$ still contains M_3 which contradicts to the fact that $H = \{H_1, H_2, \cdots, H_k\}$ is a maximal M_3−packing. So C_n is M_3−equipackable.

From above, C_n is M_3−equipackable if and only if $n = 6, 7, 3k + 2(k \in Z, k \ge 2)$. □

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