Block-Transitive 2-$(v, k, 1)$ Designs and the Groups $E_7(q)$

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Abstract: This article is a contribution to the study of block-transitive automorphism groups of 2-$(v, k, 1)$ block designs. Let $\mathcal{D}$ be a 2-$(v, k, 1)$ design admitting a block-transitive, point-primitive but not flag-transitive group G of automorphisms. Let $k_r = (k, v-1)$ and $q = p^f$ for prime p. In this paper we prove that if G and $\mathcal{D}$ are as above and $q > (2(k_r k - k_r + 1)f)^{1/4}$ then G does not admit a Chevalley group $E_7(q)$ as its socle.
Keywords: *block design; block-transitive; point-primitive; automorphism group*

1 Introduction

A 2-$(v, k, 1)$ design $\mathcal{D}$=($\mathcal{P}$, $\mathcal{B}$) is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of k-subsets of $\mathcal{P}$, called blocks, such that each 2-subset of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq \mathrm{Aut}(\mathcal{D})$ be a group of automorphisms of a $2-(v, k, 1)$ design $\mathcal{D}$. Then G is said to be *block-transitive* if G is transitive on $\mathcal{B}$, and is said to be *point- transitive* (*point-primitive*) if G is transitive (primitive) on $\mathcal{P}$. A *flag* of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is *flag-transitive* if G is transitive on the set of flags.

In 1990, a six-person team [4] classified the pairs $(G, \mathcal{D})$ where G is a flag-transitive automorphism group of $\mathcal{D}$, with the exception of those in

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which G is a one-dimensional affine group. In this paper we contribute to the classification of designs which have an automorphism group transitive on blocks. It follows from a result of Block [2] that a block-transitive automorphism group of a 2-$(v,k,1)$ design is transitive on points. In [7] it is shown that the study of block-transitive 2-$(v,k,1)$ designs can be reduced to three cases, distinguishable by properties of the action of G on the point set $\mathcal{P}$: that in which G is of affine type in the sense that it has an elementary abelian transitive normal subgroup; that in which G is almost simple, in the sense that G has a simple nonabelian transitive normal subgroup T whose centralizer is trivial, so that $T \trianglelefteq G \leq AutT$; and that in which G has an intransitive minimal normal subgroup. Much work is needed to achieve this classification, see [5], [7], and [9]. W. Liu et al have studied the special case where $G = T := \mathrm{Soc}(G)$ is any finite group of Lie type of Lie rank 1 acting block-transitively on a design in [15]-[18]. Here we focus on the second case, that is classifying 2-$(v,k,1)$ designs with a block-transitive automorphism group of almost simple type under the conditions that G is point-primitive but not flag-transitive. We prove the following Main Theorem.

Main Theorem. *Let $\mathcal{D}$ be a 2-$(v,k,1)$ design admitting a block-transitive, point-primitive but not flag-transitive automorphism group G. Let $k_r = (k, v-1)$, $q = p^f$ for some prime p and positive integer f. If $q > (2(k_rk - k_r + 1)f)^{1/4}$ then* $\mathrm{Soc}(G) \ncong E_7(q)$.

The assumption $q > (2(k_rk - k_r + 1)f)^{1/4}$ is necessary for the proof of the Main Theorem. Our proof depends on the result of Liebeck and Saxl [14] about the classification of maximal subgroups of $T = \mathrm{Soc}(G)$, and the properties of the lengths of the suborbits of T, given in Section 3. We shall continue this work in a forthcoming paper dealing with q small and using different methods. Recently, the first author treated the case that $T \cong E_8(q)$ using a method similar to the one in this article.

Our paper is organized as follows: In Section 2 we collect some preliminary results and in Section 3 we use them to prove the Main Theorem.

2 Preliminary Results

Let $\mathcal{D}$ be a 2-$(v,k,1)$ design defined on the point set $\mathcal{P}$, and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a 2-$(v,k,1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. Also, G^B denotes the permutation group induced by the action of G_B on the points of B, and so $G^B \cong G_B/G_{(B)}$.

For a set X, we define $X^{(2)} = \{(x,y) | x \neq y \in X\}$.

Lemma 2.1 (Li [13]). *Let $\mathcal{D}$ and G be as above. If $\psi_1, \cdots, \psi_s$ are the orbits of G_B on the set $B^{(2)}$ and $\Psi_1, \cdots, \Psi_t$ are the orbits of G on $\mathcal{P}^{(2)}$, then the map σ, which maps ψ_i to Ψ_j if $\psi_i \subseteq \Psi_j$, is a bijection between $\{\psi_i | i = 1, 2, \cdots, s\}$ and $\{\Psi_j | j = 1, 2, \cdots, t\}$, and so in particular $s = t$. Moreover, the rank of G is $s+1$ and if $\psi_i^\sigma = \Psi_i$ then $|\Psi_i| = b|\psi_i|$.*

We will use the following Fang-Li parameters of 2-$(v,k,1)$ designs, introduced by Fang-Li (see [11]):

$$k_v = (k, v),\ k_r = (k, r) = (k, v-1),\ b_v = (b, v),\ \ b_r = (b, r) = (b, v-1).$$

It is easy to check that

$$k = k_v k_r,\ b = b_v b_r,\ v = k_v b_v \text{ and } r = k_r b_r.$$

Corollary 2.2. *Let $\mathcal{D}$ and G be as in Lemma 2.1, let $\alpha \in \mathcal{P}$, and let Γ be a G_α-orbit in $\mathcal{P} \setminus \{\alpha\}$. Then $b_r \| \Gamma|$.*
Proof. Let $\beta \in \Gamma$. If $(\alpha, \beta) \in \Psi_i$ and if $\sigma : \psi_i \to \Psi_i$ is as in Lemma 2.1, then

$$|\Psi_i| = |G : G_{\alpha\beta}| = |G : G_\alpha||G_\alpha : G_{\alpha\beta}| = v|\Gamma|.$$

Therefore, $v|\Gamma| = b|\psi_i|$, i.e. $k_v b_v |\Gamma| = b_v b_r |\psi_i|$. Thus $k_v |\Gamma| = b_r |\psi_i|$. It follows that $b_r \| \Gamma|$ since $(k_v, b_r) = 1$. □

Lemma 2.3. *Let G be a transitive group on the point set $\mathcal{P}$ and $T :=$ Soc(G). Let $\alpha \in \mathcal{P}$ and let Γ be a G_α-orbit in $\mathcal{P} \setminus \{\alpha\}$. Then Γ is a union of orbits of T_α, all having the same size.*
Proof. Let Γ be an orbit of G_α. Then Γ is invariant under T_α and so $\Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \Delta_s$, where $\Delta_i (i = 1, 2, \cdots, s)$ are orbits of T_α. Suppose that $\beta \in \Delta_1$ and $\gamma \in \Delta_2$. Then there exists an element g in G_α such that $\gamma = \beta^g$, and so

$$|\Delta_2| = |\gamma^{T_\alpha}| = |\beta^{gT_\alpha}| = |\beta^{T_\alpha g}| = |\beta^{T_\alpha}| = |\Delta_1|.$$

□

Lemma 2.4 (Liebeck and Saxl [14]). *Suppose that $T :=$ Soc$(G) \cong E_7(q)$ is a simple exceptional group of Lie type over $GF(q)$, where $q = p^f$ for a prime p and positive integer f. Let M be a maximal subgroup of G not containing T, then one of the following holds:*

(a) $|M| < q^{64}|G : T|$;

(b) $T \cap M$ *is a parabolic subgroup of* T;

(c) $T \cap M$ *is isomorphic to one of* (i) $(E_6(q) \circ (q-1)/d).e_{+1}.2$; (ii) $({}^2E_6(q) \circ (q+1)/d).e_{-1}.2$; (iii) $(SL_2(q) \circ D_6(q)).d$; *or* (iv) $E_7(q^{\frac{1}{2}}).d$. *In all cases $d = (2, q-1)$, and $e_\varepsilon = (q - \varepsilon, 3)$ for $\varepsilon = \pm 1$.*

Let W be the Weyl group associated with the simple group T of Lie type, N the monomial subgroup of T, and H the diagonal subgroup of T. From [10, Theorem 7.2.2], we know that there exists a homomorphism ϕ :

$N \to W$ such that $N/H \cong W$. Let Φ be the root system corresponding to T with fundamental system Π, also Φ^+ (Φ^-) be the set of positive (negative) roots in Φ, respectively. If J is a subset of the set Π of fundamental roots and V_J is the subspace of V spanned by J, then Φ_J denotes the set of roots of Φ lying in the subspace V_J. We use the standard labelling for Dynkin diagrams with fundamental roots α_i as in [3, pp.250-275].

For the basic notions and results of design theory and finite permutation groups, the reader is referred to [1] and [19]. We will follow the notation of [10] for simple groups of Lie type. Also, if n is a positive integer and p is a prime number, then $|n|_p$ denotes the p-part of n and $|n|_{p'}$ denotes the p'-part of n. In other words, $|n|_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$, and $|n|_{p'} = n/|n|_p$.

3 Proof of the Main Theorem

Let $\mathcal{D}$ and G satisfy the hypotheses of the Main Theorem. Assume by way of contradiction that $T := \mathrm{Soc}(G) \cong E_7(q)$, where $q = p^f > (2(k_r k - k_r + 1)f)^{1/4}$ and p is prime.

In any 2-$(v, k, 1)$ design with parameters b, v, k, r,

$$kb = vr$$

and

$$k(k-1)b = v(v-1).$$

Using the Fang-Li parameters, we have $v = 1 + k_r(k-1)b_r$.

Now let $T := \mathrm{Soc}(G)$ and $T_\alpha = T \cap G_\alpha$, where $\alpha \in \mathcal{P}$. Let Δ be any T_α-orbit in $\mathcal{P} \setminus \{\alpha\}$ with size x, and Γ a nontrivial suborbit of G_α such that $\Delta \subseteq \Gamma$. Since $\frac{|G|}{|G_\alpha|} = \frac{|T|}{|T_\alpha|}$, we have

$$|G:T| = |G_\alpha : T_\alpha|,$$

and

$$|\Gamma| = \frac{|G_\alpha|}{|G_{\alpha\beta}|} \leq \frac{|G_\alpha|}{|T_{\alpha\beta}|} = \frac{|T_\alpha|}{|T_{\alpha\beta}|}\frac{|G_\alpha|}{|T_\alpha|} = x|G:T|,$$

where $\beta \in \Delta$. Since $v = 1 + k_r(k-1)b_r$, we have $\frac{v}{b_r} < 1 + k_r(k-1)$. By Corollary 2.2 we have $b_r \| \Gamma|$, and $b_r \leq |\Gamma|$. Thus

$$\frac{v}{x|G:T|} \leq \frac{v}{|\Gamma|} \leq \frac{v}{b_r} < 1 + k_r(k-1),$$

and so we have the following property:

(P$_1$) $\dfrac{v}{x} < (k_r k - k_r + 1)|G:T|$, where x is the size of a T_α-orbit in $\mathcal{P} \setminus \{\alpha\}$.

Since T is not a Frobenius group (because a Frobenius group has a regular nilpotent normal subgroup), there exist α, $\beta \in \mathcal{P}$ such that $|T_{\alpha\beta}| \neq 1$. Then $\frac{v}{x} = \frac{|T|}{|T_\alpha|^2}|T_{\alpha\beta}| \geq 2\frac{|T|}{|T_\alpha|^2}$. Combining with ($P_1$), we get the following property:

$$(P_2) \quad \frac{|T|}{|T_\alpha|^2} < (k_r k - k_r + 1)|G : T|.$$

To prove the Main Theorem, we also need the following very useful property:

(P_3) If $(v-1, q) = 1$, then there exists in $\mathcal{P} \setminus \{\alpha\}$ a T_α-orbit of size y such that $y \mid |T_\alpha|_{p'}$.

In fact, let t be the size of any T_α-orbit in $\mathcal{P} \setminus \{\alpha\}$. Suppose to the contrary that $t \nmid |T_\alpha|_{p'}$. Since $t \mid |T_\alpha|$, we have $p \mid t$. Furthermore, since $\mathcal{P} \setminus \{\alpha\}$ is a union of T_α-orbits, $p \mid v-1$. Thus $p \mid (v-1, q)$, which contradicts $(v-1, q) = 1$.

Since G is primitive on $\mathcal{P}$, G_α is a maximal subgroup of G for any $\alpha \in \mathcal{P}$. Hence $M = G_\alpha$ satisfies one of the three cases in Lemma 2.4. We will rule out these cases one by one.

Case 1: $|M| < q^{64}|G : T|$.

By (P_2),

$$|T| < (k_r k - k_r + 1)|T_\alpha|^2|G : T| < (k_r k - k_r + 1)q^{128}|G : T|. \qquad (1)$$

Since $|E_7(q)| = q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)/d$, then

$$\begin{aligned} \frac{|T|}{q^{128}} &= \frac{(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)}{dq^{65}} \\ &> \frac{q^5 - q^3 - \frac{1}{q} - \frac{121}{q^7}}{d} > q^4 > (k_r k - k_r + 1)|G : T|, \end{aligned}$$

contradicting (1).

Case 2: $T \cap M$ is a parabolic subgroup of T .

Let $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ be the fundamental root system of $E_7(q)$, let $J_i = \Pi - \{\alpha_i\}$, and P_{J_i} be the parabolic subgroup of $E_7(q)$ determined by J_i.

The following TABLE I lists the order of T_α and the value of $v = |T|/|T_\alpha|$ in the corresponding subcases.

Subcase 2.1: $T_\alpha = P_{J_1}$. By [10, Theorem 7.2.2], there exists a homomorphism $\phi : N \to W$ such that $N/H \cong W$. Let $\phi(n_1) = w_{\alpha_1}$, where $n_1 \in N$, w_{α_1} is the corresponding reflection of α_1 in the Weyl group W. Now we consider $P_{J_1} \cap P_{J_1}^{n_1}$. Since $P_{J_1} = \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_1}\rangle$, then

$$\begin{aligned} P_{J_1}^{n_1} &= \langle X_r, H | r \in (\Phi^+)^{n_1} \cup (\Phi_{J_1})^{n_1}\rangle \\ &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_1\}) \cup \{-\alpha_1\} \cup \Phi_{w_{\alpha_1}(J_1)}\rangle. \end{aligned}$$

TABLE I. Subcasos of case 2

T_α	$\lvert T_\alpha \rvert$	v
P_{J_1}	$q^{63}(q-1)(q^2-1)(q^4-1)(q^6-1)^2(q^8-1)(q^{10}-1)/d$	$\frac{(q^{18}-1)(q^{14}-1)(q^6+1)}{(q-1)(q^4-1)}$
P_{J_2}	$q^{63}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)(q^7-1)/d$	$\frac{(q^{18}-1)(q^7+1)(q^{12}-1)(q^5+1)(q^4+1)}{(q-1)(q^3-1)}$
P_{J_3}	$q^{63}(q-1)(q^2-1)^2(q^3-1)(q^4-1)(q^5-1)(q^6-1)/d$	$\frac{(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)}$
P_{J_4}	$q^{63}(q-1)(q^2-1)^3(q^3-1)^2(q^4-1)/d$	$\frac{(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^2-1)^2(q^3-1)^2(q^4-1)}$
P_{J_5}	$q^{63}(q-1)(q^2-1)^2(q^3-1)^2(q^4-1)(q^5-1)/d$	$\frac{(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^2-1)(q^3-1)^2(q^4-1)(q^5-1)}$
P_{J_6}	$q^{63}(q-1)(q^2-1)^2(q^4-1)(q^5-1)(q^6-1)(q^8-1)/d$	$\frac{(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^2-1)(q^4-1)(q^5-1)}$
P_{J_7}	$q^{63}(q-1)(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)/d$	$\frac{(q^{10}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^5-1)(q^9-1)}$

It follows that

$$\langle X_r, H \mid r \in (\Phi^+ - \{\alpha_1\}) \cup \Phi_{J'} \rangle \leq P_{J_1} \cap P_{J_1}^{n_1},$$

where $J' = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. Let

$$\tilde{P} = \langle X_r, H \mid r \in (\Phi^+ - \{\alpha_1\}) \cup \Phi_{J'} \rangle \quad \text{and} \quad \tilde{U} = \prod_{r \in (\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}} X_r \leq U_{J'}.$$

We claim that $\tilde{U} \trianglelefteq \tilde{P}$. We show that the subgroups generating $\tilde{P}$ all normalize $\tilde{U}$. It is clear that H normalizes $\tilde{U}$. Let r be a positive root. If $s \in (\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}$, then all roots of the form $ir + js$ with $i > 0, j > 0$ are also in $(\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}$. Thus the commutator formula (see [10, Chapter 5]) shows that X_r normalizes $\tilde{U}$. Now suppose that $r \in \Phi^- \cap \Phi_{J'}$. Then $-r$ is not in $(\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}$, and, if s is any root in $(\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}$, all roots of the form $ir + js$ with $i > 0, j > 0$ are in $(\Phi^+ - \{\alpha_1\}) \cap \bar{\Phi}_{J'}$. Since $ir + js$ involves some fundamental root not in J' with a positive coefficient, X_r normalizes $\tilde{U}$ in this case. Hence $\tilde{U} \trianglelefteq \tilde{P}$. Now we define $L_{J'}$ to be the subgroup of G generated by H and the root subgroups X_r for all $r \in \Phi_{J'}$. Then we have

$$\tilde{P} = \tilde{U} L_{J'}, \quad |\tilde{P}| = \frac{1}{d} q^{62}(q-1)^2(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1).$$

Thus T_α has an orbit of size

$$x = \frac{|P_{J_1}|}{|P_{J_1} \cap P_{J_1}^{n_1}|} \leq \frac{|P_{J_1}|}{|\tilde{P}|} = \frac{q(q^3+1)(q^8-1)(q^5+1)}{(q-1)}.$$

Therefore

$$\frac{v}{x} > q^{28} > (k_r k - k_r + 1)|G : T|$$

where v is given in the first line of Table 1. This contradicts property (P_1).

Subcase 2.2: $T_\alpha = P_{J_2}$. Let n_2 be the inverse image of w_{α_2} under ϕ. Since

$$\begin{aligned} P_{J_2} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_2} \rangle, \\ P_{J_2}^{n_2} &= \langle X_r, H | r \in (\Phi^+)^{n_2} \cup (\Phi_{J_2})^{n_2} \rangle \\ &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_2\}) \cup \{-\alpha_2\} \cup \Phi_{w_{\alpha_2}(J_2)} \rangle. \end{aligned}$$

Then

$$P_{J_2} \cap P_{J_2}^{n_2} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_2\}) \cup \Phi_{J'} \rangle$$

where $J' = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$. Hence

$$|P_{J_2} \cap P_{J_2}^{n_2}| > \frac{1}{d}\, q^{62}(q-1)^2(q^2-1)^2(q^3-1)^2(q^4-1),$$

and T_α has an orbit of size

$$x = \frac{|P_{J_2}|}{|P_{J_2} \cap P_{J_2}^{n_2}|} \leq \frac{q(q^5-1)(q^6-1)(q^7-1)}{(q-1)(q^2-1)(q^3-1)}.$$

It follows that

$$\frac{v}{x} > q^{26} > (k_r k - k_r + 1)|G:T|,$$

contradicting (P_1).

Subcase 2.3: $T_\alpha = P_{J_3}$. Let n_3 be the inverse image of w_{α_3} under ϕ. Since

$$\begin{aligned} P_{J_3} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_3} \rangle, \\ P_{J_3}^{n_3} &= \langle X_r, H | r \in (\Phi^+)^{n_3} \cup (\Phi_{J_3})^{n_3} \rangle \\ &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_3\}) \cup \{-\alpha_3\} \cup \Phi_{w_{\alpha_3}(J_3)} \rangle, \end{aligned}$$

then $P_{J_3} \cap P_{J_3}^{n_3} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_3\}) \cup \Phi_{J'} \rangle$ where $J' = \{\alpha_2, \alpha_5, \alpha_6, \alpha_7\}$, and

$$|P_{J_3} \cap P_{J_3}^{n_3}| > \frac{1}{d}\, q^{62}(q-1)^3(q^2-1)^2(q^3-1)(q^4-1).$$

Thus T_α has an orbit of size

$$x = \frac{|P_{J_3}|}{|P_{J_3} \cap P_{J_3}^{n_3}|} \leq \frac{q(q^5-1)(q^6-1)}{(q-1)^2}.$$

It follows that

$$\frac{v}{x} > q^{35} > (k_r k - k_r + 1)|G:T|,$$

contradicting (P_1).

Subcase 2.4: $T_\alpha = P_{J_4}$. Let n_4 be the inverse image of w_{α_4} under ϕ. Since

$$\begin{aligned} P_{J_4} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_4} \rangle, \\ P_{J_4}^{n_4} &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_4\}) \cup \{-\alpha_4\} \cup \Phi_{w_{\alpha_4}(J_4)} \rangle, \end{aligned}$$

then $P_{J_4} \cap P_{J_4}^{n_4} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_4\}) \cup \Phi_{J'} \rangle$ where $J' = \{\alpha_1, \alpha_6, \alpha_7\}$, and

$$|P_{J_4} \cap P_{J_4}^{n_4}| > \frac{1}{d}\, q^{62}(q-1)^4(q^2-1)^2(q^3-1).$$

So T_α has an orbit of size

$$x = \frac{|P_{J_4}|}{|P_{J_4} \cap P_{J_4}^{n_4}|} \leq \frac{q(q^2-1)(q^3-1)(q^4-1)}{(q-1)^3}.$$

It follows that

$$\frac{v}{x} > q^{43} > (k_r k - k_r + 1)|G : T|,$$

contradicting (P_1).

Subcase 2.5: $T_\alpha = P_{J_5}$. Let n_5 be the inverse image of w_{α_5} under ϕ. Since

$$\begin{aligned} P_{J_5} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_5} \rangle, \\ P_{J_5}^{n_5} &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_5\}) \cup \{-\alpha_5\} \cup \Phi_{w_{\alpha_5}(J_5)} \rangle, \end{aligned}$$

then $P_{J_5} \cap P_{J_5}^{n_5} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_5\}) \cup \Phi_{J'} \rangle$ where $J' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$, and

$$|P_{J_5} \cap P_{J_5}^{n_5}| > \frac{1}{d}\, q^{62}(q-1)^3(q^2-1)^3(q^3-1).$$

So T_α has an orbit of size

$$x = \frac{|P_{J_5}|}{|P_{J_5} \cap P_{J_5}^{n_5}|} \leq \frac{q(q^3-1)(q^4-1)(q^5-1)}{(q-1)^2(q^2-1)}.$$

It follows that

$$\frac{v}{x} > q^{38} > (k_r k - k_r + 1)|G : T|,$$

contradicting (P_1).

Subcase 2.6: $T_\alpha = P_{J_6}$. Let n_6 be the inverse image of w_{α_6} under ϕ. Since

$$\begin{aligned} P_{J_6} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_6} \rangle, \\ P_{J_6}^{n_6} &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_6\}) \cup \{-\alpha_6\} \cup \Phi_{w_{\alpha_6}(J_6)} \rangle, \end{aligned}$$

then $P_{J_6} \cap P_{J_6}^{n_6} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_6\}) \cup \Phi_{J'} \rangle$, where $J' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Consequently,

$$|P_{J_6} \cap P_{J_6}^{n_6}| > \frac{1}{d}\, q^{62}(q-1)^3(q^2-1)(q^3-1)(q^4-1)(q^5-1).$$

Thus T_α has an orbit of size

$$x = \frac{|P_{J_6}|}{|P_{J_6} \cap P_{J_6}^{n_6}|} \leq \frac{q(q^2-1)(q^6-1)(q^8-1)}{(q-1)^2(q^3-1)}.$$

It follows that

$$\frac{v}{x} > q^{28} > (k_r k - k_r + 1)|G:T|,$$

contradicting (P_1).

Subcase 2.7: $T_\alpha = P_{J_7}$. Let n_7 be the inverse image of w_{α_7} under ϕ. This time, we have

$$\begin{aligned} P_{J_7} &= \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_7} \rangle, \\ P_{J_7}^{n_7} &= \langle X_r, H | r \in (\Phi^+ - \{\alpha_7\}) \cup \{-\alpha_7\} \cup \Phi_{w_{\alpha_7}(J_7)} \rangle. \end{aligned}$$

Then

$$P_{J_7} \cap P_{J_7}^{n_7} \geq \langle X_r, H | r \in (\Phi^+ - \{\alpha_7\}) \cup \Phi_{J'} \rangle,$$

where $J' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Hence

$$|P_{J_7} \cap P_{J_7}^{n_7}| > \frac{1}{d}\, q^{62}(q-1)^2(q^2-1)(q^4-1)(q^5-1)(q^6-1)(q^8-1).$$

Thus T_α has an orbit of size

$$x = \frac{|P_{J_7}|}{|P_{J_7} \cap P_{J_7}^{n_7}|} \leq \frac{q(q^9-1)(q^{12}-1)}{(q-1)(q^4-1)}.$$

It follows that

$$\frac{v}{x} > q^9 > (k_r k - k_r + 1)|G:T|,$$

contradicting (P_1).

Case 3: T_α is one of the subgroups of Lemma 2.4 (c).

Subcase 3.1: $T_\alpha = (E_6(q) \circ (q-1)/d).e_{+1}.2$ where $e_{+1} = (q-1, 3)$. Then

$$|T_\alpha| = \frac{2e_{+1}}{d} q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)(q-1),$$

and

$$v = \frac{q^{27}(q^{14}-1)(q^9+1)(q^5+1)}{2e_{+1}(q-1)}.$$

Since $(v-1, q) = 1$, we know by (P_3) that T_α has an orbit of size y such that

$$y \leq |T_\alpha|_{p'} = \frac{2e_{+1}}{d}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)(q-1).$$

It follows that
$$\frac{v}{y} > \frac{1}{36}q^{12} > (k_r k - k_r + 1)|G:T|,$$
contradicting (P_1).

Subcase 3.2: $T_\alpha = ({}^2E_6(q) \circ (q+1)/d).e_{-1}.2$ where $e_{-1} = (q+1, 3)$. Then
$$|T_\alpha| = \frac{2e_{-1}}{d} q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)(q+1),$$
and
$$v = \frac{q^{27}(q^{14}-1)(q^9-1)(q^5-1)}{2e_{-1}(q+1)},$$
and so T_α has an orbit of size y such that
$$y \le |T_\alpha|_{p'} = \frac{2e_{-1}}{d}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)(q+1).$$
It follows that
$$\frac{v}{y} > q^4 > (k_r k - k_r + 1)|G:T|,$$
contradicting (P_1).

Subcase 3.3: $T_\alpha = (SL_2(q) \circ D_6(q)).d$. Then
$$|T_\alpha| = \frac{1}{d} q^{31}(q^{10}-1)(q^8-1)(q^6-1)^2(q^4-1)(q^2-1)^2,$$
and
$$v = \frac{q^{32}(q^{18}-1)(q^{14}-1)(q^{12}-1)}{(q^2-1)(q^4-1)(q^6-1)},$$
and so T_α has an orbit of size y such that
$$y \le |T_\alpha|_{p'} = \frac{1}{d}(q^{10}-1)(q^8-1)(q^6-1)^2(q^4-1)(q^2-1)^2.$$
It follows that
$$\frac{v}{y} > q^{26} > (k_r k - k_r + 1)|G:T|,$$
contradicting (P_1).

Subcase 3.4: $T_\alpha = E_7(q^{\frac{1}{2}}).d$. Then
$$|T_\alpha| = q^{\frac{63}{2}}(q^9-1)(q^7-1)(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q-1),$$
and
$$v = \frac{1}{d} q^{\frac{63}{2}}(q^9+1)(q^7+1)(q^6+1)(q^5+1)(q^4+1)(q^3+1)(q+1),$$

and so T_α has an orbit of size y such that

$$y \leq |T_\alpha|_{p'} = (q^9-1)(q^7-1)(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q-1).$$

It follows that

$$\frac{v}{y} > \frac{1}{d}q^{\frac{63}{2}} > (k_r k - k_r + 1)|G:T|,$$

contradicting (P_1).

Thus in all cases we get a contradiction. This completes the proof of the Main Theorem. □

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