Super Edge-Magic Labelings of Book Graphs B_n *

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Abstract

A graph G is called super edge-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv) = C$ is a constant for any $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. R. M Figueroa-Centeno et al. provided the following conjecture: For every integer $n \geq 5$, the book B_n is super edge-magic if and only if n is even or $n \equiv 5 (\mathrm{mod}\ 8)$. In this paper, we show that B_n is super edge-magic for even $n \geq 6$.

Keywords: *book graph, super edge-magic labeling, vertex labeling, edge labeling*

1 Introduction

Let $G = (V, E)$ be a simple graph with $p(G) = |V|$ vertices and $q(G) = |E|$ edges, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p + q\}$ is

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called an edge-magic labeling of G if there exists a constant C, called the valence of f, such that $f(u) + f(v) + f(uv) = C$ for any edge $uv \in E(G)$. An edge-magic labeling f of G is called a super edge-magic labeling if $f(V(G)) = \{1, 2, \ldots, p\}$ and $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$. We say that G is super edge-magic if there exists a super edge-magic labeling of G.

Kotzig and Rosa [1] introduced the notion of edge-magic labelings (in [1], edge-magic labelings are called magic valuations). They proved that complete bipartite graphs, cycles and caterpillars are edge-magic, and that the complete graph K_n is edge-magic if and only if $n = 1, 2, 3, 5$ or 6. They also conjectured that trees are edge-magic. Enomoto et al. [2] introduced the notion of super edge-magic labelings. They proved that the cycle C_n is super edge-magic if and only if n is odd, the complete bipartite graph $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$, and the complete graph K_n is super edge-magic if and only if $n = 1, 2$ or 3. They also conjectured that trees are super edge-magic. In addition, they proved that if $n \equiv 0 (\mathrm{mod}\ 4)$, then the wheel graph W_n of order n is not edge-magic.

For the literature on super edge-magic graphs we refer to [3] and the relevant references given in it.

Let n be a positive integers, the book graph B_n is defined by

$$V(B_n) = \{v_i \mid 1 \le i \le 2n+2\},$$
$$E(B_n) = \{v_1 v_i, v_i v_{i+n+1}, v_{n+2} v_{i+n+1} \mid 2 \le i \le n+1\} \cup \{v_1 v_{n+2}\}.$$

In [4], R.M Figueroa-Centeno et al. proved that B_n is not super edge-magic for $n \equiv 1, 3, 7 (\bmod 8)$. They show that B_n is super edge-magic for $n = 2, 5$ and B_4 is not super edge-magic. They conjectured: For every integer $n \ge 5$, the book B_n is super edge-magic if and only if n is even or $n \equiv 5 (\bmod 8)$. In this paper, we show that B_n is super edge-magic for even $n \ge 6$.

2 Main Result

Theorem 2.1 The book B_n is super edge magic for every even $n \ge 6$.

Proof. Let $C = (11n + 12)/2$. We define a function:

$$f : V(B_n) \cup E(B_n) \to \{1, 2, \ldots, 5n+3\}$$

according to the following two cases:
Case 1: $n \equiv 0 \bmod 4$, say $n = 4k$, then $p = 8k + 2$, $p + q = 20k + 3$,

$C = 22k + 6$.

Case 1.1. For $n = 8$, we give the total labeling of B_8 shown in Figure 2.1.

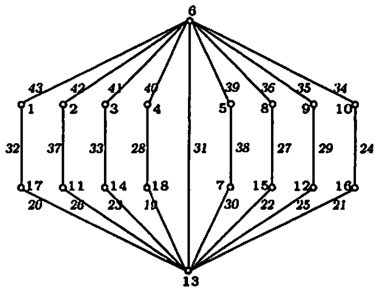

Figure 2.1: A super edge-magic labeling of the graph B_8.

According to the definition of super edge-magic labeling, it is clear that this assignment provides a super edge-magic labeling for B_8.

Case 1.2. For $n \geq 12$. We label the vertices as follows:

$$f(v_i) = \begin{cases} 2k+2, & i = 1, \\ i-1, & 2 \leq i \leq 2k+2, \\ i+1, & 2k+3 \leq i \leq 4k+1, \\ 6k+1, & i = 4k+2, \\ 8k+1, & i = 4k+3, \\ 4k+3, & i = 4k+4, \\ 16k-2i+8, & 4k+5 \leq i \leq 5k+3, \\ 8k+2, & i = 5k+4, \\ 16k-2i+9, & 5k+5 \leq i \leq 6k+2, \\ 2k+3, & i = 6k+3, \\ 20k-2i+7, & 6k+4 \leq i \leq 7k+2, \\ 20k-2i+6, & 7k+3 \leq i \leq 8k+1, \\ 8k, & i = 8k+2. \end{cases}$$

And the edges as follows:

$$f(v_i v_j) = C - f(v_i) - f(v_j).$$

Firstly, we show that f is a bijective mapping from $V(G)$ onto $\{1, 2, \ldots, 8k+2\}$.

Denote by

$$S = \{f(v_i) \mid 1 \leq i \leq 8k+2\}.$$

Then,

$$\begin{aligned}
S_1 &= \{2k+2 \mid i=1\} = \{2k+2\},\\
S_2 &= \{i-1 \mid 2 \le i \le 2k+2\} = \{1,2,\dots,2k+1\},\\
S_3 &= \{i+1 \mid 2k+3 \le i \le 4k+1\} = \{2k+4, 2k+5, \dots, 4k+2\},\\
S_4 &= \{6k+1 \mid i = 4k+2\} = \{6k+1\},\\
S_5 &= \{8k+1 \mid i = 4k+3\} = \{8k+1\},\\
S_6 &= \{4k+3 \mid i = 4k+4\} = \{4k+3\},\\
S_7 &= \{16k-2i+8 \mid 4k+5 \le i \le 5k+3\} = \{8k-2, 8k-4, \dots, 6k+2\},\\
S_8 &= \{8k+2 \mid i = 5k+4\} = \{8k+2\},\\
S_9 &= \{16k-2i+9 \mid 5k+5 \le i \le 6k+2\} = \{6k-1, 6k-3, \dots, 4k+5\},\\
S_{10} &= \{2k+3 \mid i = 6k+3\} = \{2k+3\},\\
S_{11} &= \{20k-2i+7 \mid 6k+4 \le i \le 7k+2\} = \{8k-1, 8k-3, \dots, 6k+3\},\\
S_{12} &= \{20k-2i+6 \mid 7k+3 \le i \le 8k+1\} = \{6k, 6k-2, \dots, 4k+4\},\\
S_{13} &= \{8k \mid i = 8k+2\} = \{8k\}.
\end{aligned}$$

Hence, $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12} \cup S_{13}$ is the set of labels of all vertices, and

$$\begin{aligned}
S = {} & S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12} \cup S_{13}\\
= {} & S_8 \cup S_5 \cup S_{13} \cup S_{11} \cup S_7 \cup S_4 \cup S_{12} \cup S_9 \cup S_6 \cup S_3 \cup S_{10} \cup S_1 \cup S_2\\
= {} & \{8k+2,\ 8k+1,\ 8k,\ 8k-1, 8k-3, \dots, 6k+3,\ 8k-2, 8k-4, \dots,\\
& 6k+2,\ 6k+1,\ 6k, 6k-2, \dots, 4k+4,\ 6k-1, 6k-3, \dots, 4k+5,\\
& 4k+3, 4k+2, 4k+1, \dots, 2k+4,\ 2k+3,\ 2k+2,\ 2k+1, 2k, \dots, 1\}\\
= {} & \{8k+2, 8k+1, \dots, 2, 1\}.
\end{aligned}$$

It is clear that the labels of the vertices are different. So, f is a bijection from $V(G)$ onto $\{1, 2, \dots, p\}$.

Secondly, we show that f is a bijective mapping from $E(G)$ onto $\{8k+3, 8k+4, \dots, 20k+3\}$. Denote by

$$\begin{aligned}
D &= \{f(v_iv_j) \mid v_iv_j \in E(B_n)\}\\
&= \{C - f(v_i) - f(v_j) \mid v_i, v_j \in V(B_n)\}.
\end{aligned}$$

Let $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where

$$\begin{aligned}
D_1 &= \{f(v_1v_i) \mid 2 \le i \le 4k+1\} = D_{11} \cup D_{12},\\
D_{11} &= \{f(v_1v_i) \mid 2 \le i \le 2k+2\}\\
&= \{22k+6-2k-i-1 \mid 2 \le i \le 2k+2\}\\
&= \{18k+3, 18k+4, \dots, 20k+3\},\\
D_{12} &= \{f(v_1v_i) \mid 2k+3 \le i \le 4k+1\}\\
&= \{22k+6-2k-i-3 \mid 2k+3 \le i \le 4k+1\}\\
&= \{16k+2, 16k+3, \dots, 18k\},
\end{aligned}$$

$$
\begin{aligned}
D_2 &= \{f(v_iv_{i+4k+1}) \mid 2 \le i \le 4k+1\} = D_{21} \cup D_{22} \cup D_{23} \cup D_{24} \cup \\
&\quad D_{25} \cup D_{26} \cup D_{27} \cup D_{28} \cup D_{29}, \\
D_{21} &= \{f(v_iv_{i+4k+1}) \mid i = 2\} = \{22k+6-8k-i\} = \{14k+4\}, \\
D_{22} &= \{f(v_iv_{i+4k+1}) \mid i = 3\} = \{22k+6-4k-i-2\} = \{18k+1\}, \\
D_{23} &= \{f(v_iv_{i+4k+1}) \mid 4 \le i \le k+2\} \\
&= \{22k+6-8k+i-5 \mid 4 \le i \le k+2\} \\
&= \{14k+5, 14k+6, \ldots, 15k+3\}, \\
D_{24} &= \{f(v_iv_{i+4k+1}) \mid i = k+3\} \\
&= \{22k+6-8k-i-1 \mid i = k+3\} = \{13k+2\}, \\
D_{25} &= \{f(v_iv_{i+4k+1}) \mid k+4 \le i \le 2k+1\} \\
&= \{22k+6-8k+i-6 \mid k+4 \le i \le 2k+1\} \\
&= \{15k+4, 15k+5, \ldots, 16k+1\}, \\
D_{26} &= \{f(v_iv_{i+4k+1}) \mid i = 2k+2\} \\
&= \{22k+6-2k-i-2\} = \{18k+2\}, \\
D_{27} &= \{f(v_iv_{i+4k+1}) \mid 2k+3 \le i \le 3k+1\} \\
&= \{22k+6-12k+i-6 \mid 2k+3 \le i \le 3k+1\} \\
&= \{12k+3, 12k+4, \ldots, 13k+1\}, \\
D_{28} &= \{f(v_iv_{i+4k+1}) \mid 3k+2 \le i \le 4k\} \\
&= \{22k+6-12k+i-5 \mid 3k+2 \le i \le 4k\} \\
&= \{13k+3, 13k+4, \ldots, 14k+1\}, \\
D_{29} &= \{f(v_iv_{i+4k+1}) \mid i = 4k+1\} \\
&= \{22k+6-8k-i-1 \mid i = 4k+1\} = \{10k+4\}, \\
D_3 &= \{f(v_{4k+2}v_i) \mid 4k+3 \le i \le 8k+2\} = D_{31} \cup D_{32} \cup D_{33} \cup D_{34} \cup \\
&\quad D_{35} \cup D_{36} \cup D_{37} \cup D_{38} \cup D_{39}, \\
D_{31} &= \{f(v_{4k+2}v_i) \mid i = 4k+3\} \\
&= \{22k+6-14k-2 \mid i = 4k+3\} = \{8k+4\}, \\
D_{32} &= \{f(v_{4k+2}v_i) \mid i = 4k+4\} \\
&= \{22k+6-10k-4 \mid i = 4k+4\} = \{12k+2\}, \\
D_{33} &= \{f(v_{4k+2}v_i) \mid 4k+5 \le i \le 5k+3\} \\
&= \{22k+6-22k+2i-9 \mid 4k+5 \le i \le 5k+3\} \\
&= \{8k+7, 8k+9, \ldots, 10k+3\}, \\
D_{34} &= \{f(v_{4k+2}v_i) \mid i = 5k+4\} \\
&= \{22k+6-14k-3 \mid i = 5k+4\} = \{8k+3\}, \\
D_{35} &= \{f(v_{4k+2}v_i) \mid 5k+5 \le i \le 6k+2\} \\
&= \{22k+6-22k+2i-10 \mid 5k+5 \le i \le 6k+2\} \\
&= \{10k+6, 10k+8, \ldots, 12k\}, \\
D_{36} &= \{f(v_{4k+2}v_i) \mid i = 6k+3\} \\
&= \{22k+6-8k-4 \mid i = 6k+3\} = \{14k+2\}, \\
D_{37} &= \{f(v_{4k+2}v_i) \mid 6k+4 \le i \le 7k+2\} \\
&= \{22k+6-26k+2i-8 \mid 6k+4 \le i \le 7k+2\} \\
&= \{8k+6, 8k+8, \ldots, 10k+2\}, \\
D_{38} &= \{f(v_{4k+2}v_i) \mid 7k+3 \le i \le 8k+1\} \\
&= \{22k+6-26k+2i-7 \mid 7k+3 \le i \le 8k+1\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{10k+5, 10k+7, \ldots, 12k+1\},\\
D_{39} &= \{f(v_{4k+2}v_i) \mid i = 8k+2\}\\
&= \{22k+6-14k-1 \mid i = 8k+2\} = \{8k+5\},\\
D_4 &= \{f(v_1 v_{4k+2})\} = \{22k+6-8k-3\} = \{14k+3\}.
\end{aligned}
$$

Hence, $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is the set of labels of all edges, and

$$
\begin{aligned}
D = & \; D_1 \cup D_2 \cup D_3 \cup D_4\\
= & \; D_{11} \cup D_{12} \cup D_{21} \cup D_{22} \cup D_{23} \cup D_{24} \cup D_{25} \cup D_{26} \cup D_{27} \cup D_{28} \cup\\
& \; D_{29} \cup D_{31} \cup D_{32} \cup D_{33} \cup D_{34} \cup D_{35} \cup D_{36} \cup D_{37} \cup D_{38} \cup D_{39} \cup D_4\\
= & \; D_{34} \cup D_{31} \cup D_{39} \cup D_{37} \cup D_{33} \cup D_{29} \cup D_{38} \cup D_{35} \cup D_{32} \cup D_{27} \cup\\
& \; D_{24} \cup D_{28} \cup D_{36} \cup D_4 \cup D_{21} \cup D_{23} \cup D_{25} \cup D_{12} \cup D_{22} \cup D_{26} \cup D_{11}\\
= & \; \{8k+3,\ 8k+4,\ 8k+5,\ 8k+6, 8k+8, \ldots, 10k+2,\ 8k+7, 8k+\\
& \; 9, \ldots, 10k+3, 10k+4,\ 10k+5, 10k+7, \ldots, 12k+1,\ 10k+6, 10k\\
& \; +8, \ldots, 12k,\ 12k+2,\ 12k+3, 12k+4, \ldots, 13k+1,\ 13k+2,\ 13k\\
& \; +3, 13k+4, \ldots, 14k+1,\ 14k+2,\ 14k+3,\ 14k+4,\ 14k+5, 14k\\
& \; +6, \ldots, 15k+3,\ 15k+4, 15k+5, \ldots, 16k+1,\ 16k+2, 16k+3,\\
& \; \ldots, 18k,\ 18k+1,\ 18k+2,\ 18k+3, 18k+4, \ldots, 20k+3\}\\
= & \; \{8k+3, 8k+4, \ldots, 20k+3\}.
\end{aligned}
$$

It is clear that the labels of each edge are distinct, and the edge labels are $\{p+1, p+2, \ldots, p+q\}$. According to the definition of super edge-magic labeling, we thus conclude that the graph B_n is super edge-magic for $n \equiv 0 \bmod 4$.

Case 2: $n \equiv 2 \bmod 4$, say $n = 4k+2$, then $p = 8k+6$, $p+q = 20k+13$, $C = 22k+17$.

Case 2.1. For $n = 6$, we give the total labeling of B_6 shown in Figure 2.2.

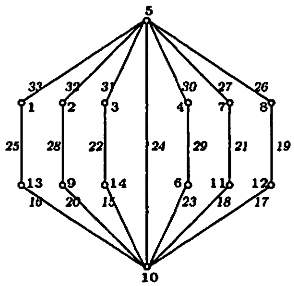

Figure 2.2: A super edge-magic labeling of the graph B_6.

According to the definition of super edge-magic labeling, it is clear that this assignment provides a super edge-magic labeling for B_6.

Case 2.2. For $n \geq 10$. We label the vertices as follows:

$$f(v_i) = \begin{cases} 2k+3, & i=1, \\ i-1, & 2 \leq i \leq 2k+3, \\ i+1, & 2k+4 \leq i \leq 4k+3, \\ 6k+4, & i=4k+4, \\ 8k+5, & i=4k+5, \\ 4k+5, & i=4k+6, \\ 16k-2i+16, & 4k+7 \leq i \leq 5k+5, \\ 8k+6, & i=5k+6, \\ 16k-2i+17, & 5k+7 \leq i \leq 6k+5, \\ 2k+4, & i=6k+6, \\ 20k-2i+17, & 6k+7 \leq i \leq 7k+6, \\ 20k-2i+16, & 7k+7 \leq i \leq 8k+5, \\ 8k+4, & i=8k+6. \end{cases}$$

And the edges as follows:

$$f(v_iv_j) = C - f(v_i) - f(v_j).$$

Similar to the proof in Case 1, we have that this assignment provides a super edge-magic labeling for $n \equiv 2 \bmod 4$.

According to the proof of Case 1 and Case 2, we thus conclude that B_n are super edge-magic for even $n \geq 6$. □

In Figure 2.3 and 2.4, we show our super edge-magic labelings for B_{12} and B_{14}.

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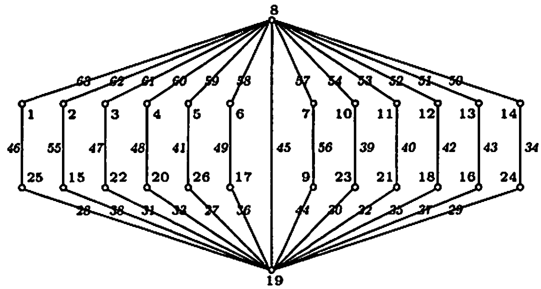

Figure 2.3: The super edge-magic labeling of the graph B_{12}.

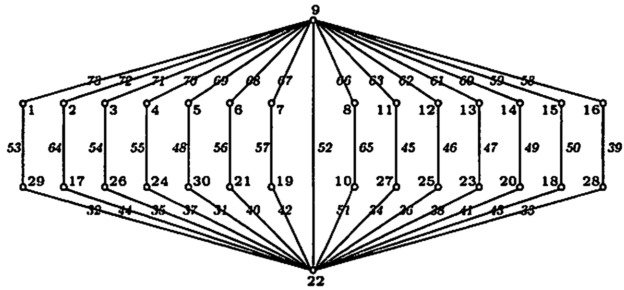

Figure 2.4: The super edge-magic labeling of the graph B_{14}.