

Multidecomposition of the complete graph into graph pairs of order 4 with various leaves

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Abstract

A *graph-pair of order t* is two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H) , we say (G, H) *divides* some graph K if the edges of K can be partitioned into copies of G and H with at least one copy of G and at least one copy of H . We will refer to this partition as a (G, H) -*multidecomposition* of K .

In this paper, we consider the existence of multidecompositions of $K_n - F$ for the graph-pair of order 4 where F is a Hamiltonian cycle, a 1-factor or almost 1-factor.

1 Introduction

The authors in [?, ?] defined a *graph-pair of order t* as two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. For a given graph-pair (G, H) , we say (G, H) *divides* K_n if the edges of K_n can be partitioned into copies of G and H with at least one copy of G and at least one copy of H . We will refer to this partition as a (G, H) -*multidecomposition*. The authors in [?, ?, ?] settled the problem of multidecomposition of K_n and λK_n into graph-pairs of order 4 and 5. We will consider simple graphs of the form $K_n - F$, where $K_n - F$ denotes the graph K_n from which a Hamiltonian cycle, a 1-factor or almost 1-factor has been removed. In this paper we will consider multidecomposition into the graph-pair G and H whose union is the complete graph K_4 .

Throughout the paper, we let G be a 4-cycle; let H be a pair of K_2 's. A 4-cycle on vertices w, x, y , and z is denoted as $\{w, x, y, z\}$.

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Furthermore, if we denote the vertices by $0, 1, \dots, n - 1$, then the *difference* of the edge joining vertices u and v is defined to be $\min\{u - v, v - u\}$, reducing modulo n .

2 Hamilton Cycle Leave

Theorem 1 *Let F be a Hamilton cycle on K_n . $K_n - F$ has a (G, H) -multidecomposition if and only if $n \equiv 0$ or $3 \pmod{4}$ and $n \geq 7$.*

Proof: Let $E = |E(K_n - F)|$. Then $E = \frac{n(n-1)}{2} - n = \frac{n^2 - 3n}{2}$.

Since G and H both have an even number of edges, E must also be even. Thus $n^2 - 3n \equiv 0 \pmod{4}$, and so $n \equiv 0$ or $3 \pmod{4}$. Clearly there cannot be a (G, H) -multidecomposition of K_3 ; for K_5 , the removal of a Hamilton cycle leaves an odd number of edges, thus $n \geq 7$.

Case $n \equiv 0 \pmod{4}$:

Let $V(K_n) = \mathbb{Z}_n$. The edges of K_n have differences $\{1, 2, \dots, \frac{n}{2}\}$. Note that there are exactly n edges of each difference, except for the difference $\frac{n}{2}$ which has $\frac{n}{2}$ edges, and that $\frac{n}{2}$ is even because $n \equiv 0 \pmod{4}$. We will begin by constructing a 4-cycle decomposition of K_n with a 1-factor leave. We will do this by exploiting the fact that a 4-cycle can be constructed by using four edges whose differences sum to n . We break the 4-cycles into Type I and Type II.

Type I: For each $\alpha \in \{1, \dots, \frac{n}{4} - 1\}$ there is a unique $\beta \in \{\frac{n}{4} + 1, \dots, \frac{n}{2} - 1\}$ such that $\alpha + \beta = \frac{n}{2}$. The vertices $\{i, i + \alpha, i + \frac{n}{2}, i + \alpha + \frac{n}{2}\}$ form a 4-cycle that uses exactly two edges of difference α and two of difference β . This 4-cycle is a base block in the decomposition, and generates $\frac{n}{2}$ 4-cycles, one for each value of $i \in \{0, \dots, \frac{n}{2} - 1\}$. This uses all the edges of K_n except for those with difference $\frac{n}{2}$ or $\frac{n}{4}$.

Type II: The edges of difference $\frac{n}{4}$ can be partitioned into 4-cycles by using the base block $\{i, i + \frac{n}{4}, i + \frac{n}{2}, i + \frac{3n}{4}\}$. This base block generates a 4-cycle for each value of $i \in \{0, \dots, \frac{n}{4} - 1\}$, using all n edges of difference $\frac{n}{4}$.

The only edges that remain are the $\frac{n}{2}$ edges of difference $\frac{n}{2}$. These form a 1-factor leave. This decomposition can easily be modified to give a (G, H) -decomposition of K_n with a Hamilton cycle leave. This is accomplished by breaking up the Type I 4-cycles generated by the base block $\{i, i + 1, i + \frac{n}{2}, i + \frac{n}{2} + 1\}$. These 4-cycles consist of edges of difference 1 and $\frac{n}{2} - 1$. The edges of difference 1 form a Hamilton cycle, and the edges of differences $\frac{n}{2} - 1$ can be trivially partitioned into copies of H . Similarly, the $\frac{n}{2}$ edges of the 1-factor leave can be partitioned into $\frac{n}{4}$ copies of H .

Note that in this case the construction gives $\frac{n^2 - 6n}{8}$ copies of G . Since each copy of G can be decomposed into two copies of H , one can decompose all but one copy of G into copies of H .

Case $n \equiv 3 \pmod{4}$:

If $n \equiv 3 \pmod{8}$, then the number of differences is $\frac{n-1}{2} \equiv 1 \pmod{4}$. The difference 1 edges form the Hamilton cycle and we group the remaining differences into sets of four, each of which will generate copies of G . The groups of differences are defined by $\{4i+2, 4i+3, 4i+4, 4i+5\}$ for $0 \leq i \leq \frac{n-11}{8}$. Each group is used to build the base block $\{j, j+(4i+2), j+(8i+7), j+(4i+4)\}$. For $0 \leq j \leq n-1$, each base block will give n different copies of G .

The construction gives $\frac{n(n-3)}{8}$ copies of G . Since any copy of G can be decomposed into two copies of H , one can decompose any number (between 1 and $\frac{n^2-3n-8}{8}$) copies of G into copies of H .

If $n \equiv 7 \pmod{8}$, then the number of differences is $\frac{n-1}{2} \equiv 3 \pmod{4}$. The difference 1 edges form the Hamilton cycle leave. The differences 2 and $\frac{n-1}{2}$ edges will each form a Hamilton cycle. Each of these Hamilton cycles can be decomposed into $\frac{n-1}{2}$ copies of H with one remaining edge. Without loss of generality we can assume that the two remaining edges have distinct vertices. Hence they can be paired to form another copy of H . We group the remaining differences into sets of four, each of which will generate copies of G . The groups of differences are defined by $\{4i+3, 4i+4, 4i+5, 4i+6\}$ for $0 \leq i \leq \frac{n-15}{8}$. Each group is used to build the base block $\{j, j+(4i+3), j+(8i+9), j+(4i+5)\}$. For $0 \leq j \leq n-1$, each base block will give n different copies of G .

The construction gives $\frac{n(n-7)}{8}$ copies of G . Since any copy of G can be decomposed into two copies of H , one can decompose any number (between 0 and $\frac{n^2-7n-8}{8}$) copies of G into copies of H . ■

3 1-Factor Leave

Theorem 2 *Let F be a 1-factor subgraph of K_n . $K_n - F$ has a (G, H) -multidecomposition if and only if $n \equiv 0$ or $2 \pmod{4}$ and $n \geq 6$. Moreover, if x is an integer between 1 and $\frac{1}{8}(n^2 - 2n) - 1$, then there exists a (G, H) -multidecomposition of $K_n - F$ having exactly x copies of G .*

Proof: (Necessity) In order for K_n to have a 1-factor, n must be even. $K_2 - F$ has no edges, and $K_4 - F$ consists of only a single copy of G .

(Sufficiency) In [?], Alspach and Gavlas show that $K_n - F$ has a G -decomposition if and only if $|E(K_n - F)|$ is divisible by 4.

The number of edges in $K_n - F$ is

$$\frac{n(n-1)}{2} - \frac{n}{2} = \frac{1}{2}(n^2 - 2n)$$

If $n = 2j$, for some $j \geq 3$, then

$$|E(K_n - F)| = \frac{1}{2}((2j)^2 - 2(2j)) = 2(j^2 - j).$$

Now $j^2 - j$ must be even, so $|E(K_n - F)| \equiv 0 \pmod{4}$ and hence there is a G -decomposition of $K_n - F$ consisting of exactly $\frac{n^2-2n}{8}$ copies of G . Since any copy of G can be decomposed into two copies of H , one can decompose any number (between 1 and $\frac{n^2-2n}{8} - 1$) of copies of G into two copies of H . ■

4 Almost 1-Factor Leave

For odd $n \geq 3$, we define an *almost 1-factor* of K_n to be a 1-factor of K_{n-1} .

Theorem 3 *Let F be an almost 1-factor of K_n . Then $K_n - F$ has a (G, H) -multidecomposition if and only if n is odd and $n \geq 7$. Moreover, if x is an integer between 1 and $\frac{n^2-4n+3}{8}$ then there exists a (G, H) -multidecomposition of $K_n - F$ having exactly x copies of G .*

Proof:

(Necessity) $K_3 - F$ is a path on three vertices, which contains no copy of G or H as a subgraph. It can be shown that $K_5 - F$ has no (G, H) -multidecomposition.

(Sufficiency) Let $V(K_n - F) = \{0, 1, \dots, n-2\} \cup \{\infty\}$. The edges of $K_n \setminus \{\infty\}$ have differences $\{1, \dots, \frac{n-1}{2}\}$. In both cases the half difference $h = \frac{n-1}{2}$ makes up the almost 1-factor.

Case $n \equiv 3 \pmod{4}$: In this case there is no quarter difference, so h is odd and the remaining $h-1$ differences can be paired up as $(i, h-i)$ for $2 \leq i \leq \frac{h-1}{2}$. Each such pair gives a base block $(0, i, h, i+h)$ that can be rotated to generate $\frac{n-1}{2}$ copies of G .

We now partition the edges of differences 1 and $h-1$, and the edges incident with ∞ into copies of G and H .

The base block $(\infty, 0, 1, h)$ is rotated halfway around to generate $\frac{n-1}{2}$ copies of G .

The remaining edges form a cycle of length h , with pendant edge at each vertex on the cycle. Those edges can be easily decomposed into copies of H .

Case $n \equiv 1 \pmod{4}$: In this case h is even, so there is a quarter difference. Take the quarter difference $\frac{h}{2}$ to create the base block $(0, \frac{h}{2}, h, \frac{3h}{2})$, which is rotated one quarter of the way around to generate $\frac{n-1}{4}$ copies of G . The remaining edges are partitioned the same way as in case $n \equiv 3 \pmod{4}$.

Note that the construction gives $\frac{n^2-4n+3}{8}$ copies of G . Since each copy of G can be decomposed into two copies of H , one can decompose all but one of the copies of G into copies of H . ■

In this paper we settled the existence of the multidecomposition of the complete graph into graph pairs of order 4 with a leave consisting of a Hamiltonian cycle, a 1-factor or almost 1-factor. A nice consequent result will be to show that there exists a multidecomposition of the complete graph into graph pairs of order

4 with a general 2-factor leave.

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