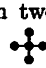
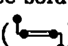
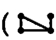
Resolvable decompositions of λK_n into the union of two 2-paths

Peter Danziger *
Department of Mathematics
Ryerson University
Toronto, ON M5B 2K3
Canada
danziger@ryerson.ca

Eric Mendelsohn †
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
Canada
mendelso@math.utoronto.ca

Gaetano Quattrocchi ‡
Dipartimento di Matematica
Università di Catania
Catania, Italia
quattrocchi@dmi.unict.it

Abstract

We give necessary and sufficient conditions for a resolvable $\mathcal{H}$-decomposition of λK_n in the case where $\mathcal{H}$ is one of the 10 graphs obtained by the union of two paths of length 2, with two possible exceptions. In particular, we complete the 4-star () and T () for higher lambda and give complete solutions for resovable decompositions into Fish (), Mulinetto () and Kites (). In the cases of the Fish and Mulinetto the solution is obtained 1-rotationally.

1 Introduction

Given two graphs $\mathcal{G}$ and $\mathcal{H}$, an *$\mathcal{H}$-decomposition of $\mathcal{G}$* is a decomposition of the edges of $\mathcal{G}$ into isomorphic copies of $\mathcal{H}$, the copies of $\mathcal{H}$ are called *blocks*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ (called *parallel or resolution classes*) such that every point of $\mathcal{G}$ appears exactly once in each $\mathcal{P}_i$. A resolvable $\mathcal{H}$-decomposition

*supported by NSERC research grant #OGP0170220 and MURST, Italy
†supported by NSERC research grant #OGP007621 and Dipartimento di Matematica, Università di Catania
‡supported by MURST, Italy

of $\mathcal{G}$ is sometimes also referred to as an $\mathcal{H}-$ factorization of $\mathcal{G}$, a class can be called an $\mathcal{H}$ factor of $\mathcal{G}$.

A (resolvable) $\mathcal{H}-$decomposition of λK_n, the lambda-fold complete graph on n vertices, is also known as a (resolvable) design. A (resolvable) $\mathcal{H}-$decomposition of the complete multipartite graph with u parts each of size g (λK_{g^u}) is known as a (resolvable) group divisible design (R)GDD$_\lambda$, the parts of size g are called the groups of the design. We also allow for groups of differing sizes and use an exponential notation ($g_1^{u_1} \cdots g_i^{u_i} \cdots g_n^{u_n}$) to specify that there are u_i groups of size g_i, this is the (R)GDD's *type*. When $\mathcal{H} = K_n$ we will call it an $n-$(R)GDD$_\lambda$. If the blocks of an $\mathcal{H}-$GDD$_\lambda$ of type g^u can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a $\mathcal{H}-$*frame*$_\lambda$. In the case where the parameter $\lambda = 1$ it is often omitted.

We denote a path of length two on three points by P_3. In this paper we consider resolvable $\mathcal{H}-$decompositions of λK_n, where $\mathcal{H}$ is the union of two paths of length two. There are, up to isomorphism, 10 different ways to form the union of two P_3's (see Figure 1). They are given by the following table of second paths where the first path is *ab bc*:

A1	*de ef*						
B1	*ad de*	B2	*db be*	B3	*bd de*		
C1	*ab ad*	C2	*bc cd*	C3	*ac ad*	C4	*cd da*
D1	*bc ca*	D2	*ab bc*				

(see Figure 1).

We now consider the necessary conditions for a resolvable $\mathcal{G}-$decomposition of λK_v, where $\mathcal{G} = (V, E)$ and $|V| = n$, $|E| = e$. Clearly we must have that $n \mid v$ for resolvability. The number of blocks is $b = \frac{\lambda v(v-1)}{2e}$ and the number of resolution classes

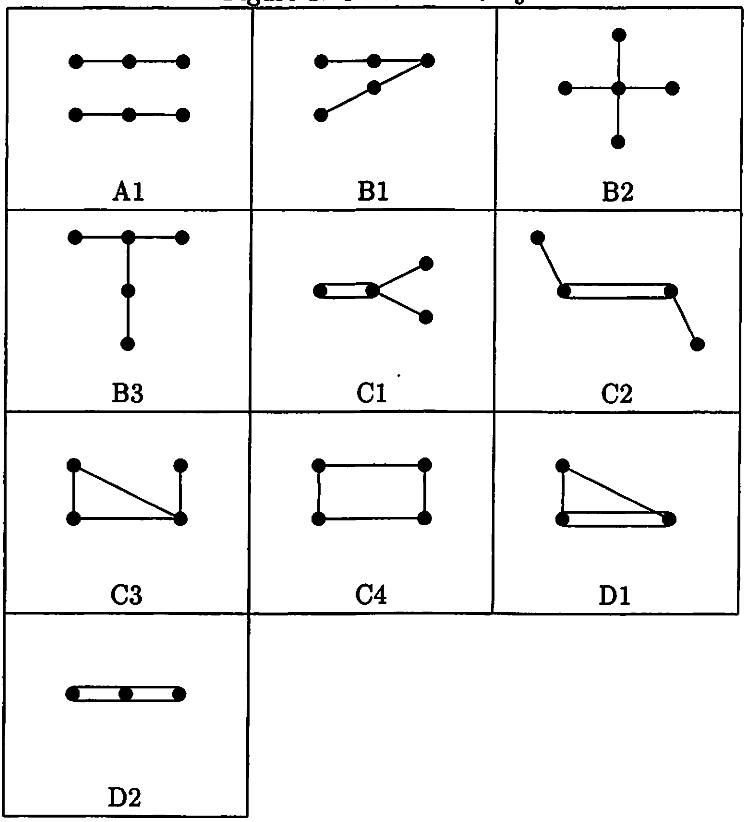
$$r = \frac{\lambda n(v-1)}{2e}.$$

Thus we require $2e \mid \lambda n(v-1)$. These conditions depend only on n and e, in our case $e = 4$, and each of the cases in Figure 1 may be enumerated by the number of points. This is summarized in Figure 2.

In certain cases it is possible to get more stringent conditions by considering the degrees of points. Let d_i, $i = 1, \ldots, s$, be the degrees of $\mathcal{G}$ in some order and let $r_i(x)$, $i = 1, \ldots, s$, be the number of times a point x appears with degree d_i. Then we have that

$$\sum_{i=1}^{s} r_i(x) = r = \frac{\lambda n(v-1)}{2e}, \tag{1}$$

Figure 1: Union of two P_3



$$\sum_{i=1}^{s} d_i r_i(x) = \lambda(v-1). \tag{2}$$

In [1, 7] it is shown that a resolvable P_k−decomposition of λK_v exists if and only if $v \equiv 0 \bmod k$ and $\lambda k(v-1) \equiv 0 \bmod 2(k-1)$. This deals with B1 above as this is a resolvable P_5−decomposition. The case A1 is obtained by a resolvable P_3−decomposition where the number of blocks per class is even. For the case D2 we note that we must clearly have $\lambda \equiv 0 \bmod 2$; we obtain the design by placing 2 identical resolvable P_3−decompositions on the same point set.

The case C4 is solved in [6] (see also Theorem IV.29.8 in [2]), a cyclic solution is given in [4]. The case D1 is completely solved in [3] with two possible exceptions in v. Existence for B2 and B3 with $\lambda = 1$ was settled in [8, 9]. For higher λ the case B2 with $v \equiv 5 \bmod 10$ can be found in [8].

Figure 2: Necessary Conditions

Case	v	λ
$n = 6$ (A)	$v \equiv 0 \bmod 6$	$\lambda \equiv 0 \bmod 4$
$n = 5$ (B)	$v \equiv 0, 10, 20, 30 \bmod 40$	$\lambda \equiv 0 \bmod 8$
	$v \equiv 15, 35 \bmod 40$	$\lambda \equiv 0 \bmod 4$
	$v \equiv 5 \bmod 40$	$\lambda \equiv 0 \bmod 2$
	$v \equiv 25 \bmod 40$	any λ
$n = 4$ (C)	$v \equiv 0 \bmod 4$	$\lambda \equiv 0 \bmod 2$
$n = 3$ (D)	$v \equiv 0, 6, 12, 18 \bmod 24$	$\lambda \equiv 0 \bmod 8$
	$v \equiv 3, 15 \bmod 24$	$\lambda \equiv 0 \bmod 4$
	$v \equiv 21 \bmod 24$	$\lambda \equiv 0 \bmod 2$
	$v \equiv 9 \bmod 24$	any $\lambda > 1$

In this paper we provide the remaining higher λ cases for B2 and B3 and find complete solutions for the remaining cases: C1, C2, and C3, which we call Fish, Mulinetto and Kites respectively.

We will make use of the following well known constructions, we refer the reader to [2] for further results, references and explanations of designs and these constructions.

Given an $\mathcal{H}-\text{RGDD}_\lambda$ of type $\prod g_i^{u_i}$, if there is a resolvable $\mathcal{H}-$decomposition of λK_{g_i} for each i then we may obtain a resolvable $\mathcal{H}-$decomposition of λK_v, where $v = \sum g_i u_i$, by placing copies of the requisite decomposition on each of the groups. This well known construction is called *filling in the groups*.

Given an $n-\text{RGDD}_{\lambda_1}$ of type $\prod g_i^{u_i}$, and an $\mathcal{H}-\text{RGDD}_{\lambda_2}$ of type w^n, we may apply Wilson's construction, giving each point a weight w and placing a copy of the $\mathcal{H}-\text{RGDD}_{\lambda_2}$ of type w^n on each expanded block to obtain a $\mathcal{H}-\text{RGDD}_{\lambda_1\lambda_2}$ of type $(wg_i)^{u_i}$. This is known as *expanding by w*.

We note that expanding a frame using an RGDD results in a frame, but an even more powerful construction is Wilson's fundamental frame construction. In this case given an $n-$GDD of type $\prod g_i^{u_i}$, the result of expanding by a $\mathcal{H}-$frame of type w^n is again a frame, which can be seen by considering the blocks through a point in the master $n-$GDD.

Given a graph $\mathcal{G}$, a (resolvable) $\mathcal{H}-$decomposition of $\mathcal{G} - K_h$ is said to have a *hole* of size h. If in addition we have a (resolvable) $\mathcal{H}-$decomposition of K_h we may obtain a (resolvable) decomposition with *a subsystem* of order h, given such a system the existence of the holey system is assumed. In the resolvable case the holey system will contain partial classes which miss the hole. Given an $\mathcal{H}-\text{frame}_\lambda$ of type $\prod g_i^{u_i}$, and for each g_i there is a resolvable $\mathcal{H}$−decomposition of λK_{g_i+h} with a subsystem of size h then there is a resolvable $\mathcal{H}$−decomposition of λK_{v+h}, where $v = \sum g_i u_i$, the number of points in the frame.

We will find the following well known designs useful.

Lemma 1.1 ([2]). *There exists a*

- *2-frame of type* 1^{2t+1} *for all* $t \geq 1$ *(a near 1-factorization);*
- *2-frame of type* 2^t *for all* $t \geq 3$*;*
- 2−*RGDD of type* 1^{2t} *for all* t *(a 1-factorization)*

2 4-star (B2)

Blocks of 4−star () designs are written $(a; b, c, d, e)$, where a has degree 4. We write $X(v, \lambda)$ for a resolvable 4−star design on v points.

Theorem 2.1. *The necessary conditions for an* $X(v, \lambda)$ *are sufficient.*

Proof. All cases are covered by results in [8, 9] (See also [2] p. 362), except when $\lambda \equiv 0 \bmod 8$ and $v \equiv 0 \bmod 10$. We give an $X(v, 8)$ to solve these remaining cases.

Expand a 2−RGDD of type 1^{2t} by 5, using the X−GDD of type 5^2 given below. Fill in the groups with a $X(5, 8)$ [8] to get an $X(10t, 8)$.

To get an X−GDD_8 of type 5^2 on point set $\{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\} \cup \mathbb{Z}_5$ with groups $\{\infty_i \mid i = 0, \ldots, 4\}$ and $\mathbb{Z}_5$, develop the following class mod5:

$$(\infty_0; 0, 1, 2, 3),\ (4; \infty_1, \infty_2, \infty_3, \infty_4)$$

with the convention that $\infty_i + x = \infty_i$. □

3 T (B3)

Blocks of T () designs are written $(a; b, c, d; e)$, where a has degree 3, joined to b, c and d, and de is an edge. We write $T(v, \lambda)$ for a resolvable B3 design on v points. The case $\lambda = 1$ is done in [9], in this section we complete the other cases in λ.

Lemma 3.1. *A* $T(5, 2)$ *and a* $T(15, 4)$ *with a hole of size* 5 *both exist.*

Proof. • A $T(5, 2)$. Point set $\mathbb{Z}_5$, develop the following mod 5:

$$(0; 1, 2, 3; 4)$$

- A $T(15,4)$ with a hole of size 5.
  Point set is $(\mathbb{Z}_5 \times \{0,1\}) \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. Develop the following two short classes mod5:

$$(0_0; 1_0, 2_0, 3_0; 4_0) \quad (0_1; 1_1, 2_1, 3_1; 4_1)$$
$$(0_0; 1_1, 2_0, 3_0; 4_0) \quad (0_1; 1_0, 2_1, 3_1; 4_1)$$

  Develop the following five long classes mod5:

$$(4_0; 0_1, \infty_3, \infty_0; 2_0) \quad (1_1; 1_0, \infty_4, \infty_1; 3_0) \quad (0_0; 3_1, 4_1, \infty_2; 2_1)$$
$$(4_0; 0_1, \infty_4, \infty_1; 2_0) \quad (1_1; 1_0, \infty_3, \infty_0; 3_0) \quad (0_0; 3_1, 4_1, \infty_2; 2_1)$$
$$(3_0; 0_1, \infty_1, \infty_4; 2_1) \quad (1_1; 1_0, \infty_2, \infty_0; 3_1) \quad (4_1; 2_0, 0_0, \infty_3; 4_0)$$
$$(0_0; 2_1, \infty_2, \infty_3; 0_1) \quad (1_1; 1_0, \infty_0, \infty_1; 3_1) \quad (4_1; 2_0, 3_0, \infty_4; 4_0)$$
$$(2_0; 0_1, \infty_0, \infty_3; 3_0) \quad (1_1; 2_1, \infty_1, \infty_4; 4_0) \quad (0_0; 1_0, 3_1, \infty_2; 4_1)$$

□

Lemma 3.2. *A $T(45,2)$ with a hole of size 5 exists.*

Proof. Point set is $\mathbb{Z}_5 \times \mathbb{Z}_9$. The 5 classes below give a T−RGDD of type 5^9 when developed mod$(5,-)$, with groups $\mathbb{Z}_5 \times \{i\}$, $0 \leq i \leq 8$. Taking each block twice and filling in all but one of the groups with a $T(5,2)$ gives the required structure.

$$(0_0; 2_6, 4_8, 3_1; 4_0) \quad (0_1; 4_2, 4_6, 3_2; 4_4) \quad (0_2; 1_3, 4_1, 3_3; 1_2)$$
$$(0_3; 1_0, 4_5, 3_4; 1_1) \quad (0_4; 0_6, 2_5, 3_5; 4_7) \quad (0_5; 2_2, 2_3, 3_6; 1_7)$$
$$(1_4; 2_1, 2_7, 3_7; 4_3) \quad (0_7; 1_8, 2_8, 3_8; 2_4) \quad (0_8; 1_6, 2_0, 3_0; 1_5)$$

$$(0_0; 1_2, 2_2, 3_2; 2_3) \quad (0_1; 4_3, 4_0, 3_3; 2_1) \quad (0_2; 1_0, 1_6, 3_4; 4_2)$$
$$(1_1; 1_7, 2_4, 3_5; 4_4) \quad (0_4; 2_6, 1_3, 3_6; 1_5) \quad (0_5; 2_7, 4_6, 3_7; 4_5)$$
$$(0_6; 1_8, 2_8, 3_8; 1_4) \quad (0_7; 4_1, 2_0, 3_0; 4_7) \quad (0_8; 0_3, 2_5, 3_1; 4_8)$$

$$(2_7; 1_8, 4_4, 3_4; 2_8) \quad (0_1; 4_5, 1_6, 3_5; 4_6) \quad (0_2; 1_5, 2_6, 3_6; 4_2)$$
$$(0_3; 1_7, 2_4, 3_7; 1_1) \quad (0_4; 2_1, 4_0, 3_8; 2_5) \quad (0_5; 0_0, 4_1, 3_0; 1_4)$$
$$(0_6; 4_3, 2_0, 3_1; 1_0) \quad (0_7; 1_2, 2_2, 3_2; 4_7) \quad (0_8; 1_3, 2_3, 3_3; 4_8)$$

$$(0_0; 1_8, 3_3, 0_8; 2_1) \quad (0_1; 4_8, 0_3, 0_6; 4_4) \quad (0_2; 3_1, 4_5, 0_7; 0_5)$$
$$(1_3; 3_7, 3_5, 1_2; 1_6) \quad (1_4; 2_0, 4_0, 1_0; 1_7) \quad (1_5; 3_8, 0_4, 1_1; 4_6)$$
$$(2_6; 4_7, 4_3, 2_5; 3_0) \quad (2_7; 3_6, 4_1, 2_3; 3_4) \quad (2_8; 3_2, 4_2, 2_4; 2_2)$$

$$(0_3; 3_1, 1_5, 0_0; 0_2) \quad (0_6; 1_4, 1_7, 0_7; 1_1) \quad (0_8; 0_1, 3_2, 0_5; 0_4)$$
$$(2_6; 1_0, 3_0, 2_3; 2_5) \quad (2_0; 3_5, 3_3, 2_1; 2_4) \quad (1_2; 3_4, 1_8, 2_8; 2_7)$$
$$(4_0; 3_7, 1_3, 4_6; 4_8) \quad (4_3; 1_6, 3_6, 4_4; 4_7) \quad (4_5; 2_2, 3_8, 4_2; 4_1)$$

□

Lemma 3.3. *A $T-RGDD_\lambda$ of type g^2 exists for*

$$(g,\lambda) \in \{(5,8),(10,4),(20,2)\}.$$

Proof. • A $T-$RGDD$_8$ of type 5^2 ($\lambda = 8$). Point set $\mathbb{Z}_5 \times \mathbb{Z}_2$, groups $\mathbb{Z}_5 \times \{i\}$, $i \in \mathbb{Z}_2$. Develop the following classes mod$(5,-)$.

$$\begin{array}{ll}
(0_1;0_0,1_0,2_0;3_1), & (3_0;1_1,2_1,4_1;4_0)\\
(0_0;0_1,1_1,2_1;3_0), & (3_1;1_0,2_0,4_0;4_1)\\
(0_0;1_1,3_1,2_1;4_0), & (4_1;1_0,3_0,2_0;0_1)\\
(0_1;1_0,3_0,2_0;4_1), & (4_0;1_1,3_1,2_1;0_0)\\
(0_0;0_1,1_1,4_1;4_0), & (2_1;1_0,2_0,3_0;3_1)
\end{array}$$

- A $T-$RGDD$_4$ of type 10^2 ($\lambda = 4$). Point set $\mathbb{Z}_5 \times \mathbb{Z}_4$, groups $\mathbb{Z}_5 \times \{i, i+1\}$, $i = 0,2$. Develop the following five classes mod$(5,-)$.

$$\begin{array}{llll}
(0_2;0_0,1_0,2_0;3_2), & (3_0;1_2,2_2,4_2;4_0), & (0_3;0_1,1_1,2_1;3_3), & (3_1;1_3,2_3,4_3;4_1)\\
(0_0;0_3,1_3,2_3;3_0), & (3_3;1_0,2_0,4_0;4_3), & (0_1;0_2,1_2,2_2;3_2), & (3_2;1_1,2_1,4_1;4_2)\\
(0_0;1_3,3_3,2_2;4_0), & (4_2;1_1,3_1,2_0;0_2), & (0_1;1_2,3_2,2_3;4_1), & (4_3;1_0,3_1,2_1;0_3)\\
(0_2;1_1,3_0,2_1;4_2), & (4_1;1_2,3_2,2_2;0_0), & (0_3;1_0,3_1,2_0;4_3), & (4_0;1_3,3_3,2_3;0_1)\\
(0_0;0_3,1_2,4_2;4_0), & (2_2;1_0,2_1,3_0;3_2), & (0_1;0_2,1_3,4_3;4_1), & (2_3;1_1,2_0,3_1;3_3)
\end{array}$$

- A $T-$RGDD$_2$ of type 20^2 ($\lambda = 2$). Point set $\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, groups $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \{i\}$, $i \in \mathbb{Z}_2$. Develop each of the following pairs of blocks mod$(-,4,-)$ to obtain five classes, develop each of these classes mod$(5,-,-)$ to get the rest.

$$\begin{array}{ll}
(001;000,120,200;301), & (300;101,221,401;400)\\
(000;011,111,211;300), & (311;100,200,400;411)\\
(000;121,311,201;410), & (401;130,320,200;0212)\\
(001;110,320,210;401), & (410;101,301,201;020)\\
(000;021,131,401;410), & (201;110,220,300;331)
\end{array}$$

□

Theorem 3.4. *There exists a $T(v,\lambda)$ if and only if the conditions of Table 2 are met.*

As noted above the case $v \equiv 25 \bmod 40$ is dealt with in [9], we consider the other cases below.

- $v \equiv 0 \bmod 10$, $\lambda \equiv 0 \bmod 8$.

  Use the $T-$RGDD$_8$ of type 5^2 above to expand a $2-$RGDD of type 1^{2t} to obtain a $T-$RGDD$_8$ of type 5^{2t}. Fill in the groups of this factorization with a $T(5,8)$ from Lemma 3.1.

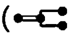
- $v \equiv 15 \bmod 20$, $\lambda \equiv 0 \bmod 4$.

  Expand a 2−frame of type 1^{2t+1} (a near 1-factorization) by 10, using the T−RGDD$_4$ of type 10^2 above to obtain a T−frame$_4$ of type 10^{2t+1}. Add 5 ideal points and fill each group along with the 5 new points with a $T(15,4)$ with a hole of size 5 from Lemma 3.1.

- $v \equiv 5 \bmod 40$, $\lambda \equiv 0 \bmod 2$.

  Expand a 2−frame of type 2^t $(t \geq 3)$ by 20, using the T−RGDD$_2$ of type 20^2 above to obtain a T−frame$_2$ of type 40^t. Add 5 ideal points and fill each group along with the 5 new points with a $T(45,2)$ with a hole of size 5 from Lemma 3.2. This leaves the case $v = 85$ which is given in the appendix.

4 Fish (C1)

Blocks of fish () are written $(a:b;c,d)$, where a has degree 4, b has degree 2 and c and d each have degree one. We write $F(v,\lambda)$ for a resolvable fish design on v points.

Theorem 4.1. *There exists a* $F(v,\lambda)$ *if and only if* $v \equiv 0 \bmod 4$ *and* $\lambda \equiv 0 \bmod 2$.

Proof. As noted above these conditions are necessary. We give a construction for $F(v,2)$. We consider three modular classes, in each case the point set is $\mathbb{Z}_{v-1} \cup \{\infty\}$. Develop the given class mod $v-1$.

- $v \equiv 4 \bmod 8$

  $F(4)$: $(0:\infty;1,2)$.

  For $v > 4$:

$$\begin{array}{l}(0:\infty;1,v-2),\\ \left.\begin{array}{l}(2+2i:v-3-6i;v-5-6i,v-8-6i)\\ (3+2i:v-6-6i;v-4-6i,v-7-6i)\end{array}\right\} 0 \leq i < \frac{v-4}{8}\end{array}$$

- $v \equiv 8 \bmod 16$

  $F(8)$: $(0:\infty;2,6)$, $(1:5;3,4)$.

  For $v > 8$:

$$\begin{array}{l}(0:\infty;1,v-2),\\ (\frac{v}{8}:\frac{5v}{8}+3;\frac{5v}{8}+5,\frac{5v}{8}-1),\quad (\frac{v}{8}+1:\frac{5v}{8}+2;\frac{5v}{8}-2,\frac{5v}{8}),\\ (3+2i:v-6-6i;v-4-6i,v-7-6i)\quad 0 \leq i < \frac{v-24}{16}\\ \left.\begin{array}{l}(2+2i:v-3-6i;v-5-6i,v-8-6i)\\ (\frac{v}{8}+2+2i:\frac{5v}{8}-3-6i;\frac{5v}{8}-5-6i,\frac{5v}{8}-8-6i)\\ (\frac{v}{8}+3+2i:\frac{5v}{8}-6-6i;\frac{5v}{8}-4-6i,\frac{5v}{8}-7-6i)\end{array}\right\} 0 \leq i < \frac{v-8}{16}\end{array}$$

- $v \equiv 0 \bmod 16$

  $F(16)$: $(0:\infty;1,14)$, $(2:13;11,8)$, $(4:7;6,9)$, $(10:3;5,12)$.

  For $v > 16$:

$$
\begin{array}{ll}
(0:\infty;1,v-2), & (\frac{v}{8}:\frac{5v}{8}+3;\frac{5v}{8}+1,\frac{5v}{8}), \\
(\frac{v}{8}+1:\frac{5v}{8}-2;\frac{5v}{8}-1,\frac{5v}{8}-5), & (\frac{v}{8}+2:\frac{5v}{8}-3;\frac{5v}{8}-4,\frac{5v}{8}+2),
\end{array}
$$

$$
\left.
\begin{array}{l}
(2+2i:v-3-6i;v-5-6i,v-8-6i) \\
(3+2i:v-6-6i;v-4-6i,v-7-6i) \\
(\frac{v}{8}+4+2i:\frac{5v}{8}-9-6i;\frac{5v}{8}-7-6i,\frac{5v}{8}-10-6i) \\
(\frac{v}{8}+3+2i:\frac{5v}{8}-6-6i;\frac{5v}{8}-8-6i,\frac{5v}{8}-11-6i)
\end{array}
\right\} 0 \le i < \frac{v}{16}-1
$$

□

5 Mulinetto (C2)

Blocks of mulinetto() are written $(a,b;c,d)$, where a and c have degree 3, b and d have degree 1 and ab, cd are edges. We write $M(v,\lambda)$ for a resolvable mulinetto design on v points.

In this case the equations 1 and 2 give further conditions. For some particular point x, let $r_1(x)$ be the number of times x appears as a point of degree 1, and $r_2(x)$ be the number of times x appears as a point of degree 3. Then

$$r_1(x)+r_2(x)=\frac{\lambda(v-1)}{2},$$
$$r_1(x)+3r_2(x)=\lambda(v-1).$$

Thus

$$r_1(x)=r_2(x)=\frac{\lambda(v-1)}{4}.$$

So $\lambda \equiv 0 \bmod 4$.

Theorem 5.1. *There exists a $M(v,\lambda)$ if and only if $v \equiv 0 \bmod 4$ and $\lambda \equiv 0 \bmod 4$.*

Proof. As noted above $v,\lambda \equiv 0 \bmod 4$ is necessary. We give a construction for $M(v,4)$. The point set is $\mathbb{Z}_{v-1} \cup \{\infty\}$. We give two classes, each of which is developed $\bmod (v-1)$.

Class I:

$$
\begin{array}{ll}
(\infty,1;0,v-1) & \\
(2+2i,3+2i;v-3-2i,v-4-2i) & 0 \le i < \frac{v}{4}-1
\end{array}
$$

Class II:

$$
\begin{array}{ll}
(0,1;v-1,\infty) & \\
(2+2i,3+2i;v-3-2i,v-4-2i) & 0 \le i < \frac{v}{4}-1
\end{array}
$$

□

6 Kite (C3)

Blocks of Kite () designs are written $(a, b, c; d)$, where abc is a triangle and $a\ d$ is an edge. We write $\mathfrak{K}(v, \lambda)$ for a resolvable kite design on v points. We will find the following designs useful.

Lemma 6.1 ([2]). *There exists a*

- *3-RGDD of type* 3^{2t+1} *for all* $t \geq 1$*;*
- *3-RGDD of type* 6^t *for all* $t \geq 4$*;*
- *resolvable* K_4*−decomposition of* K_v *(an RBIBD*$(v, 4, 1)$*) for all* $v \equiv 4 \bmod 12$*;*
- *4-GDD of type* 6^t *for all* $t \geq 5$*.*

Lemma 6.2. *The following all exist:*

1. *a* $\mathfrak{K}(v, 2)$ *for* $v \in \{4, 8, 12, 24, 32, 44\}$*;*
2. *a* $\mathfrak{K}$*−frame of type* 4^4*;*
3. $\mathfrak{K}$*−RGDD of type* 4^3 *and a* $\mathfrak{K}$*−RGDD of type* 2^4*.*

Further the $\mathfrak{K}(32, 2)$ *can be constructed so that it contains a subsystem of order* 8.

Proof.

- A $\mathfrak{K}(4, 2)$. Point set $\mathbb{Z}_3 \cup \{\infty\}$. Develop the following class mod 3.

$$(0, 1, \infty; 2)$$

- A $\mathfrak{K}(8, 2)$. Point set $\mathbb{Z}_7 \cup \{\infty\}$. Develop the following class mod 7.

$$(0, 3, \infty; 2),\ (5, 6, 1; 4)$$

- A $\mathfrak{K}(24, 2)$. Point set $\mathbb{Z}_{23} \cup \{\infty\}$. Develop the following class mod 23.

$$\begin{array}{lll} (0, 1, \infty; 2), & (4, 2, 7; 12), & (6, 3, 10; 17) \\ (5, 9, 14; 16), & (8, 15, 21; 18), & (11, 13, 19; 20) \end{array}$$

- A $\mathfrak{K}(44, 2)$. Point set $\mathbb{Z}_{43} \cup \{\infty\}$. Develop the following class mod 43.

$$\begin{array}{llll} (0, 1, \infty; 2), & (3, 4, 6; 28), & (5, 8, 12; 33), & (7, 11, 16; 24), \\ (9, 14, 20; 38), & (10, 17, 23; 26), & (13, 21, 32; 31), & (15, 25, 39; 35), \\ (18, 27, 40; 34), & (19, 29, 41; 36), & (22, 30, 42; 37) & \end{array}$$

- A $\mathfrak{K}$–frame of type 4^4. Point set is $\mathbb{Z}_4 \times \mathbb{Z}_4$, groups are $\mathbb{Z}_4 \times \{i\}$, $i \in \mathbb{Z}_4$, develop the following class mod(4, 4):

$$((1,2),(0,0),(0,1);(2,1)), \quad ((1,0),(3,1),(3,2);(0,2)),$$
$$((1,1),(2,0),(2,2);(3,0))$$

- $\mathfrak{K}$–RGDD of type 4^3. Point set is $\mathbb{Z}_4 \times \mathbb{Z}_3$, groups are $\mathbb{Z}_4 \times \{i\}$, $i \in \mathbb{Z}_3$, develop the following two classes mod(4, -):

$$(0_0,0_1,0_2;1_1), \quad (1_0,2_1,3_2;2_2), \quad (1_2,3_0,3_1;2_0)$$
$$(0_1,1_0,0_2;3_2), \quad (1_1,2_0,2_2;3_0), \quad (1_2,0_0,2_1;3_1)$$

This also gives a $\mathfrak{K}(12,2)$ by filling in the groups with $\mathfrak{K}(4,2)$'s.

- A $\mathfrak{K}$–RGDD of type 2^4. Point set $\mathbb{Z}_4 \times \mathbb{Z}_2$, groups are $\{i\} \times \mathbb{Z}_2$, $i \in \mathbb{Z}_4$. Develop the following three classes mod(-, 2):

$$((0,0),(1,0),(2,0);(1,1)), \quad ((0,1),(2,1),(3,0);(3,1))$$
$$((1,0),(0,0),(2,1);(0,1)), \quad ((1,1),(2,0),(3,1);(3,0))$$
$$((3,0),(0,1),(2,0);(0,0)), \quad ((3,1),(1,1),(2,1);(1,0))$$

Expand a 4-RGDD of type 4^4 (an RTD(4,4) [2]) by 2 apply the $\mathfrak{K}$–RGDD of type 2^4 above to obtain a $\mathfrak{K}$–RGDD of type 8^4. Fill in the groups of this design with $\mathfrak{K}(8,2)$ to obtain a $\mathfrak{K}(32,2)$ with a sub $\mathfrak{K}(8,2)$.

□

Theorem 6.3. *There exists a $\mathfrak{K}(v,\lambda)$ if and only if $v \equiv 0 \bmod 4$ and $\lambda \equiv 0 \bmod 2$,*

Proof. By lemma 6.1 there exists a RBIBD$(v,4,1)$ for all $v \equiv 4 \bmod 12$. Replace the blocks (K_4) of this design with the $\mathfrak{K}(4,2)$ above to obtain a $\mathfrak{K}(v,2)$, for every $v \equiv 4 \bmod 12$. We consider the remaining cases mod 24.

- $v \equiv 0 \bmod 24$

  By lemma 6.1 there exists a 3-RGDD of type 6^t for all $t \geq 4$. Expand this design using the $\mathfrak{K}$–RGDD of type 4^3 above to obtain a $\mathfrak{K}$–GDD of type 24^t, $t \geq 4$. Fill in the groups of the resulting design with $\mathfrak{K}(24,2)$'s to obtain a $\mathfrak{K}(24t,2)$, $t \geq 4$. This leaves the cases $v = 48, 72$ which are done in the appendix.

- $v \equiv 8 \bmod 24$

  By lemma 6.1 there exists a 4-GDD of type 6^t for all $t > 4$. Expand this design using the $\mathfrak{K}$–frame of type 4^4 above to get a $\mathfrak{K}$–frame of type 24^t. Use this design with 8 extra points and a $\mathfrak{K}(32,2)$ with a sub $\mathfrak{K}(8,2)$ to obtain a $\mathfrak{K}(8+24t,2)$ for all $t \geq 5$. This leaves the cases $v = 56, 80, 104$ which are done in the appendix.

- $v \equiv 12 \bmod 24$

  By lemma 6.1 there exists a 3-RGDD of type 3^{2t+1} for all $t \geq 1$. Expand this design using the $\mathfrak{K}$–RGDD of type 4^3 above to obtain a $\mathfrak{K}$–RGDD of type 12^{2t+1}, $t \geq 1$. Fill in the groups of the resulting design with $\mathfrak{K}(12, 2)$'s to obtain a $\mathfrak{K}(12 + 24t, 2)$.

- $v \equiv 20 \bmod 24$

  In [5] it is shown that a 4-GDD of type $6^t 9^1$ exists for all $t \geq 4$, $t \neq 10$, 12, 13, 14, 15, 17, 18, 19, 23. Expand this design using the $\mathfrak{K}$–frame of type 4^4 above to get a $\mathfrak{K}$–frame of type $24^t 36^1$. We will use this design with 8 extra *infinite* points.

  Place a $\mathfrak{K}(32, 2)$ with a sub $\mathfrak{K}(8, 2)$ on each group of size 24, with the hole over the 8 infinite points. There are 24 classes of the frame which miss each group of this type, pair these with the 24 long classes form the $\mathfrak{K}(32, 2)$. On the group of size 36 and the 8 infinite points place a $\mathfrak{K}(44, 2)$, 36 of the classes from this design are paired with the 36 classes from the frame which miss this group. The remaining 7 classes are paired with the the short classes from the $\mathfrak{K}(32, 2)$. This gives a $\mathfrak{K}(20 + 24(t + 1), 2)$ for all $t \geq 4$ and the exceptional values above. This leaves the cases $v \in$ { 20, 68, 92, 116, 284, 332, 356, 380, 404, 452, 476, 500, 596} which are done in the appendix.

□

We note that it should be possible to obtain a cyclic solution for the Kite as was done with the Fish and Mulinetto. However the existence of a cycle (triangle) within the design complicates any such solution.

7 Appendix: Small Cases

We first give a $T(85, 2)$, we then give a $\mathfrak{K}(v, 2)$ for each $v \in$ { 20, 48, 56, 68, 72, 80, 92, 104, 116, 140, 284, 332, 356, 380, 404, 452, 476, 500, 596}. In each of these latter cases the point set is $\mathbb{Z}_{v-1} \cup \{\infty\}$, develop the given class mod $v - 1$.

A $T(85, 2)$. Point set is $\mathbb{Z}_5 \times \mathbb{Z}_{17}$. The 10 classes below give a T–RGDD of type 5^{17} when developed mod$(5, -)$, with groups $\mathbb{Z}_5 \times \{i\}$, $0 \leq i \leq 8$. Taking each block twice and filling in all of the groups with a $T(5, 2)$ gives the required structure.

$(2_2; 1_5, 1_{13}, 1_0; 0_3)$
$(0_1; 2_3, 3_7, 3_{10}; 2_9)$
$(1_6; 4_2, 4_{11}, 4_{12}; 1_2)$
$(3_2; 1_3, 2_{14}, 0_{15}; 0_4)$
$(1_{12}; 3_{11}, 2_7, 0_9; 0_{10})$

$(2_8; 0_6, 1_1, 1_9; 4_4)$
$(0_{16}; 2_{13}, 1_8, 0_0; 4_{14})$
$(3_{12}; 1_{15}, 3_8, 2_5; 3_9)$
$(3_{13}; 4_3, 4_8, 1_{11}; 1_{16})$

$(4_{16}; 1_{10}, 2_6, 3_{15}; 3_1)$
$(4_0; 0_{12}, 0_5, 4_7; 0_2)$
$(4_{13}; 0_{11}, 2_{10}, 2_{12}; 2_4)$
$(4_{15}; 3_4, 3_5, 0_{13}; 4_9)$

$(3_0; 0_{14}, 3_{14}, 2_{11}; 0_7)$
$(2_{16}; 3_6, 1_4, 4_6; 2_1)$
$(1_7; 4_1, 0_8, 1_{14}; 3_{16})$
$(3_3; 2_{15}, 2_0, 4_5; 4_{10})$

$(0_8; 3_2, 2_{14}, 4_2; 1_{16})$
$(0_{15}; 1_8, 0_7, 3_6; 1_{10})$
$(4_{11}; 3_3, 2_5, 1_4; 3_1)$
$(0_{11}; 3_0, 1_1, 2_8; 0_{12})$
$(4_4; 3_{11}, 2_{13}, 1_{11}; 0_{14})$

$(1_3; 3_5, 1_{13}, 0_4; 4_5)$
$(0_9; 0_{16}, 4_{10}, 4_8; 4_7)$
$(2_3; 2_7, 4_0, 3_{15}; 0_{13})$
$(0_0; 2_9, 2_{12}, 0_5; 4_{15})$

$(4_{14}; 4_3, 1_7, 3_{16}; 0_3)$
$(2_0; 2_{10}, 2_{11}, 1_6; 2_{16})$
$(0_2; 1_{15}, 3_4, 1_{12}; 3_{10})$
$(0_{10}; 4_1, 2_{15}, 1_0; 1_9)$

$(4_6; 4_{16}, 1_5, 4_{13}; 2_6)$
$(3_9; 0_1, 2_4, 3_8; 4_{12})$
$(0_6; 1_2, 3_7, 3_{14}; 4_9)$
$(2_1; 2_2, 3_{13}, 3_{12}; 1_{14})$

$(0_{10};4_7,2_{11},3_{11};4_9)$
$(0_7;4_{13},2_{12},1_{16};3_0)$
$(4_3;4_{11},0_{14},1_7;3_2)$
$(1_3;0_{12},4_0,4_4;1_8)$
$(0_{16};0_4,3_5,3_{12};2_{14})$

$(1_5;4_2,4_8,1_{12};2_2)$
$(0_8;4_{12},4_6,1_1;0_2)$
$(0_0;2_{16},2_8,1_4;1_{11})$
$(1_9;2_3,3_8,4_{16};2_4)$

$(1_2;4_{14},2_{13},3_3;2_7)$
$(2_9;1_{15},2_1,0_5;1_6)$
$(1_{14};0_6,4_{10},4_1;3_{16})$
$(3_{10};0_3,1_{13},4_5;2_{10})$

$(3_9;3_6,2_0,4_{15};3_{13})$
$(3_7;1_0,2_{15},0_{13};3_{14})$
$(3_{15};2_6,0_1,0_{11};0_9)$
$(0_{15};1_{10},2_5,3_4;3_1)$

$(4_{15};3_0,4_0,2_8;2_1)$
$(3_{15};3_3,2_{14},0_{16};1_0)$
$(1_{11};2_{15},3_9,0_{10};3_7)$
$(0_8;3_{14},2_9,1_6;3_{12})$
$(2_5;4_7,2_{13},1_4;4_{14})$

$(2_3;2_{10},2_0,4_{12};2_{11})$
$(4_{11};4_8,3_{16},3_2;4_3)$
$(0_9;4_6,0_{14},2_2;0_6)$
$(0_{11};0_{13},1_3,3_8;0_2)$

$(2_7;4_{16},2_{16},2_4;1_9)$
$(1_{12};2_6,1_{16},0_7;3_{10})$
$(4_2;3_1,3_{11},4_9;2_{12})$
$(1_7;3_5,1_1,3_4;1_{15})$

$(0_4;1_2,4_{13},1_{13};0_{12})$
$(0_1;0_3,4_4,3_6;1_5)$
$(1_{14};0_5,4_5,0_0;1_{10})$
$(4_{10};1_8,4_1,3_{13};0_{15})$

$(1_1;4_{13},0_0,2_5;3_{11})$
$(4_{16};1_{12},1_5,4_3;3_8)$
$(2_7;0_4,3_4,0_6;0_1)$
$(2_{14};4_{15},4_5,4_{11};0_{14})$
$(3_9;1_{15},2_2,0_{15};2_8)$

$(1_3;4_7,3_{16},3_{14};3_1)$
$(1_6;1_2,2_4,4_4;3_{10})$
$(4_9;2_{10},1_{14},1_0;4_{12})$
$(2_{11};3_{13},1_9,1_8;2_{15})$

$(0_5;4_6,3_{12},0_{11};3_2)$
$(0_9;0_7,2_6,2_{12};1_{16})$
$(0_{13};3_3,4_{10},0_{12};0_2)$
$(0_3;2_1,4_{14},0_8;2_0)$

$(3_6;2_3,2_9,3_0;1_{10})$
$(4_1;1_{11},3_7,1_{13};1_4)$
$(0_{16};0_{10},1_7,4_8;3_5)$
$(2_{13};4_2,4_0,2_{16};3_{15})$

$(3_9;4_5,2_{13},0_7;2_9)$
$(3_{14};4_7,2_{15},3_2;2_{16})$
$(4_4;0_0,1_{12},1_{10};2_8)$
$(3_{10};4_{16},4_{15},1_2;0_9)$
$(4_6;0_{11},4_{14},0_1;4_{10})$

$(0_{12};4_3,3_8,4_{13};1_6)$
$(4_{12};4_1,1_1,2_{10};0_3)$
$(2_5;1_{13},1_{11},1_{16};0_5)$
$(1_{15};3_6,2_{11},2_4;1_{14})$

$(4_{11};1_5,3_{12},3_1;4_9)$
$(4_2;3_{15},4_8,2_0;3_7)$
$(0_6;3_0,1_3,0_4;1_7)$
$(3_4;2_2,2_3,3_3;1_9)$

$(2_7;3_{11},0_{15},3_5;3_{16})$
$(0_2;2_{12},2_{14},1_0;2_6)$
$(0_{16};3_{13},2_1,1_4;1_8)$
$(0_8;0_{13},4_0,0_{10};0_{14})$

$(1_9;3_4,4_{13},2_0;2_2)$
$(0_{13};4_5,4_3,4_8;4_{15})$
$(2_5;0_0,1_2,2_7;3_9)$
$(3_1;3_5,0_8,0_5;3_{13})$
$(2_{15};4_2,4_{10},2_{14};3_6)$

$(3_{14};4_{16},0_3,1_{13};3_{11})$
$(3_{12};3_{15},1_1,0_{14};3_7)$
$(3_0;1_7,2_{12},0_1;2_9)$
$(3_2;3_{10},2_3,0_7;0_{11})$

$(4_{11};2_6,1_{15},4_{14};4_4)$
$(1_4;4_0,2_{11},0_{12};0_6)$
$(1_{14};0_4,0_{10},0_9;2_{16})$
$(3_{16};4_9,3_8,1_{11};1_{10})$

$(2_1;1_5,2_{13},0_2;2_4)$
$(1_3;0_{16},2_8,1_6;2_{10})$
$(4_1;1_{16},3_3,1_0;1_8)$
$(0_{15};1_{12},4_{12},4_7;4_6)$

$(4_5;2_4,1_{10},2_9;1_3)$
$(1_{13};0_7,2_1,3_7;2_6)$
$(3_2;0_{13},3_4,4_{16};3_0)$
$(2_{12};1_6,4_4,3_{16};2_{13})$
$(0_1;1_4,1_0,4_{12};1_{15})$

$(2_{15};2_5,2_2,4_{14};2_{16})$
$(0_6;4_{13},4_{15},4_7;2_{11})$
$(4_1;0_{16},4_0,0_{15};0_{11})$
$(4_{10};3_{12},2_{14},3_{14};0_9)$

$(3_6;3_8,2_{10},3_{11};0_3)$
$(3_{10};4_3,3_{15},1_8;3_1)$
$(0_5;0_2,1_7,2_8;1_{11})$
$(1_5;2_0,3_{13},0_8;0_{14})$

$(2_7;1_2,1_1,1_9;4_6)$
$(4_{11};1_{14},0_{12},2_3;4_9)$
$(0_4;0_0,3_5,3_3;1_{12})$
$(0_{10};4_2,3_9,4_8;1_{16})$

$(4_3;0_7,1_6,2_{15};2_9)$
$(0_{16};0_{15},2_7,0_2;3_5)$
$(1_2;4_{11},3_9,1_7;4_{12})$
$(3_0;0_{13},3_{12},1_{15};2_1)$
$(1_{16};3_{15},0_9,2_{13};1_0)$

$(4_6;3_1,1_{10},1_3;1_5)$
$(4_{14};2_6,1_1,0_1;0_{11})$
$(3_{13};0_3,1_9,2_{14};4_0)$
$(3_{10};2_{16},2_{11},3_7;4_8)$

$(2_5;2_8,4_1,1_{12};1_{11})$
$(0_{14};0_5,0_{12},4_{13};4_7)$
$(2_3;4_{15},2_{12},0_8;3_3)$
$(4_5;0_{10},4_9,3_{14};3_{16})$

$(2_{10};4_4,1_4,3_2;3_{11})$
$(1_{13};1_{14},3_8,0_6;4_2)$
$(1_8;4_{16},2_2,2_0;4_{10})$
$(3_6;2_4,0_4,0_0;3_4)$

$(4_0;2_5,3_{15},2_{11};3_{16})$
$(3_{10};4_{12},3_6,2_3;1_1)$
$(1_4;1_5,1_{10},3_{14};2_2)$
$(1_{13};1_{15},4_8,2_0;3_{11})$
$(0_{13};2_{12},0_0,3_4;2_8)$

$(1_8;4_7,2_{14},0_{14};4_1)$
$(0_{12};1_{14},1_9,0_{10};1_7)$
$(2_{10};2_{13},4_2,4_{16};1_{11})$
$(4_9;4_3,4_{13},0_1;0_{16})$

$(3_{12};3_9,2_4,3_7;4_{14})$
$(0_7;2_{15},1_0,4_{11};2_9)$
$(0_4;0_9,3_8,3_1;0_2)$
$(0_{15};3_5,0_6,3_0;1_2)$

$(3_2;3_{13},3_3,1_{16};0_3)$
$(2_1;4_{10},4_{15},0_{11};1_6)$
$(4_5;1_{12},4_6,1_3;0_5)$
$(0_8;2_7,2_6,4_4;2_{16})$

The Kites

In each case the point set is $\mathbb{Z}_{v-1} \cup \{\infty\}$, develop the given class mod $v-1$.

$v = 20$
$(0, 1, \infty; 18)$, $(3, 13, 16; 14)$, $(2, 4, 11; 6)$, $(12, 8, 5; 17)$, $(15, 9, 7; 10)$

$v = 48$
$(0, 1, \infty; 46)$, $(11, 18, 40; 16)$, $(30, 3, 15; 39)$, $(37, 12, 31; 27)$, $(29, 8, 26; 43)$, $(4, 34, 10; 35)$,
$(32, 13, 28; 41)$, $(5, 17, 19; 36)$, $(20, 7, 9; 44)$, $(6, 2, 23; 42)$, $(25, 33, 38; 45)$, $(14, 24, 21; 22)$,

$v = 56$
$(0, 1, \infty; 54)$, $(38, 42, 51; 27)$, $(45, 37, 20; 13)$, $(34, 2, 12; 52)$, $(3, 24, 40; 9)$, $(41, 44, 46; 32)$,
$(22, 25, 50; 8)$, $(31, 5, 48; 17)$, $(26, 7, 47; 33)$, $(39, 10, 23; 29)$, $(30, 49, 18; 19)$, $(15, 43, 35; 11)$,
$(21, 16, 36; 14)$, $(4, 28, 6; 53)$

$v = 68$
$(0, 1, \infty; 66)$, $(2, 39, 15; 63)$, $(23, 7, 51; 49)$, $(4, 46, 38; 45)$, $(52, 36, 25; 18)$, $(59, 19, 21; 47)$,
$(53, 17, 24; 43)$, $(14, 60, 5; 34)$, $(54, 35, 57; 31)$, $(41, 20, 26; 65)$, $(8, 61, 13; 40)$, $(9, 37, 27; 44)$,
$(6, 42, 56; 64)$, $(12, 16, 29; 32)$, $(33, 48, 11; 22)$, $(58, 55, 50; 62)$, $(28, 10, 3; 30)$

$v = 72$
$(0, 1, \infty; 70)$, $(43, 51, 63; 3)$, $(30, 19, 36; 13)$, $(14, 54, 44; 39)$, $(24, 57, 38; 68)$, $(64, 28, 9; 6)$,
$(42, 46, 12; 34)$, $(32, 55, 52; 7)$, $(4, 60, 53; 66)$, $(26, 58, 15; 50)$, $(23, 2, 61; 17)$, $(33, 31, 5; 29)$,
$(47, 62, 65; 10)$, $(25, 41, 67; 48)$, $(20, 18, 11; 59)$, $(27, 49, 22; 37)$, $(69, 40, 45; 16)$, $(21, 56, 35; 8)$,

$v = 80$
$(0, 1, \infty; 78)$, $(44, 76, 23; 67)$, $(60, 16, 3; 30)$, $(66, 49, 56; 70)$, $(45, 48, 8; 2)$, $(19, 64, 69; 37)$,
$(73, 63, 61; 17)$, $(53, 33, 74; 14)$, $(59, 41, 35; 28)$, $(55, 5, 43; 71)$, $(26, 9, 72; 34)$, $(38, 6, 31; 51)$,
$(11, 39, 54; 42)$, $(10, 29, 24; 40)$, $(15, 75, 4; 21)$, $(50, 65, 13; 47)$, $(27, 25, 52; 18)$, $(58, 62, 7; 36)$,
$(68, 22, 57; 77)$, $(46, 12, 32; 20)$

$v = 92$
(0, 1, ∞; 90), (8, 73, 66; 74), (59, 76, 46; 25), (65, 10, 2; 89), (38, 7, 57; 23), (4, 81, 42; 40),
(9, 82, 52; 14), (43, 88, 69; 63), (80, 39, 18; 31), (84, 87, 78; 26), (77, 35, 75; 54), (12, 5, 36; 27),
(56, 70, 53; 34), (67, 3, 11; 22), (6, 28, 16; 62), (49, 37, 21; 45), (13, 15, 47; 86), (44, 83, 60; 33),
(61, 17, 71; 24), (85, 64, 58; 41), (55, 30, 50; 51), (20, 79, 68; 29), (19, 32, 72; 48)

$v = 104$
(0, 1, ∞; 102), (26, 54, 11; 77), (43, 20, 91; 64), (52, 74, 2; 73), (27, 71, 25; 66), (3, 40, 75; 90),
(69, 72, 60; 89), (92, 56, 39; 44), (9, 37, 12; 15), (61, 38, 21; 14), (94, 36, 68; 24), (98, 42, 35; 19),
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