LATTICES ASSOCIATED WITH VECTOR SPACES OVER A FINITE FIELD

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ABSTRACT. *Let V denote the n-dimensional row vector space over a finite field F_q, and fix a subspace W of dimension $n-d$. Let $L(n,d) = P \cup \{0\}$, where $P = \{A|A$ is a subspace of V, $A+W = V\}$. Partially ordered by ordinary or reverse inclusion, two families of finite atomic lattices are obtained. This article discusses their geometricity, and computes their characteristic polynomials.*

Key words: Finite field; Vector space; Lattices; Characteristic polynomials.

AMS classification: 20G40; 51D25

1. INTRODUCTION

Let F_q be a finite field with q elements, where q is a prime power. For a positive integer n, let V be the n-dimensional row vector space over F_q. For a fixed $(n-d)$-subspace W of V, let $L(n,d) = P \cup \{0\}$, where $P = \{A|A$ is a subspace of V, $A+W = V\}$. Partially ordered by ordinary or reverse inclusion, $L(n,d)$ is a finite poset, denoted by $L_O(n,d)$ or $L_R(n,d)$, respectively. For any two elements $A, B \in L_O(n,d)$,

$$A \wedge B = \begin{cases} A \cap B & \text{if } W + (A \cap B) = V, \\ \{0\} & \text{otherwise.} \end{cases}$$

$$A \vee B = A + B.$$

Similarly, for any two elements $A, B \in L_R(n,d)$,

$$A \vee B = \begin{cases} A \cap B & \text{if } W + (A \cap B) = V, \\ \{0\} & \text{otherwise.} \end{cases}$$

$$A \wedge B = A + B$$

Therefore, both $L_O(n,d)$ and $L_R(n,d)$ are finite lattices. This article discusses their geometricity, and computes their characteristic polynomials.

The results on the lattices generated by distance-regular graphs can be found in Guo, Gao and Wang(2007), the lattices generated by orbits of subspaces under finite nonsingular classical groups can be found in Wang

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and Feng(2006), Wang and Guo(2007), Wang and Li(in press), Huo et al.(1992a,b,1993),Huo and Wan (2001,2002a), and under finite singular symplectic group and singular unitary group can be found in Gao and You(2003a ,b).

2. PRELIMIANRIES

In the following we recall some definitions and facts on ordered sets and lattices(see Aigher,1979)

Let P denote a finite set. A *partial order* on P is a binary relation $\leq$ on P such that
(1) $a \leq a$ for any $a \in P$.
(2) $a \leq b$ and $b \leq a$ implies $a = b$.
(3) $a \leq b$ and $b \leq c$ implies $a \leq c$.
By a *partial ordered set* (or *poset* for short), we mean a pair $(P, \leq)$, where P is a finite set and $\leq$ is a partial order on P. As usual, we write $a < b$ whenever $a \leq b$ and $a \neq b$. By abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P be a poset and let R be a commutative ring with the identical element. A binary function $\mu(a, b)$ on P with values in R is said to be the *Möbius function* of P if

$$\sum_{a \leq c \leq b} \mu(a, c) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

For any two elements $a, b \in P$, we say a *covers* b, denoted by $b < \cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. An element m of P is said to be *minimal* (resp. *maximal*) whenever there is no element $a \in P$ such that $a < m$ (resp. $a > m$). If P has a unique minimal (resp. maximal) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the non-negative integers such that
(1) $r(0) = 0$.
(2) $r(a) = r(b) + 1$ whenever $b < \cdot a$.
Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)},$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an *atom* in P, we mean an element in P covering 0. We say P is *atomic lattice* if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \forall a, b \in P.$$

For any two positive integers $n \geq m$, let

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{\prod_{i=n-m+1}^{n}(q^i - 1)}{\prod_{i=1}^{m}(q^i - 1)}.$$

For convenience, we assume that $\begin{bmatrix} n \\ i \end{bmatrix}_q = 0$ whenever $n < i$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$.

By Wan(2002b,Th. 1.7) we know that $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is the number of m-dimensionl subspaces in the n-dimensional row space over a finite field F_q.

Let V denote the n-dimensional row space over a finite field F_q. Denote by $GLn(F_q)$ the set of all the $n \times n$ nonsingular matrices over F_q. Then $GLn(F_q)$ forms a group under matrix multiplication, and acts on V as followings

$$V \times GLn(F_q) \longrightarrow V$$
$$((x_1, x_2, \cdots, x_n), T) \longmapsto (x_1, x_2, \cdots, x_n)T.$$

If U is an m-subspace of V with a basis $u_1, u_2, \cdots, u_m$, the $m \times n$ matrix

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

is said to be a matrix representation of U. We usually denote a matrix representation of the m-subspace U still by U. The above action induces an action on the set of all the subspaces. The above action is transitive on the set of all the subspaces with the same dimension by Wan(2002b,Th. 1.3).

Lemma 2.1. *Let V denote the n-dimensional row vector space over a finite field F_q, and fix an $(n-d)$-subspace W of V. Then the number of i-subspaces U of V satisfying $U + W = V$ is $\begin{bmatrix} n-d \\ i-d \end{bmatrix}_q q^{d(n-i)}$.*

Proof. By the transitivity of $GLn(F_q)$ on the set of all the subspaces with the same dimension, we may assume that W has the matrix representation of the form

$$W = (I^{(n-d)} \qquad 0^{(n-d,d)})$$

Let U has a matrix representation of the form

$$\begin{pmatrix} X^{(i-d,n-d)} & 0^{(i-d,d)} \\ Y^{(d,n-d)} & I^{(d)} \end{pmatrix}.$$

where X is an $(i-d) \times (n-d)$ matrix of rank $(i-d)$, Y is a $d \times (n-d)$ matrix. Then X is a $(i-d)$-subspace which contained in $I^{(n-d)}$. By Wan (2002b, Theorem 1.7), there are $\begin{bmatrix} n-d \\ i-d \end{bmatrix}_q$ choices for X. By the transitivity of $GL_n(F_q)$ we may take $X = (I^{(i-d)}\ 0^{(i-d,n-d)})$. Then U has the unique matrix representation of the form

$$\begin{pmatrix} I^{(i-d)} & 0^{(i-d,n-i)} & 0^{(i-d,d)} \\ 0 & Y_1 & 0^{(d)} \end{pmatrix}.$$

Hence the number of i-subspace U of V satisfying $U+W = V$ is $\begin{bmatrix} n-d \\ i-d \end{bmatrix}_q q^{d(n-i)}$.

Lemma 2.2. *Let V denote the n-dimensional row vector space over a finite field F_q, and fix an $(n-d)$-subspace W of V . For a given l_2-subspace U_2 of V satisfying $U_2 + W = V$, let $u(n,d;l_1,l_2)$ denote the number of l_1-subspaces U_1 of V satisfying $U_1 + W = V$ and $U_1 \subseteq U_2$. Then*

$$u(n,d;l_1,l_2) = \begin{bmatrix} l_2-d \\ l_1-d \end{bmatrix}_q q^{d(l_2-l_1)}.$$

Proof. Since the subgroup $GLn(F_q)_W$ of $GLn(F_q)$ fixing W acts transitively on the set$\{U|U+W = V, \dim U = l_2\}$, the number $u(n,d;l_1,l_2)$ depend only on l_1 and l_2. by Lemma 2.1 we obtain

$$u(n,d;l_1,l_2) = \begin{bmatrix} l_2-d \\ l_1-d \end{bmatrix}_q q^{d(l_2-l_1)}.$$

Hence the desired result follows. □

3. THE LATTICE $L_O(n,d)$

The lattice $L_O(n,d)$ has the unique minimal element $\{0\}$-subspace, and the unique maximal element V.

Theorem 3.1 Let $1 \le d \le n-1$. Then (1) $L_O(n,d)$ is an atomic lattice;
(2) *$L_O(n,d)$ is a geometric lattice if and only if $d = 1$ or $n-1$.*

Proof. (1) By the transitivity of $GL_n(F_q)$ we may take $W = (I^{(n-d)}0^{(n-d,d)})$.

For any $\{0\} \neq B \in L_O(n,d)$, $\dim B = i$, By the transitivity of $GLn(F_q)$ on the set of all the subspaces with the same dimension, we may assume that B has the matrix representation of the form

$$B = \begin{pmatrix} 0^{(d,i-d)} & 0^{(d,n-i)} & I^{(d)} \\ I^{(i-d)} & 0^{(i-d,n-i)} & 0^{(i-d,d)} \end{pmatrix}$$

Let A and $C_1, C_2, \dots, C_{i-d}$ be the subspaces of V with the following matrix representations

$$A = \begin{pmatrix} 0^{(d,i-d)} & 0^{(d,n-i)} & I^{(d)} \end{pmatrix}$$

$$C_k = \begin{pmatrix} L_k^{(d,i-d)} & 0^{(d,n-i)} & I^{(d)} \end{pmatrix}$$

where $1 \leq k \leq i-d$ and $L_k = (x_{st}^{(k)})_{d\times(i-d)}$ satisfying

$$x_{st}^{(k)} = \begin{cases} 1 & \text{if } s=1, t=k \\ 0 & \text{otherwise.} \end{cases}$$

Then A and $C_1, C_2, \dots, C_{i-d}$ are atoms of $L_O(n,d)$ satisfying $A \vee C_1 \vee C_2 \vee \cdots \vee C_d = B$. Therefore, $L_O(n,d)$ is an atomic lattice.

(2) For any $A \in L_O(n,d)$, define

$$r(A) = \begin{cases} 0 & \text{if } A = \{0\}, \\ \dim A - d + 1 & \text{otherwise.} \end{cases}$$

Then r is the rank function of $L_O(n,d)$.

If $d = 1$ or $n-1$, it is clear that $L_O(n,d)$ is a geometric lattice. Now suppose that $2 \leq d \leq n-2$, let

$$A = \begin{pmatrix} I^{(d,n-d)} & I^{(d)} \end{pmatrix}, \quad B = \begin{pmatrix} I^{(2)} & 0^{(2,n-d-2)} & I^{(2)} & 0 \\ 0 & 0 & 0 & I^{(d-2)} \end{pmatrix},$$

then $\dim(A \vee B) = d+2$ and $r(A \vee B) = 3$. It follows that

$$r(A \wedge B) + r(A \vee B) = r(A \vee B) = 3 > r(A) + r(B) = 2.$$

Hence $L_O(n,d)$ is not a geometric lattice whenever $2 \leq d \leq n-2$. □

Lemma 3.2. *The Möbius function of* $L_O(n,d)$ *is*

$$\mu(A,B) = \begin{cases} (-1)^{\dim B - \dim A} q^{\binom{\dim B - \dim A}{2}} & \text{if } \{0\} \neq A \leq B \text{ or } A = B = \{0\}, \\ \sum_{i=0}^{\dim B - d} (-1)^{i+1} q^{\binom{i}{2}} u(n,d;\dim B - i, \dim B) & \text{if } \{0\} = A \leq B \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Möbius function of $L_O(n,d)$ is

$$\mu(A,B)=\begin{cases}(-1)^{\dim B-\dim A}q^{\binom{\dim B-\dim A}{2}} & \text{if } \{0\}\neq A\leq B \text{ or } A=B=\{0\}\\ -\sum_{A<C\leq B}\mu(C,B) & \text{if } \{0\}=A\leq B,\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 2.2, we have

$$\begin{aligned}-\sum_{A<C\leq B}\mu(C,B) &= -\sum_{A<C\leq B}(-1)^{\dim B-\dim C}q^{\binom{\dim B-\dim A}{2}}\\ &= -\sum_{i=0}^{\dim B-d}(-1)^{i}q^{\binom{i}{2}}u(n,d;\dim B-i,\dim B)\\ &= \sum_{i=0}^{\dim B-d}(-1)^{i+1}q^{\binom{i}{2}}u(n,d;\dim B-i,\dim B).\end{aligned}$$

Hence the desired result follows. □

Theorem 3.3. *The character polynomial of $L_O(n,d)$ is*

$$\chi(L_O(n,d),x)=x^{n-d+1}+\sum_{j=d}^{n}\sum_{i=0}^{j-d}(-1)^{i+1}q^{\binom{i}{2}}u(n,d;j-i,j)\begin{bmatrix}n-d\\ i-d\end{bmatrix}_q q^{d(n-j)}x^{n-j}.$$

Proof.

$$\begin{aligned}\chi(L_O(n,d),x) &= \sum_{\{0\}\leq B\leq V}\mu(\{0\},B)x^{r(V)-r(B)}\\ &= \mu(\{0\},\{0\})x^{r(V)-r(\{0\})}+\sum_{\{0\}<B\leq V}\mu(\{0\},B)x^{r(V)-r(B)}\\ &= x^{n-d+1}+\sum_{\{0\}<B\leq V}\mu(\{0\},B)x^{n-\dim B}\\ &= x^{n-d+1}+\sum_{j=d}^{n}\sum_{i=0}^{j-d}(-1)^{i+1}q^{\binom{i}{2}}u(n,d;j-i,j)\begin{bmatrix}n-d\\ j-d\end{bmatrix}_q q^{d(n-j)}x^{n-j}.\end{aligned}$$

4. THE LATTICE $L_R(n,d)$

The lattice $L_R(n,d)$ has the unique minimal element V, and the unique maximal element $\{0\}$-subspace.

Theorem 4.1. *Let $1 \le d \le n-1$. Then (1) $L_R(n,d)$ is an atomic lattice;*

(2) $L_R(n,d)$ is a geometric lattice if and only if $n-d=1$.

Proof. (1) By the transitivity of $GL_n(F_q)$ we may take $W = (I^{(n-d)} 0^{(n-d,d)})$. For any $V \neq B \in L_R(n,d)$, dim$B = i$, By the transitivity of $GLn(F_q)$ on the set of all the subspaces with the same dimension, we may assume that B has the matrix representation of the form

$$B = \begin{pmatrix} 0^{(d,i-d)} & 0^{(d,n-i)} & I^{(d)} \\ I^{(i-d)} & 0^{(i-d,n-i)} & 0^{(i-d,d)} \end{pmatrix}$$

Let A and $C_1, C_2, \ldots, C_{n-1-i}$ be the subspaces of V with the following matrix representations

$$A = \begin{pmatrix} 0^{(d,i-d)} & 0^{(d,n-1-i)} & 0^{(d,1)} & I^{(d)} \\ I^{(i-d)} & 0^{(i-d,n-1-i)} & 0^{(i-d,1)} & 0^{(i-d,d)} \\ 0^{(n-1-i,i-d)} & I^{(n-1-i)} & 0^{(n-1-i,1)} & 0^{(n-1-i,d)} \end{pmatrix}$$

$$C_k = \begin{pmatrix} 0^{(d,i-d)} & 0^{(d,n-1-i)} & 0^{(d,1)} & I^{(d)} \\ I^{(i-d)} & 0^{(i-d,n-1-i)} & 0^{(i-d,1)} & 0^{(i-d,d)} \\ 0^{(n-1-i,i-d)} & L_k^{(n-1-i)} & M_k^{(n-1-i,1)} & 0^{(n-1-i,d)} \end{pmatrix}$$

where $1 \le k \le n-1-i$ and $L_k = (x_{st}^{(k)})_{(n-1-i)\times(n-1-i)}$ satisfying

$$x_{st}^{(k)} = \begin{cases} 1 & \text{if } s = t \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

$M_k = (y_{s1}^{(k)})_{(n-1-i)\times 1}$ satisfying

$$y_{s1}^{(k)} = \begin{cases} 1 & \text{if } s = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then A and $C_1, C_2, \ldots, C_{n-1-i}$ are atoms of $L_R(n,d)$ satisfying $A \vee C_1 \vee C_2 \vee \cdots \vee C_{n-1-i} = B$. Therefore, $L_R(n,d)$ is an atomic lattice.

(2)For any $A \in L_R(n,d)$, define

$$r'(A) = \begin{cases} n-d+1 & \text{if } A = \{0\}, \\ n - \text{dim}A & \text{otherwise.} \end{cases}$$

Then r' is the rank function of $L_R(n,d)$.

Clearly, $L_R(n,n-1)$ is a geometric lattice. Now suppose that $1 \le d \le n-2$. Let

$$A = \begin{pmatrix} I^{(n-d-1)} & 0^{(n-d-1,1)} & 0^{(n-d-1,d)} \\ 0^{(d,n-d-1)} & 0^{(d,1)} & I^{(d)} \end{pmatrix}$$

$$B = \begin{pmatrix} I^{(n-d-1)} & 0^{(n-d-1,1)} & 0^{(n-d-1,1)} & 0^{(n-d-1,d-1)} \\ 0^{(1,n-d-1)} & 1^{(1,1)} & 1^{(1,1)} & 0^{(1,d-1)} \\ 0^{(d-1,n-d-1)} & 0^{(d-1,1)} & 0^{(d-1,1)} & I^{(d-1)} \end{pmatrix}$$

then $A, B \in L_R(n,d)$,dim$A$ =dim$B = n-1$, $A \vee B = \{0\}, A \wedge B = V$. It follows that

$$r'(A \wedge B) + r'(A \vee B) = n - d + 1 \geq 3 > r'(A) + r'(B) = 2.$$

Hence $L_R(n,d)$ is not a geometric lattice whenever $1 \leq d \leq n-2$. □

Lemma 4.2. *The Möbius function of* $L_R(n,d)$ *is*

$$\mu(A,B) = \begin{cases} (-1)^{\dim A - \dim B} q^{\binom{\dim A - \dim B}{2}} & \text{if } A \leq B \neq \{0\} \text{ or } A = B = \{0\}, \\ \sum_{i=0}^{\dim A - d} (-1)^{i+1} q^{\binom{i}{2}} u(n,d;\dim A - i, \dim A) & \text{if } A \leq B = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Möbius function of $L_R(n,d)$ is

$$\mu(A,B) = \begin{cases} (-1)^{\dim A - \dim B} q^{\binom{\dim A - \dim B}{2}} & \text{if } A \leq B \neq \{0\} \text{ or } A = B = \{0\}, \\ -\sum_{A \leq C < B} \mu(A,C) & \text{if } A \leq B = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2, we have

$$\begin{aligned} -\sum_{A \leq C < B} \mu(A,C) &= -\sum_{A \leq C < B} (-1)^{\dim A - \dim C} q^{\binom{\dim A - \dim C}{2}} \\ &= -\sum_{i=0}^{\dim A - d} (-1)^i q^{\binom{i}{2}} u(n,d;\dim A - i, \dim A) \\ &= \sum_{i=0}^{\dim A - d} (-1)^{i+1} q^{\binom{i}{2}} u(n,d;\dim A - i, \dim A). \end{aligned}$$

Hence the desired result follows. □

Theorem 4.3. *The character polynomial of* $L_R(n,d)$ *is*

$$\chi(L_R(n,d),x)=\sum_{i=0}^{n-d}(-1)^{i+1}q^{\binom{i}{2}}\begin{bmatrix}n-d\\n-i-d\end{bmatrix}_q q^{di}x^{n-d+1}+$$
$$\sum_{j=d}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\begin{bmatrix}n-d\\j-d\end{bmatrix}_q q^{d(n-j)}x^{j-d+1}.$$

Proof.

$$\begin{aligned}\chi(L_R(n,d),x) &= \sum_{V\leq B\leq\{0\}}\mu(V,B)x^{r(\{0\})-r(B)}\\ &= \mu(V,\{0\})x^{r(\{0\})-r(V)}+\sum_{V\leq B<\{0\}}\mu(V,B)x^{\dim B-d+1}\\ &= \sum_{i=0}^{\dim V-d}(-1)^{i+1}q^{\binom{i}{2}}u(n,d;n-i,n)x^{n-d+1}+\\ &\sum_{j=d}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\begin{bmatrix}n-d\\j-d\end{bmatrix}_q q^{d(n-j)}x^{j-d+1}.\end{aligned}$$

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China under Grant No. 60776810 and the Natural Science Foundation of Tianjin City in China under Grant No. 08JCYBJC13900.

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