Θ-graphs of partial cubes and strong edge colorings

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Abstract

It was conjectured in [10] that the upper bound for the strong chromatic index $s'(G)$ of bipartite graphs is $\Delta(G)^2$, where $\Delta(G)$ is the largest degree of vertices in G. In this note we study the strong edge coloring of some classes of bipartite graphs that belong to the class of partial cubes. We introduce the concept of Θ-graph $\Theta(G)$ of a partial cube G, and show that $s'(G) \leq \chi(\Theta(G))$ for every tree-like partial cube G. As an application of this bound we derive that $s'(G) \leq 2\Delta(G)$ if G is a p-expansion graph.

Key words: median graph, tree-like partial cube, p-expansion graph, strong chromatic index.

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1 Introduction

A *strong edge-coloring* of a graph is an edge-coloring in which every color class is a strong matching, that is, the subgraph induced by the end vertices

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of the edges of any color class is a 1-regular graph. The *strong chromatic index* $s'(G)$ is the minimum number of colors in a strong edge-coloring of G. Since in ordinary edge colorings color classes form (ordinary) matchings, it is clear that for any graph G, $\chi'(G) \leq s'(G)$.

There are several papers studying this invariant (see e.g. [10, 18, 21]) many of which are concerned with finding the best possible upper bounds on $s'(G)$ in terms of the largest degree of a graph $\Delta(G)$, yielding an analogue of Vizing's theorem on the strong chromatic index. Several conjectures are known in this respect, one of which claims that for bipartite graphs $s'(G) \leq \Delta^2(G)$ [10]. Whether the bound is indeed correct for all bipartite graphs is open, yet on the other hand, one can expect better upper bounds when restricting to some particular classes of bipartite graphs. In this paper we shall present such a result for the class of p-expansion graph, introduced in [4], that are related to the well-known median graphs. We will show that $s'(G) \leq 2\Delta(G)$ for these graphs, and obtain some partial results for strong edge colorings of median graphs and tree-like partial cubes. Let us present the classes of bipartite graphs that we will consider in this paper, along with their main properties that motivated the study of their strong chromatic indices.

For $u, v \in V(G)$, let $d_G(u, v)$ denote the length of a shortest path (also called *geodesic*) in G from u to v. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of a graph G is *convex* if for any two vertices u, v of H all shortest paths between u and v in G are already in H.

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever either $ab \in E(G)$ and $x = y$ or $a = b$ and $xy \in E(H)$. The Cartesian product is commutative and associative. The n-cube (or hypercube) Q_n is the Cartesian product of n copies of K_2. Isometric subgraphs of hypercubes are also called *partial cubes*. They have been extensively studied, in particular in recent years [6, 7, 8, 13, 15]. The structure of partial cubes can be well understood by using the so-called *Djoković-Winkler relation* Θ, defined on the edge set of a graph.

Edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler relation Θ [9, 22] if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x, u) = d(y, v)$ and $d(x, v) = d(y, u)$. Winkler's well-known result from [22] states that a bipartite graph G is a partial cube if and only if relation Θ is transitive in G (and so Θ is an equivalence relation). Clearly in a partial cube G edges of a Θ class form an edge cut set, and the subgraph induced by these edges is isomorphic to $K_2 \Box P$, where P is a subgraph of G. (Note that by the expression 'the subgraph induced by the set of edges E' we always mean the subgraph induced by the end vertices of the edges from E.) Now, since P is also bipartite, one can derive that edges of each Θ-class can be colored by two colors in such a way that each color class is an induced matching (see the proof of Theorem 2). This observation lead us to study the strong chromatic index of (some classes of) partial cubes.

In the next section we consider tree-like partial cubes, introduced in [5], a natural class of graphs that lies between median graphs and partial cubes. We prove that $s'(G) \leq 2\chi(\Theta(G))$ for a tree-like partial cube G. In Section 3 we first show that for a p-expansion graph G we have $\omega(\Theta(G)) = \Delta(G)$. Then we prove our main theorem that Θ-graphs of p-expansion graphs are chordal, which implies the above mentioned bound for $s'(G)$ for these graphs. In the last section we present some open problems.

2 Θ-graph of a tree-like partial cube

We start with the definition of median graphs, a very important and well studied class of graphs, see [16] and the references therein.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u,v-path, a shortest u,w-path, and a shortest w,v-path [19]. The vertex x is called the *median* of the triple u,v,w. It follows immediately from the definition that median graphs are bipartite. It is well-known that median graphs are isometric subgraphs of hypercubes [19].

One of the main characterizations of median graphs involves a so-called peripheral expansion procedure, which is a special case of an expansion procedure as introduced by Mulder [19]. This graph operation has been used to characterize and define several classes of graphs that are related to median graphs, see [3].

Let G be a (bipartite) graph, and H an isometric subgraph of G. We say that G' is *obtained from G by the peripheral expansion of H* if G' is the graph obtained from the disjoint union of graphs G and H, by addition of $|V(H)|$ edges between H and the subgraph of G isomorphic to H that correspond to an isomorphism between the copies of H. We also say that we obtained G' from G by *expanding* H. Note: if we expand a subgraph H of a partial cube G then exactly one new Θ-class appears in G' whose edges induce an isometric subgraph of G' isomorphic to $H \Box K_2$.

It was proved by Mulder that median graphs are precisely those graphs that can be obtained by successive use of peripheral expansion from K_1 in which at every step a convex subgraph is expanded [20]. With the aim to get a deeper understanding of median graphs, a more general class of so called *tree-like partial cubes* was introduced as the graphs that can be obtained from K_1 by successive use of (ordinary) peripheral expansions. Several common properties of tree-like partial cubes with median graphs were established, see [5]. Some other classes of partial cubes, defined by expansion procedure, were defined and studied in [6].

Similarly as above, we note that in a tree-like partial cube G, the subgraph induced by edges of a Θ-class E is isomorphic to $K_2 \Box M$, where M is an isometric (and thus connected) subgraph of G. For an edge $ab \in E$ the set

$$U_{ab} = \{u \in V(G) \,|\, uv \in E \text{ and } d(u,a) < d(u,b)\}$$

induces the subgraph isomorphic to M (as also the set U_{ba}). We also need the following observation about the relation Θ.

Lemma 1 *[12] If a walk P connects the end vertices of an edge e but does not contain it, then P contains an edge f with $e\Theta f$.*

Let $T(G)$ denote the set of Θ-classes of a (tree-like) partial cube G. We say that Θ-classes $F_1, F_2 \in T(G)$ are *adjacent* if there exist edges $e_1 \in F_1$ and $e_2 \in F_2$ which are incident (that is, e_1 and e_2 have an end vertex in common). By $\Theta(G)$ we denote the intersection graph of Θ-classes of a graph G. That is, the vertex set of $\Theta(G)$ is $T(G)$ and two vertices in $\Theta(G)$ are adjacent whenever the respective Θ-classes are adjacent. Thus the Θ-graph presents an intersection concept in the sense of [17].

The distance $d(e, f)$ between edges $e = xy$ and $f = uv$ is defined as the distance between their closest end vertices:

$$d(e, f) = \min\{d(x,u), d(x,v), d(y,u), d(y,v)\}.$$

Note that $\chi(H)$ denotes the chromatic number of a graph H.

Theorem 2 *For a tree-like partial cube G,*

$$s'(G) \leq 2\chi(\Theta(G)).$$

Proof. Let G be a tree-like partial cube. Note that G is a connected bipartite graph. If G is K_1, the theorem is trivially true. Otherwise, G has at least one Θ-class. We claim that edges of each Θ-class can be colored by two colors α_1 and α_2 such that the subgraph A_i induced by edges of α_i is isomorphic to 1-regular graph. (We will use this coloring of Θ-classes also in the final construction.)

Let E be a Θ-class of G. Edges of E induce a subgraph isomorphic to $K_2 \Box M$ where M is a connected subgraph of G, and all edges of E project to K_2 by this representation. Since M is also bipartite, we may partition its vertices into two independent sets X_1, X_2, and color an edge of E by α_i if and only if their end vertices are in X_i. Thus we clearly obtained a strong coloring c' of edges of a Θ-class E using two colors. Note that by this coloring for any two edges e, f in the Θ-class E, we have that $d(e, f)$ is even if and only if $c'(e_i) = c'(f_i)$

Choose a minimum coloring c of $\Theta(G)$ (that is, a coloring with $\chi(\Theta(G))$ colors). Let $E_1, E_2, \ldots, E_k$ be the Θ-classes of G that are colored by the same color with respect to c. Thus no two Θ-classes E_i, E_j are adjacent in $\Theta(G)$, and so no two edges $e_i \in E_i, e_j \in E_j$ are incident in G. We claim that we can color all edges of $E_1 \cup \ldots \cup E_k$ by two colors in such a way that no two edges at distance 1 are colored by the same color. (This claim clearly suffices for the proof of the theorem.)

Without loss of generality choose an edge $e_1 \in E_1$ and set $c'(e_1) = \alpha_1$ (let $e_1 = x_1y_1$). First of all, by this choice, colors of all edges of E_1 are determined (by the construction from the second paragraph of this proof). Colors of edges in other Θ classes $E_2, \ldots, E_k$ are defined recursively as follows. If a Θ-class E_i has the property that no Θ-class E_j exists such that E_i and E_1 lie in different connected components of $G - E_j$, then for every $f \in E_i$ we set $c'(f) = \alpha_1$ if $d(f, e_1)$ is even, and $c'(f) = \alpha_2$ if $d(f, e_1)$ is odd. On the other hand if there is such a Θ-class E_j, then colors of edges from E_i are determined in one of the next steps, after the edges of E_j are colored. Clearly, by repeating this coloring procedure, all colors of edges from $E_1 \cup \ldots \cup E_k$ are determined. We claim that this coloring is strong, that is, any two edges from $E_1 \cup E_2 \cup \ldots \cup E_k$ at distance 1 are colored differently.

This claim clearly holds for edges of E_1. So assume first that E_i is any other Θ-class, and let $e \in E_1$, $e' \in E_i$ be edges at distance 1. Clearly E_i lies in one component of $G - E_1$, and assume x_1 also lies in this component. Since G is bipartite, and e and e' are from different Θ-classes, we infer that precisely one end vertex of e' is at distance 1 to precisely one end vertex of e. Suppose that $d(e_1, e)$ and $d(e_1, e')$ are of the same parity. Then we find that $d(e_1, e) = d(e_1, e') + 2$. Now, using Lemma 1 we infer the existence of an edge in relation Θ with e' which lies on the path between e_1 and e in $U_{x_1y_1}$ (since G is a tree-like partial cube, $U_{x_1y_1}$ is connected, and so such path exists). We deduce that there is an edge from E_i which connects two vertices of $U_{x_1y_1}$. This is a contradiction with E_1 and E_i being non adjacent in $\Theta(G)$. Thus distances of e and e' from e_1 are of different parity, and so they are colored by different colors by c'.

Now, let $e \in E_i$, $f \in E_j$ (both i and j different from 1 but not necessarily distinct) be two edges at distance 1. As noted earlier, they lie in the same component of $G - E_1$, and let x_1 be an end vertex of $e_1 = x_1y_1$ which also lies in this component. (We may assume that no Θ-class E_k exists such that E_i or E_j would lie in the other component of $G - E_k$ as E_1. Namely, if this is the case, we can translate the problem by observing the coloring of E_k, and then use the part of the definition that involves distances.) Again we infer that one end vertex of $e = xy$, say x, is closer to x_1 than the other, and analogously, let $d(u, x_1) < d(v, x_1)$ where $f = uv$. Suppose $d(u, x) = 1$. Then, as G is bipartite we find $d(x_1, x) \neq d(x_1, u)$, moreover $|d(x_1, x) - d(x_1, u)| = 1$. This readily implies that e and f are colored differently. If $d(u, x) = 2$, we may assume without loss of generality that v and x are adjacent (the possibility when u and y are adjacent can be treated analogously). Then, since x is closer to x_1 than y, we find by Lemma 1 that there is an edge of E_j that lies on a shortest path between x and x_1. Hence the Θ-class E_i lies in the other component of $G - E_j$ as E_1 contrary to our assumption. The final case is $d(u, x) = 3$ which implies that v and y are adjacent. Since G is bipartite we infer $d(x_1, u)$ is of different parity as $d(x_1, x)$ which implies e and f are colored differently. By the observation from the end of the third paragraph of the proof we infer that c' is a strong edge coloring of G. □

Corollary 3 *For a median graph G,*

$$s'(G) \leq 2\chi(\Theta(G)).$$

3 Strong edge coloring of p-expansion graphs

Another subclass of tree-like partial cubes are the so-called *p-expansion graphs*, treated in [4]. They are the graphs that can be obtained by successive use of peripheral expansions from K_1 in such a way that at each step the expanded graph H is

- one vertex, or
- a union of maximal hypercubes in G with nonempty common intersection.

It turns out that also in the second case H is an isometric subgraph, and so the expansion is well-defined. This class was introduced in relation with the cube graph transformation. More precisely, it was proved that chordal graphs are precisely the cube graphs (that is, intersection graphs of maximal hypercubes) of p-expansion graphs [4]. This result was established to complement a similar theorem for dually chordal graphs. In fact it is the rather close relation of p-expansion graphs with chordal graphs that will help us to prove the bound for their strong chromatic index.

By *clique* we mean a maximal complete subgraph of a graph.

Lemma 4 *Let G a nontrivial p-expansion graph. For every clique K_r in $\Theta(G)$ there exists a family of maximal hypercubes $H_1, H_2, \ldots, H_k$ in G, such that every vertex of $H_1 \cap H_2 \cap \ldots \cap H_k$ is incident with all Θ-classes of K_r.*

Proof. The proof is by induction on the order of G. If $G = K_2$ the assertion is trivially correct. Let G' be a p-expansion graph of order n, and let it be obtained from a (smaller) p-expansion graph G by one of two the expansion operations.

First, assume we expand a vertex x in G (to obtain an edge xx'). Then by induction hypothesis the assertion is true for all cliques in $\Theta(G)$. Clearly, $\Theta(G')$ can be obtained from $\Theta(G)$ by adding a vertex (that corresponds to Θ-class (edge) xx'), that is adjacent to some vertices – Θ-classes – which are all adjacent. Hence by one larger clique appears in $\Theta(G')$, and there is a unique family of hypercubes (including xx') that has a nonempty intersection (vertex x) and x is incident to all Θ-classes of this clique.

Assume now that G' is obtained from G by expanding a union of maximal hypercubes from $U = \{H'_1, H'_2, \ldots, H'_p\}$ in G with $H'_1 \cap H'_2 \cap \ldots \cap H'_p \neq \emptyset$. Then a new Θ-class E appers that is incident with all Θ-classes $E_1, E_2, \ldots, E_r$ that lie in hypercubes from U, and also with some other "border" Θ-classes $F_1, F_2, \ldots, F_s$, which are incident (but not lie) in the hypercubes from U. Consider an arbitrary clique $\mathcal{K}$ in $\Theta(G')$. If $\mathcal{K}$ does not contain vertex E, then the family of maximal hypercubes which enjoy the assertion of the theorem appears also in G. Note this family does not contain any hypercube from U (otherwise E would be incident to the intersection of these hypercubes, and hence it would be in $\mathcal{K}$). Thus this family of hypercubes is the same in G' as in G, and so it also enjoys the assertion of the theorem in G'.

Let us now assume that the clique $\mathcal{K}$ contains E. Hence all its vertices are Θ-classes from $\{E_1, E_2, \ldots, E_r, F_1, F_2, \ldots, F_s, E\}$. Without loss of generality we may denote the vertices of $\mathcal{K}$ by $E_1 \ldots, E_k, F_1, \ldots, F_m, E$, where $0 \leq m \leq s$ and $k \leq r$. If $m = 0$, then $\mathcal{K}$ is clearly induced by $\{E_1, E_2, \ldots, E_r, E\}$. In this case hypercubes from U that are all expanded in G' form the family of maximal hypercubes, and the new intersection is obtained as expanded intersection of expanded hypercubes (that is nonempty by definition of p-expansion graphs). Clearly all vertices of the intersection are incident with all Θ-classes from $\mathcal{K}$.

In the remaining case ($m > 0$) note that vertices $E_1, \ldots, E_k, F_1, \ldots, F_m$ formed a clique already in $\Theta(G)$. Hence by induction there is a family of maximal hypercubes $H_1, \ldots, H_\ell$ in G such that every vertex from $H_1 \cap \ldots \cap H_\ell$ is incident with an edge from any of the Θ-classes $E_1 \ldots, E_k, F_1, \ldots, F_m$. Since every hypercube that is incident with $H_1 \cap \ldots \cap H_\ell$ must in fact be one of $H_1, \ldots, H_\ell$ (because the clique is maximal complete subgraph), we derive that at least one of these hypercubes is from U. Indeed, otherwise $H_1 \cap \ldots \cap H_\ell$ would not be incident with $H'_1 \cup H'_2 \cup \ldots \cup H'_p$, but then it would be impossible that E is in $\mathcal{K}$, which is a contradiction.

Denote by $H_1, \ldots H_t$, where $t \leq \ell$, the hypercubes that are from U. Obviously hypercubes $H_{t+1} \ldots, H_\ell$ are in the "border" – incident with hypercubes from U, and $H_1 \cap \ldots \cap H_\ell$ is also in U. Now in G' hypercubes $H_1, \ldots, H_t$ become by one dimension larger. The vertices in the intersection of hypercubes $H_1, \ldots, H_\ell$ in G' are incident with all Θ-classes as in G and to E. The proof is complete. □

Corollary 5 *For a p-expansion graph G,*

$$\omega(\Theta(G)) = \Delta(G).$$

Proof. Clearly $\omega(\Theta(G)) \geq \Delta(G)$. Set $r = \omega(\Theta(G))$. Let K_r be the largest clique in $\Theta(G)$. By Lemma 4 there exist vertices that are incident to all Θ-classes of K_r. Hence, $\Delta(G) \geq r$. □

Figure 1 shows that the above assertion does not hold for all tree-like partial cubes. In the graph G on this figure $\Theta(G) = K_6$, hence $\omega(\Theta(G)) = 6$, while $\Delta(G) = 5$.

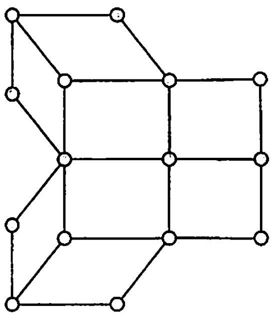

Figure 1: Tree-like partial cube

Recall that a graph is called *chordal* if it does not have any induced cycle of length greater than 3. Note that chordal graphs are perfect which implies $\chi(G) = \omega(G)$ for every chordal graph G, cf. [11].

Theorem 6 *Let G be a p-expansion graph. Then $\Theta(G)$ is a chordal graph, and so*

$$s'(G) \leq 2\omega(\Theta(G)) = 2\Delta(G).$$

Proof. Let G be a p-expansion graph. In the proof we will use the result from [4] that the cube graph $Q(G)$ of a p-expansion graph G is always chordal. Assume $\Theta(G)$ is not a chordal graph, and let C be an induced cycle in $\Theta(G)$ of length k, where $k > 3$. Let $E_1, \ldots, E_k$ be the Θ-classes corresponding to this cycle, such that $E_iE_{i+1} \in E(\Theta(G))$ for all $i = 1, \ldots, n-1$, $E_1E_k \in E(\Theta(G))$, and all other Θ-classes E_i and E_j of this cycle are not adjacent.

Let H_i and H_j be maximal hypercubes such that H_i contains some edges of E_i, and H_j contains some edges of E_j, where E_i and E_j are nonadjacent Θ-classes. Clearly by the structure of Θ-classes in hypercubes, every vertex of H_i is incident to an edge of E_i and similarly every vertex of H_j is incident to an edge of E_j. Thus hypercubes H_i and H_j have no vertex in common, and are nonadjacent in $Q(G)$.

Consider Θ-classes E_1 and E_2 from the cycle C. Since they are adjacent, there exist edges $e_1 \in E_1$ and $e_2 \in E_2$ that are incident, but they may as well lie in some 4-cycle. Suppose first that they lie in some 4-cycle. Denote by H_1 a maximal hypercube that contains this 4-cycle, and note that every vertex of H_1 is incident with edges from both Θ-classes. Hence the Θ-class E_3 is not incident with the hypercube H_1 (because E_1 and E_3 are not adjacent), yet it is incident with an edge from E_2. Denote by H_3 a maximal hyper cube that contains edges of E_3 and is incident or contains an edge from E_2. It is clear that H_1 and H_3 do not have a common vertex. It is possible that there exists a maximal hypercube H_2, that contains edges from E_2, and is incident with H_1 as well as with H_3. If not, then there is a sequence of maximal hypercubes which we denote by $H_2^1, \ldots H_2^{r_2}$ that all contain edges from E_2 and H_2^i has nonempty intersection with H_2^{i+1} for all $i = 1, \ldots r_2 - 1$, H_2^i has an empty intersection with H_2^j if $|i-j| > 1$, H_2^1 is the only hypercube among these that has a nonempty intersection with H_1 and $H_2^{r_2}$ is the only hypercube among these that has a nonempty intersection with H_3. In other words hypercubes $H_1, H_2^1, \ldots, H_2^{r_2}, H_3$ form an induced path in $Q(G)$. It is clear that the hypercubes $H_2^1, \ldots, H_2^{r^2}$ are all pairwise disjoint with any hypercube from $H_4, \ldots, H_k$, because they all posses edges from E_2. We started with the case that hypercubes E_1 and E_2 cross in some cycle. In the other case simply define H_1 as a maximal hypercube that contains e_1 and H_2 a maximal hypercube that contains e_2. (Note that if $H_1 = H_2$ then we are in the previous case when Θ-classes cross.)

We continue this way so that for each Θ-class E_i we either define a maximal hypercube H_i that contains edges from E_i, or, if necessary, more maximal hypercubes $H_i^1, \ldots H_i^{r_i}$ that all contain edges from E_i and that they form an induced path in $Q(G)$ together with E_{i-1} on one side and E_{i+1} (or E_1 if $i = n$) on the other side. It is easy to see that these hypercubes together form an induced cycle in $Q(G)$ that is of length at least k. This is a contradiction with the fact that $Q(G)$ is chordal. The proof is complete. $\square$

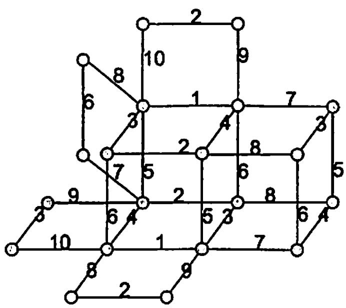

Figure 2: Median graph with a strong edge coloring

At first we suspected that a similar result can be proved for median graphs. In fact it was posed as a problem in [2] whether $\chi(\Theta(G)) = \Delta(G)$ for every median graph G. However, the answer is negative as one can see in the example G depicted in Figure 2. For this graph $\Theta(G)$ is the join of K_3 and C_5, which clearly has the chromatic number 6, while $\Delta(G) = 5$.

On the other hand, the strong edge coloring of this graph (see Figure 2) shows $s'(G) \leq 10$, which is thus not greater than $2\Delta(G)$. (Of course this coloring is different from the "canonical coloring" from the proof of Theorem 6, where in each Θ-class there are only two colors.) The question remains whether there is a median graph such that $s'(G) > 2\Delta(G)$.

4 Concluding remarks

1. The concept of the Θ-graph of a partial cube seems to be interesting in its own right. For instance in [13, 14] some properties of partial cubes in relation with Θ-graphs have been established. In particular, partial cubes whose Θ-graphs are 2-connected, trees, and complete graphs, respectively, have been characterized.

In [1] the so-called graphs of acyclic cubical complexes (ACC graphs for short) were introduced as the graphs that can be obtained by successive use of peripheral expansions from K_1 such that in each step a hypercube is expanded. It has been proved that the cube graph $Q(G)$ of an ACC graph G is a dually chordal graph [1], and conversely, every dually chordal graph is the cube graph of some ACC graph [4]. In Theorem 6 we proved that Θ-graph of a p-expansion graph is chordal, by using a result from [4]. However, it is not true that every chordal graph can be obtained as the Θ-graph of some p-expansion graph, since in [13] it was shown that among trees only paths can be obtained as Θ-graphs of partial cubes.

We pose a related problem concerning ACC graphs and dually chordal graphs.

Problem 1 *Let G be ACC graph. Is $\Theta(G)$ a dually chordal graph?*

2. We found a strong edge coloring for p-expansion graphs that is bounded by a linear function in Δ. The natural question is whether there are any other such interesting previously known classes of (bipartite) graphs. Related question, concerning the structure of the tree-like partial cubes, is also natural.

Problem 2 *Is there some other class of tree-like partial cubes such that $\chi(\Theta(G)) = \omega(\Theta(G))$ and/or $\omega(\Theta(G)) = \Delta(G)$?*

A partial answer can be obtained by the following observations. (Let $A \oplus B$ denote the join graphs A and B, that is, the graph obtained from the disjoint union of A and B by joining every vertex of A with every vertex of B by an edge.)

Lemma 7 *[14] Let G and H be partial cube. Then*

$$\Theta\left(G \square H\right) = \Theta\left(G\right) \oplus \Theta\left(H\right).$$

Lemma 8 *Let G and H be partial cubes such that $\omega\left(\Theta(G)\right) = \Delta(G)$ and $\omega\left(\Theta(H)\right) = \Delta(H)$, then*

$$\omega\left(\Theta(G \square H)\right) = \Delta\left(G \square H\right).$$

Proof. This assertion follows from the above formula $\Theta\left(G \square H\right) = \Theta\left(G\right) \oplus \Theta\left(H\right)$. Indeed, we get $\omega\left(\Theta(G \square H)\right) = \omega\left(\Theta(G)\right) + \omega\left(\Theta(H)\right)$ and by definition of the Cartesian product $\Delta\left(G \square H\right) = \Delta(G) + \Delta(H)$. □

Combining the lemma, the fact that tree-like partial cubes are preserved by Cartesian multiplication [5, Corollary 3.4], and results of this paper we derive

Proposition 9 *Let G and H be tree-like partial cubes such that $\chi\left(\Theta\left(G\right)\right) = \omega\left(\Theta(G)\right) = \Delta(G)$ and $\chi\left(\Theta\left(H\right)\right) = \omega\left(\Theta(H)\right) = \Delta(H)$. Then*

$$s'(G \square H) \leq 2\Delta\left(G \square H\right).$$

Hence we can find new families of graphs with the property $s'(G) \leq 2\Delta(G)$ from the existing such families.

3. We pose the question from Section 3 about the strong chromatic index of median graphs as follows.

Problem 3 *Is $s'(G) \leq 2\Delta(G)$ for every median graph G?*

If the question has a negative answer, one can ask if there is some other constant c such that the strong chromatic index of median graphs is bounded above by $c\Delta$.

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