On 3-chromatically unique and 3-chromatically equivalent hypergraphs

Ewa Drgas-Burchardt
Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
ul. prof. Z.Szafrana 4a, 65-516 Zielona Góra, Poland
e-mail: E.Drgas-Burchardt@wmie.uz.zgora.pl

May 13, 2009

Abstract

We introduce notions of k-chromatic uniqueness and k-chromatic equivalence in the class of all Sperner hypergraphs. They generalize the chromatic uniqueness and equivalence defined in the class of all graphs [10] and hypergraphs [2, 4, 8]. Using some known facts, concerning a k-chromatic polynomial of a hypergraph [5], a set of hypergraphs whose elements are 3-chromatically unique is indicated. A set of hypergraphs characterized by a described 3-chromatic polynomial is also shown. The application of the investigated notions can be found in [5].

Keywords: colorings of hypergraphs, chromatic polynomials of hypergraphs, chromatic uniqueness and equivalence
2000 Mathematics Subject Classification: 05C15, 05C65

1 Notation

For terminology not explicitly given in the article we follow [1]. A hypergraph $\mathcal{H}$ consists of a finite non-empty set $X(\mathcal{H})$ of *vertices*, and a family $\mathcal{E}(\mathcal{H})$ of non-empty subsets of $X(\mathcal{H})$, called *edges*. We often write $\mathcal{H} = (X, \mathcal{E})$, which means that $X = X(\mathcal{H})$ and $\mathcal{E} = \mathcal{E}(\mathcal{H})$.
An edge of cardinality h is called h-edge. If all edges of a hypergraph $\mathcal{H}$ are h-edges, then $\mathcal{H}$ is said to be *h-uniform*. A hypergraph whose no edge is a subset of another one is said to be *Sperner*.

In this article all hypergraphs are Sperner and considered up to isomorphism. It means that $\mathcal{H}_1 = \mathcal{H}_2$ ($\mathcal{H}_1 \neq \mathcal{H}_2$) denotes that $\mathcal{H}_1$ is isomorphic to $\mathcal{H}_2$ ($\mathcal{H}_1$ is non-isomorphic to $\mathcal{H}_2$).
For an n-vertex hypergraph $\mathcal{H}$, where $\mathcal{H} = (X, \mathcal{E})$, and positive integers $r, a_1, \ldots, a_r \in \mathbb{N}$, by a symbol $\mathcal{H}[a_1, \ldots, a_r]$, we denote a hypergraph with the vertex set X and the edge set $\{e \in \mathcal{E} : |e| \in \{a_1, \ldots, a_r\}\}$. Next, for $\mathcal{E}_1 \subseteq \mathcal{E}$ a symbol $\mathcal{H} - \mathcal{E}_1$ stands for a hypergraph $(X, \mathcal{E} - \mathcal{E}_1)$.
Let $\lambda \in \mathbb{N}$. λ*−coloring* of a hypergraph $\mathcal{H}$ is a mapping from the set $X(\mathcal{H})$ into the set $\{1, \ldots, \lambda\}$.
For a fixed $\lambda, k \in \mathbb{N}$ and a hypergraph $\mathcal{H}$ we denote by $f_k(\mathcal{H}, \lambda)$ the number of different λ-colorings of $\mathcal{H}$ satisfying that the image of each edge $e \in \mathcal{E}(\mathcal{H})$ is a multiset containing at least k different elements.
Let $\mathcal{H}$ be a hypergraph and let $Par_k^p(\mathcal{H})$ denote the number of partitions of $X(\mathcal{H})$ into exactly p non-empty parts, such that there exists an edge $e \in \mathcal{E}(\mathcal{H})$ that is contained in the union of at most $k-1$ partition parts. It is a known fact that $f_k(\mathcal{H}, \lambda)$ is a polynomial in λ of the following form [5]

$$f_k(\mathcal{H}, \lambda) = \lambda^n - \sum_{j=0}^{n} \lambda^j \sum_{p=j}^{n} s(p,j) Par_k^p(\mathcal{H}), \tag{1}$$

where $n = |X(\mathcal{H})|$ and $s(p,j)$ is the Stirling number of the first kind with parameters p, j. It may be observed that $f_k(\mathcal{H}, \lambda) \neq 0$ is equivalent to the condition that $|e| \geq k$ for every $e \in \mathcal{E}(\mathcal{H})$.
The partitions counted by $Par_k^p(\mathcal{H})$ for each permissible $k, p, \mathcal{H}$ are called *bad*. Other partitions of $X(\mathcal{H})$ will be called *good*.

2 Preliminaries

Below we recall the theorem, which will be useful in the next part of the article. It is a special case of the general result, that was shown in [5].

Theorem 1 *[5] Let $\mathcal{H}$ be a hypergraph with n vertices satisfying that each edge of $\mathcal{H}$ has the cardinality at least three and let $l \in \mathbb{N}$. The sets of bad partitions counted by the numbers $Par_3^p(\mathcal{H})$ and $Par_3^p(\mathcal{H}[3, 4, \ldots, l+2])$ are equal for $p \in \{n-l, \ldots, n\}$.*

Let $a_1, \ldots, a_n \in \mathbb{N}$. A partition of an n-element set has a *type* $(a_1, \ldots, a_n)$ or is called an $(a_1, \ldots, a_n)$*-type partition* if it contains a_i parts of size i for $i \in \{1, \ldots, n\}$. By $\binom{X}{n}$ we denote the set of all n-element subsets of the set X.

Let $\mathcal{E}_q$ denote a family of 4-element subsets of a set $\{x_1, y_1, x_2, y_2, \ldots, x_q, y_q\}$, which has the form $\{\{x_i, y_i, x_j, y_j\} : i, j \in \{1, \ldots, q\}, i \neq j\}$. By $\mathcal{H}_n^4$ we mean an n-vertex hypergraph with the set $\binom{V(\mathcal{H}_n^4)}{4}$ taken as a set of edges. A symbol $\mathcal{H}_n^{4,q}$, $n \geq 2q$, stands for an n-vertex, 4-uniform hypergraph whose vertices can be labeled in such a way that $\mathcal{H}_n^{4,q} = \mathcal{H}_n^4 - \mathcal{E}_q$. For instance the set of vertices of $\mathcal{H}_7^{4,3}$ can be labeled $x_1, y_1, x_2, y_2, x_3, y_3, x_4$ and in such case $\mathcal{E}(\mathcal{H}_7^{4,3}) = \binom{V(\mathcal{H}_7^{4,3})}{4} \setminus \{\{x_1,x_2,y_1,y_2\},\{x_1,x_3,y_1,y_3\},\{x_2,x_3,y_2,y_3\}\}$.

Lemma 2 *Let $\mathcal{H}$ be a 4-uniform n-vertex hypergraph with m edges. Now, $Par_3^{n-2}(\mathcal{H}) = 3m + \binom{n}{3} - t$, where t is the number of 3-element sets in $V(\mathcal{H})$ which are not included in any edge.*

Proof. Partitions of an n-element set into $n-2$ parts can be only of two types: $\underbrace{(n-3,0,1,0,\ldots,0)}_{n}$, $\underbrace{(n-4,2,0,\ldots,0)}_{n}$. It is clear that among $\binom{n}{3}$ partitions of the first kind, $\binom{n}{3} - t$ are bad. Moreover, for each 4-element edge we can find three partitions of the second type which are bad with respect to this edge. Obviously different edges correspond to different partitions in that case, which completes the proof. ∎

Corollary 3 *Let $\mathcal{H}$ be a 4-uniform n-vertex hypergraph with m edges, satisfying that $m \geq \binom{n}{4} - n + 4$. Each partition containing a part of the cardinality at least three is counted by $Par_3^l(\mathcal{H})$ for corresponding l. Moreover, $Par_3^{n-2}(\mathcal{H}) = 3m + \binom{n}{3}$.*

Proof. The assumption $m \geq \binom{n}{4} - n + 4$ implies that each 3-element subset of $V(\mathcal{H})$ is included in at least one edge. The last sentence of the corollary follows Lemma 2. Consider a partition with an s-element part A. Next analyze a fixed 3-element subset B of A. It was stated that B is included in at least one edge of $\mathcal{H}$, and 4-uniformity of $\mathcal{H}$ gives the assertion. ∎

Recall that the *Stirling* number of a second kind $S(n,k)$ with parameters n, k stands for the number of partitions of an n-element set into exactly k non-empty parts.

Lemma 4 *If $n, q \in \mathbb{N}$, $q \geq 3$ and $\binom{q}{2} \leq n-4$, then*

$$Par_3^p(\mathcal{H}_n^{4,q}) = \begin{cases} 0 & p \in \{n-1, n\} \\ 3(\binom{n}{4} - \binom{q}{2}) + \binom{n}{3} & p = n-2 \\ S(n,p) - \binom{q}{n-p} & p \in \{n-q, n-q+1, \ldots, n-3\} \\ S(n,p) & p \in \{1, 2, \ldots, n-q-1\}. \end{cases}$$

Proof. Because $\mathcal{H}_n^{4,q}[2,3]$ is an edgeless hypergraph, the application of Theorem 1 gives $Par_3^n(\mathcal{H}_n^{4,q}) = Par_3^{n-1}(\mathcal{H}_n^{4,q}) = 0$. By Corollary 3 we know the number $Par_3^{n-2}(\mathcal{H}_n^{4,q})$. Let $p \in \{1, \ldots, n-3\}$, Corollary 3 used again implies that each partition into p parts, that has at least one s-element part, $s \geq 3$, is bad. Hence, it is enough to consider only $(2p-n, n-p, 0, \ldots, 0)$-type partitions. Such a partition is good if and only if any two of 2-element parts are not included in an edge. It means that in each good partition, 2-element parts create the set $\{\{x_i, y_i\},\ i \in \{1, \ldots, n-p\}\}$ with labeling suggested by the definition of $\mathcal{H}_n^{4,q}$. The construction of $\mathcal{H}_n^{4,q}$ implies that the number of such partitions is equal to $\binom{q}{n-p}$, and consequently $Par_3^p(\mathcal{H}_n^{4,q}) = S(n,p) - \binom{q}{n-p}$. The equality $\binom{q}{n-p} = 0$, for p satisfying $1 \leq p \leq n-q-1$, completes the proof. ∎

3 Uniqueness and equivalence

Hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ are called *k-chromatically equivalent* if $f_k(\mathcal{H}_1, \lambda) = f_k(\mathcal{H}_2, \lambda)$. A hypergraph $\mathcal{H}$ is said to be *k-chromatically unique* if $\mathcal{H} = \mathcal{H}_1$ for each hypergraph $\mathcal{H}_1$ such that $f_k(\mathcal{H}_1, \lambda) = f_k(\mathcal{H}, \lambda)$.
The notion of 2-chromatically unique graphs was introduced by Chao and Whitehead [3]. Next it was generalized to the notion of 2-chromatically unique hypergraphs studied by Borowiecki and Lazuka [2], Dhomen [4] and Tomescu [7, 8]. The number of results in this field is still not impressive. Below we recall all of the ones we know.
Let C_m^h denote a linear, h-uniform cycle with m edges. Tomescu proved that a cycle C_m^h is 2-chromatically unique for all m, h being numbers not less than three. The symbol $SH(n,p,h)$ stands for an h-uniform hypergraph of order n and size k, where $n = h + (k-1)p$ and $1 \leq p \leq h-1$, $h \geq 3$. Each edge of C_m^h consists of p distinct vertices and a common subset to all edges with $h-p$ vertices. In [8] it was shown that this hypergraph is 2-chromatically unique in the set of all h-uniform hypergraphs for every p, that satisfies $1 \leq p \leq h-2$, but this is not true for p, h satisfying $p = h-1$ and $k \geq 3$. Also $SH(n,p,h)$ is not 2-chromatically unique for all p, k, where $p, k \geq 2$.
Borowiecki and Lazuka proved that $SH(n,1,h)$ is 2-chromatically unique. Their result deals with the class of all hypergraphs (not only h-uniform). The proof of the result was corrected in [9]. It is worth mentioning that $SH(n,1,4)$ is 3-chromatically unique for $n \in \{3, \ldots, 6\}$ [6], which was pointed out using a computer. Moreover, $SH(n,1,h)$ is not h-chromatically unique for any n greater than two, which can be verified immediately by Theorem 1. Drgas-Burchardt and Kmiecik stated a hypothesis that $SH(n,1,4)$ is 3-chromatically unique for all n, where $n \geq 3$. They have conjectured that there exists the dependence between k-chromatic uniqueness and l-chromatic uniqueness of a given hypergraph for different parameters l and k.
Below, we present a new contribution to the forward development of this area. The next lemma follows immediately by equality (1).

Lemma 5 *Let $\mathcal{H}_1$ be a hypergraph satisfying $f_k(\mathcal{H}_1, \lambda) \neq 0$. Hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ are k-chromatically equivalent if and only if $|V(\mathcal{H}_1)| = |V(\mathcal{H}_2)|$*

and $Par_k^p(\mathcal{H}_1) = Par_k^p(\mathcal{H}_2)$ *for* $p \in \{1,\dots,n\}$, *where* n *is the common cardinality of the vertex sets* $V(\mathcal{H}_1)$ *and* $V(\mathcal{H}_2)$.

Proof. Obviously, from formula (1), it follows that if for $n = |V(\mathcal{H}_1)| = |V(\mathcal{H}_2)|$ and $p \in \{1,\dots,n\}$ we have $Par_k^p(\mathcal{H}_1) = Par_k^p(\mathcal{H}_2)$ then $f_k(\mathcal{H}_1,\lambda) = f_k(\mathcal{H}_2,\lambda)$.
Assume that $f_k(\mathcal{H}_1,\lambda) = f_k(\mathcal{H}_2,\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda$. Evidently, using the formula (1) once more, $n = |V(\mathcal{H}_1)| = |V(\mathcal{H}_2)|$ and $Par_k^n(\mathcal{H}_1) = Par_k^n(\mathcal{H}_2) = 0$. For fixed j, $1 \le j \le n-1$, we can observe the equality $Par_k^j(\mathcal{H}_1) = Par_k^j(\mathcal{H}_2)$ as a consequence of the equalities: $a_{n-j} = -\sum_{p=j}^{n} s(p,j)Par_k^p(\mathcal{H}_1) = -\sum_{p=j}^{n} s(p,j)Par_k^p(\mathcal{H}_2)$ and $Par_k^l(\mathcal{H}_1) = Par_k^l(\mathcal{H}_2)$ for $l \in \{j+1,\dots,n\}$. The last $n-j$ equalities are based on analysis of the coefficients $a_1,\dots,a_{n-(j+1)}$. It means that after $n-1$ steps, for $j = n-1,\dots,1$ respectively, the above considerations yield the assertion in this case. ■

Theorem 6 *Let* $n, q \in \mathbb{N}$ *and* $\binom{q}{2} \le n-4$. *The hypergraph* $\mathcal{H}_n^{4,q}$ *is* 3-*chromatically unique.*

Proof. Conversely, suppose that there exists a hypergraph $\mathcal{H}^*$, different from $\mathcal{H}_n^{4,q}$, that is 3-chromatically equivalent to $\mathcal{H}_n^{4,q}$. Of course $|V(\mathcal{H}^*)| = n$ because $f_3(\mathcal{H}^*,\lambda) \neq 0$ is a polynomial of degree n. By Theorem 1 we have that $0 = P_3^{n-1}(\mathcal{H}^*) = P_3^n(\mathcal{H}^*) = P_3^n(\mathcal{H}^*[2,3])$, so for every $e \in \mathcal{E}(\mathcal{H}^*)$ must hold $|e| \ge 4$. Moreover, Theorem 1 and Lemmas 2,4 give us $Par_3^{n-2}(\mathcal{H}^*) = 3((\binom{n}{4} - \binom{q}{2})) + \binom{n}{3} = Par_3^{n-2}(\mathcal{H}^*[4]) = 3m + \binom{n}{3} - t$, where m is the number of 4-edges in $\mathcal{H}^*$ and t stands for the number of 3-element vertex subsets in $V(\mathcal{H}^*)$, that are not included in any 4-edge of $\mathcal{H}^*$. Next, it is possible to write the following sequence of equalities

$$3(\binom{n}{4} - \binom{q}{2}) + \binom{n}{3} = 3m + \binom{n}{3} - t$$
$$3(\binom{n}{4} - \binom{q}{2}) = 3m - t.$$

Notice, that if $t > 0$ then $m \le \binom{n}{4} - n + 3$ and it follows

$$3(\binom{n}{4} - \binom{q}{2} \le 3(\binom{n}{4} - n + 3) - t$$
$$\text{and } \binom{q}{2} \ge n - 3 + \tfrac{t}{3}$$

which is impossible with respect to the assumption $\binom{q}{2} \le n-4$. Hence $t = 0$, $m = \binom{n}{4} - \binom{q}{2}$ and, in consequence, each partition of $X(\mathcal{H}^*)$ into $n-q$ non-empty parts, that contains a partition part being at least a 3-element set is bad. The only good partition into $n-q$ parts can be of the type $\underbrace{(n-2q, q, 0, \dots, 0)}_{n}$. Such a partition is good if and only if $\binom{q}{2}$ missed 4-edges form a structure $\mathcal{E}_q$. We thus get $\mathcal{H}^*[4] = \mathcal{H}_n^{4,q}$. It is enough to decide if $\mathcal{H}^*$ possesses any edges of size greater than four. According to the construction of $\mathcal{H}^*[4]$ we can easily check that each s-element vertex subset of $X(\mathcal{H}^*)$, where $s \geq 5$, contains a 4-edge of $\mathcal{H}^*$. It guarantees $\mathcal{H}^* = \mathcal{H}^*[4]$ and completes the proof because $\mathcal{H}^*$ is a Sperner hypergraph. ■

Let Q denote a class of hypergraphs $\mathcal{H}$ satisfying the condition: if $x_{i_1}, \ldots, x_{i_6} \in X(\mathcal{H})$, then at least one of sets: $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$, $\{x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ $\{x_{i_1}, x_{i_2}, x_{i_5}, x_{i_6}\}$ is an edge of $\mathcal{H}$.

Lemma 7 *If $\mathcal{H} \in Q$ is a 4-uniform, n-vertex hypergraph with at least $\binom{n}{4} - n + 4$ edges, then $Par_3^p(\mathcal{H}) = S(n,p)$ for $p \in \{1, \ldots, n-3\}$.*

Proof. By Corollary 3 the only type of a good partition could be $\underbrace{(n-2q, q, 0, \ldots, 0)}_{n}$, where $q \geq 3$. Let us consider arbitrary six vertices creating three of 2-element parts $\{x_i, y_i\}$, $1 \leq i \leq 3$. According to the condition $\mathcal{H} \in Q$ we are able to find an edge $e \in \mathcal{E}$, that is included in at most two parts of the partition. Hence it is the bad partition. ■

Theorem 8 *If $n, m \in \mathbb{N}$ and $m \geq \binom{n}{4} - n + 4$, then $f_3(\mathcal{H}, \lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda$ where*

$$a_i = \begin{cases} 0 & i = 1 \\ -(3m + \binom{n}{3}) & i = 2 \\ -s(n-2, n-i)(3m + \binom{n}{3}) - \sum_{k=n-i}^{n-3} S(n,k)s(k, n-i) & 3 \leq i \leq n-1 \end{cases}$$

if and only if $\mathcal{H}$ has exactly n vertices, m 4-edges and $\mathcal{H} = \mathcal{H}[4,5] \in Q$.

Proof. Let $\mathcal{H}$, where $\mathcal{H} = \mathcal{H}[4,5] \in Q$, be an n-vertex hypergraph with m 4-edges. Next, by Lemma 7, $S(n,p) \geq Par_3^p(\mathcal{H}) \geq Par_3^p(\mathcal{H}[4]) = S(n,p)$ for $p \in \{1, \ldots, n-3\}$. Moreover, $Par_3^n(\mathcal{H}) = Par_3^n(\mathcal{H}[2]) = 0$ and $Par_3^{n-1}(\mathcal{H}) = Par_3^{n-1}(\mathcal{H}[2,3]) = 0$, by Theorem 1. Corollary 3 implies that $Par_3^{n-2}(\mathcal{H}[4]) = 3m + \binom{n}{3}$. Because of Theorem 1, $Par_3^{n-2}(\mathcal{H}) = Par_3^{n-2}(\mathcal{H}[3,4]) = Par_3^{n-2}(\mathcal{H}[4])$. Using (1) we obtain the expected form of the polynomial.

Now, we assume that $f_3(\mathcal{H}_1, \lambda)$, where $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$, has the given form with specified n, m. The fact $f_3(\mathcal{H}_1, \lambda) \neq 0$ implies that every edge of $\mathcal{H}_1$ has the cardinality at least three. Theorem 1 determines the order of $\mathcal{H}_1$ ($|X_1| = n$), and according to $Par_3^{n-1}(\mathcal{H}_1) = Par_3^{n-1}(\mathcal{H}_1[3]) = 0$ we have that every edge of $\mathcal{H}_1$ consists of at least four vertices. Let $|\mathcal{E}(\mathcal{H}_1[4])| = m_1$. Theorem 1 and the pattern (1) imply that the coefficient of λ^{n-2} in $f_3(\mathcal{H}_1, \lambda)$ satisfies: $-a_2 = Par_3^{n-2}(\mathcal{H}_1) = Par_3^{n-2}(\mathcal{H}_1[4]) = 3m + \binom{n}{3}$. On the other hand by Lemma 2, we can see that $Par_3^{n-2}(\mathcal{H}[4]) = 3m_1 + \binom{n}{3} - t$, where t is the number of 3-element subsets of X_1, that are not included in any 4-edge of $\mathcal{H}[4]$. We thus get

$$3m + \binom{n}{3} = 3m_1 + \binom{n}{3} - t$$

and $m_1 = m + \frac{t}{3}$. It follows $m_1 \geq \binom{n}{4} - n + 4$ and by Corollary 3 we obtain the equalities $t = 0$ and $m_1 = m$. The form of $f_3(\mathcal{H}_1, \lambda)$ and Theorem 1 gives

$$Par_3^{n-3}(\mathcal{H}_1) = S(n, n-3) + Par_3^{n-3}(\mathcal{H}_1[3,4,5]) = Par_3^{n-3}(\mathcal{H}_1[4,5]).$$

Hence, each $\underbrace{(n-6, 3, 0, \ldots, 0)}_{n}$-partition is bad in a hypergraph $\mathcal{H}_1[4,5]$. We consider such a partition with 2-element partition parts $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$. The partition has to be bad with respect to some 4-edges or some 5-edges but it is not possible for such the partition to destroy any 5-edge. Thus at least one of sets $\{v_1, v_2, v_3, v_4\}$, $\{v_3, v_4, v_5, v_6\}$, $\{v_1, v_2, v_5, v_6\}$ is a 4-edge of $\mathcal{H}_1$ and $\mathcal{H}_1 \in Q$. Finally assume that there exists a p-edge in $\mathcal{H}_1$ for $p \geq 6$. We consider six vertices belonging to this edge and because of $\mathcal{H}_1 \in Q$ we know that this set contains at least one 4-edge of $\mathcal{H}_1$, contrary to the claim that $\mathcal{H}_1$ is Sperner. ∎

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

[2] M. Borowiecki and E. Łazuka, *On chromaticity of hypergraphs*, Discrete Mathematics, 307 (2007) 1418–1429.

[3] C.Y. Chao and E.G. Whitehead Jr., *On chromatic equivalence of graphs*, in:Y. Alavi, D.R. Lick, Theory and Application of Graphs, Lecture Notes in Math., vol. 642, Springer New York, Berlin, (1978), 121–131.

[4] K. Dohmen, *Chromatische Polynome von Graphen und Hypergraphen*, Dissertation, Düsseldorf, (1993).

[5] E. Drgas-Burchardt and E. Łazuka, *Chromatic polynomials of hypergraphs*, Applied Mathematics Letters 20 (2007) 1250–1254.

[6] Ł. Kmiecik, M.S.Dissertation, Zielona Góra, (2006)

[7] I. Tomescu, *Chromatic coefficients of linear uniform hypergraphs*, J. Combin. Theory (B) 72 (1998) 229–235.

[8] I. Tomescu, *On the chromaticity of sunflower hypergraphs SH(n,p,h)*, Discrete Mathematics 307 (2007) 781–786.

[9] I. Tomescu, *Sunflower hypergraphs are chromatically unique*, Discrete Mathematics 285 (2004) 355–757.

[10] H. Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. 38 (1932) 572–579.