SIGNED EDGE DOMINATION NUMBERS IN TREES

H. KARAMI
DEPARTMENT OF MATHEMATICS
SHARIF UNIVERSITY OF TECHNOLOGY
P.O. BOX 11365-9415
TEHRAN, I.R. IRAN

ABDOLLAH KHODKAR*
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WEST GEORGIA
CARROLLTON, GA 30118
AKHODKAR@WESTGA.EDU

S.M. SHEIKHOLESLAMI†
DEPARTMENT OF MATHEMATICS
AZARBAIJAN UNIVERSITY OF TARBIAT MOALLEM
TABRIZ, I.R. IRAN
S.M.SHEIKHOLESLAMI@AZARUNIV.EDU

ABSTRACT. The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum_{x\in N[e]} f(x) \geq 1$ for each $e \in E(G)$, then f is called a signed edge dominating function of G. The minimum of the values $\sum_{e\in E(G)} f(e)$, taken over all signed edge dominating function f of G, is called the signed edge domination number of G and is denoted by $\gamma'_s(G)$. It has been conjectured that $\gamma'_s(T) \geq 1$ for every tree T. In this paper we prove that this conjecture is true and then classify all trees T with $\gamma'_s(T) = 1, 2$ and 3.

Keyword: Tree, Signed edge domination function; Signed edge domination number

1. INTRODUCTION

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [1] for terminology and notation which are not defined here. Two

* Research supported by the Faculty Research Grant, University of West Georgia.
† Research supported by the Research Office of Azarbaijan University of Tarbiat Moallem.

edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. If $S = N_G[e]$ for some $e \in E$, then we denote $f(S)$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in I(v)} f(e)$, where $I(v)$ is the set of all edges at vertex v. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f[e] \geq 1$ for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all signed edge dominating functions f of G, is called *signed edge domination number* of G. The signed edge domination number was introduced by B. Xu in [2] and denoted by $\gamma'_s(G)$. The signed edge dominating function f of G with $f(E(G)) = \gamma'_s(G)$ is called $\gamma'_s(G)$*-function.*

In 2002, it was conjectured [3] that for all trees T, $\gamma'_s(T) \geq 1$. In Section 2, we first prove that this conjecture is true. Then we characterize all trees T for which $\gamma'_s(T) = 1, 2$. In Section 3, we characterize all trees T with $\gamma'_s(T) = 3$. All connected graphs G with $\gamma'_s(G) = |E(G)|$ were characterized in [2].

Here are some well-known results on $\gamma'_s(G)$.

Theorem A. *(See [3]) Let G be a graph with m edges. Then $\gamma'_s(G) \equiv m \pmod 2$.*

Theorem B. *(See [3]) Let u, v, w be three vertices of a tree T such that u is a pendant vertex of T and v is adjacent to exactly two vertices u, w. Let f be an SEDF of T . Then*

$$f(uv) = f(vw) = 1.$$

Theorem C. *(See [3]) Let T be a star with m edges. If m is odd, then $\gamma'_s(T) = 1$. If m is even, then $\gamma'_s(T) = 2$.*

Theorem D. *(See [2]) Let G be a connected graph. Then $\gamma'_s(G) = |E(G)|$ if and only if either $G \cong P_n$ for some n $(1 \leq n \leq 5)$ or G is the subdivision of some star $K_{1,n}$ $(n \geq 3)$.*

2. A PROOF OF THE CONJECTURE

In 2002, Bohdan Zelinka and Liberec [3] showed that for some special classes of trees T, $\gamma'_s(T) \geq 1$, and they conjectured that $\gamma'_s(T) \geq 1$ for every tree T. In this section we prove that this conjecture is true. We also characterize all trees T for which $\gamma'_s(T) = 1, 2$. Throughout this paper $\ell(v)$ denotes the number of pendant edges at vertex v. For $i = 1, 2$, define $\mathcal{T}_i$ to be the collection of all trees of order $n \geq 2$ with exactly $i - 1$ vertices of even degree and $\ell(v) \geq \lfloor (\deg(v) - 1)/2 \rfloor$ for every vertex v.

Theorem 1. *For any tree $T = (V, E)$, $\gamma'_s(T) \geq 1$ and, for $i = 1, 2$, $\gamma'_s(T) = i$ if and only if $T \in \mathcal{T}_i$. Also, $T \in \mathcal{T}_1$ implies $f[e] = 1$, for every γ'_s-function f and every edge $e \in E$.*

Proof. The statements hold for all trees of order $n = 2, 3, 4$. Assume T is an arbitrary tree of order $n \geq 5$ and that the statements hold for all trees with smaller order. Let f be a γ'_s-function for T.

Case 1. There is a non-pendant edge $e = uv \in E$ for which $f(e) = -1$. Let T_1 and T_2 be the subtrees of $T - e$ with $u \in T_1$. Then, $\gamma'_s(T) = f(E(T_1)) - 1 + f(E(T_2))$. For $i = 1, 2$, the function f, restricted to T_i is an SEDF for T_i, hence, $\gamma'_s(T_i) \leq f(E(T_i))$. By the induction hypothesis, $\gamma'_s(T_i) \geq 1$ and, thus, $\gamma'_s(T) \geq 1$. Notice that if $\gamma'_s(T) \leq 2$ then $2 \leq f(E(T_1)) + f(E(T_2)) \leq 3$. So we may assume without lose of generality that $f(E(T_1)) = 1$. That is, f restricted to T_1 is a γ'_s-function for T_1. Let e' be any edge in T_1 incident to vertex u. Again by the induction hypothesis, $T_1 \in \mathcal{T}_1$ and, hence, $f[e'] = 1$ in T_1. This implies $f[e'] = 0$ in T, a contradiction. Therefore, when $\gamma'_s(T) \leq 2$, all edges e for which $f(e) = -1$ are pendant edges.

Case 2. The only edges e for which $f(e) = -1$ are pendant edges.
Let $M = \{e \in E \mid f(e) = -1\}$. Let $V_M = \{v_1, v_2, \ldots, v_k\}$ be the degree one end-vertices of the edges in M and let $W_M = \{w_1, w_2, \ldots, w_r\}$ be the remaining end-vertices of the edges in M. Since $n \geq 5$, we may assume k is positive and that $1 \leq r \leq k$. Further, for $1 \leq i \leq r$, we may assume w_i has $k_i \geq 1$ neighbors in V_M. Then, each w_i must have at least $k_i + 1$ neighbors in $V \setminus V_M$, where the adjoining edge e has $f(e) = 1$. Let t be the number of edges whose end-vertices are both in W_M. Then the number of edges e with $f(e) = 1$ and which are incident to vertices in W_M is at least $(k_1 + 1) + (k_2 + 1) + \ldots + (k_r + 1) - t = k + r - t$. Since W_M induces a forest, $t \leq r - 1$. Thus, T has at least $k + r - t \geq k + 1$ distinct edges e for which $f(e) = 1$. That is, $\gamma'_s(T) \geq (k + r - t) - k \geq 1$.
Now, suppose $\gamma'_s(T) = 1$. Then, we must have that $r - t = 1$. Therefore, W_M induces a tree and, for any γ'_s-function f, every vertex w_i in W_M must have exactly $k_i + 1$ incident edges e for which $f(e) = 1$. Therefore, since $f(E(T)) = 1$, every edge e' in T must have an end-vertex in W_M. Moreover, $\deg(v) = 1$ for every vertex $v \notin W_M$. That is, $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ and $\deg(v)$ is odd for every vertex $v \in V$. Therefore, $T \in \mathcal{T}_1$. Further, the construction enforces that $f[e] = 1$, for every edge e in T.
When $\gamma'_s(T) = 2$, we must have $r - 2 \leq t \leq r - 1$. If $t = r - 2$, W_M induces a forest of two subtrees, say T_1 and T_2, and $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ for each vertex v in W_M. Let T_3 and T_4 be the subtrees induced by the vertices of V which are adjacent to a vertex of T_1 or T_2, respectively. Since $f(E(T)) = 2$ and the fact that $f(e) = 1$ for every edge $e \notin E(T_3) \cup E(T_4)$, it follows that

$f(E(T_3)) = f(E(T_4)) = 1$, hence, $E(T) = E(T_3) \cup E(T_4)$. Therefore, two vertices in W_M, one in T_1 and one in T_2, must have a common degree two neighbor, say w, in $V \setminus (V_M \cup W_M)$. That is, $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ and $\deg(v)$ is odd for every vertex $v \in V \setminus \{w\}$. Therefore, $T \in \mathcal{T}_2$. If $t = r - 1$, then W_M induces a tree and exactly one vertex w_i in W_M must have $k_i + 2$ incident edges e for which $f(e) = 1$. Moreover, every vertex $w_j \in W_M$, $i \neq j$, must have $k_i + 1$ incident edges e for which $f(e) = 1$. Again, $T \in \mathcal{T}_2$.

Conversely, let $T \in \mathcal{T}_1$. By first part of the proof we have $\gamma'_s(T) \geq 1$. If $n = 2$ then obviously $\gamma'_s(T) = 1$. Let $n \geq 3$. Define $f : E \to \{1, -1\}$ by: $f(e) = -1$ for exactly $\lfloor(\deg(v) - 1)/2\rfloor$ pendant edges e at v if $\deg(v) \geq 3$ and $f(e') = 1$ for the remaining edges e' at v. It is easy to see that $f(E(T)) = 1$. Therefore, $\gamma'_s(T) = 1$. The case $T \in \mathcal{T}_2$ is similar. □

The following result is an immediate corollary of the structure of γ'_s-functions of $T \in \mathcal{T}_i$, $i = 1, 2$.

Corollary 2. *Let $T \in \mathcal{T}_i$, $i = 1, 2$, and let f be a γ'_s-function of T. Then $f(v) = 1$ if $\deg(v) \geq 3$ and odd, and $f(v) = 2$ if $\deg(v)$ is even.*

3. Trees with signed edge domination number 3

In this section we characterize the trees T with $\gamma'_s(T) = 3$. First, we study trees T with $\gamma'_s(T) = 3$ for which there is a γ'_s-function, say f, such that $f(e) = 1$ for every non-pendant edge e in T.

Let $\mathcal{B}_1$ be the collection of trees, T, which satisfy one of the following properties:

Type 1: T has exactly two vertices of even degree and $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ for each $v \in V(T)$, or

Type 2: each vertex of T has odd degree and there exists exactly one vertex $v \in V(T)$ such that $\ell(v) = (\deg(v) - 3)/2$, and $\ell(u) \geq (\deg(u) - 1)/2$ for each $u \in V(T - v)$.

If $T \in \mathcal{B}_1$, then $\gamma'_s(T) \geq 3$ by Theorem 1. Assume T is of Type 1 and u and w are the vertices of even degree. Let T' be obtained from T by adding two pendant edges uu' and ww'. Then $\gamma'_s(T') = 1$, by Theorem 1. Moreover, if f is a $\gamma'_s(T')$-function then $f(u) = f(w) = 1$ in T' by Corollary 2. Since $\deg(u), \deg(w) \geq 3$ in T', there is a pendant edge e at u and a pendant edge e' at w with $f(e) = f(e') = -1$. So we may assume $f(uu') = f(ww') = -1$. Therefore, $f(E(T)) = 3$ and hence $\gamma'_s(T) = 3$. Similarly, if T is of Type 2 then $\gamma'_s(T) = 3$. The following lemma shows that, under a certain condition, the inverse is also true.

Lemma 3. *Let T be a tree of order $n \geq 4$ with $\gamma'_s(T) = 3$. If T has a γ'_s-function, say f, such that $f(e) = 1$ for every non-pendant edge e in T then $T \in \mathcal{B}_1$.*

Proof. If $n = 4$ then the result is trivial. Now let $n \geq 5$. Following the notations in the proof of Theorem 1, Case 2, since $\gamma'_s(T) = 3$ we have $r-3 \leq t \leq r-1$.

Case 1. $t = r-1$.
Then W_M induces a tree. Now either exactly two distinct vertices w_i and w_j in W_M must have k_i+2 and k_j+2 incident edges e, respectively, for which $f(e) = 1$. So $T \in \mathcal{B}_1$ (Type 1). Or exactly one vertex w_i in W_M must have k_i+3 incident edges e for which $f(e) = 1$. So $T \in \mathcal{B}_1$ (Type 2).

Case 2. $t = r-2$.
Then W_M induces a forest of two subtrees, say T_1 and T_2, and $\ell(v) \geq \lfloor(\deg(v)-1)/2\rfloor$ for each vertex v in W_M. Let T_3 and T_4 be the subtrees induced by the vertices of V which are adjacent to a vertex of T_1 or T_2, respectively. Since $f(E(T)) = 3$ and the fact that $f(e) = 1$ for every edge $e \notin E(T_3) \cup E(T_4)$, it follows that $2 \leq f(E(T_3)) + f(E(T_4)) \leq 3$. If $f(E(T_3)) = f(E(T_4)) = 1$, then there is precisely one edge $e' \notin E(T_3) \cup E(T_4)$ with $f(e') = 1$ and with end-vertices in T_3 and T_4. So $T \in \mathcal{B}_1$ (Type 1). If $f(E(T_3)) = 1$ and $f(E(T_4)) = 2$ (the case $f(E(T_3)) = 2$ and $f(E(T_4)) = 1$ is similar) then exactly one vertex in T_2, say w_j, must have precisely k_j+2 incident edges e with $f(e) = 1$ and for each vertex $w_i \in W_M$, $i \neq j$, there are precisely k_i+1 incident edges e with $f(e) = 1$. Finally, two vertices in W_M, one in T_1 and one in T_2, must have a common degree two neighbor in $V \setminus (V_M \cup W_M)$. So, $T \in \mathcal{B}_1$ (Type 1).

Case 3. $t = r-3$.
Then W_M induces a forest of three subtrees T_i, $i = 1,2,3$, and $\ell(v) \geq \lfloor(\deg(v)-1)/2\rfloor$ for each vertex v in W_M. Let T_{i+3}, $i = 1,2,3$, be the tree induced by the vertices of V which are adjacent to a vertex of T_i. Since $f(E(T)) = 3$ we have $f(E(T_4)) = f(E(T_5)) = f(E(T_6)) = 1$. Now two cases are possible. Either two vertices in W_M, one in each T_i (without loss of generality we may assume i=1,2) and two vertices in W_M, one in each T_i (i=2,3), must have common degree two neighbors in $V \setminus (V_M \cup W_M)$. So $T \in \mathcal{B}_1$ (Type 1). Or three vertices in W_M, one in each subtree T_i, $i = 1,2,3$, must have a common degree three neighbor in $V \setminus (V_M \cup W_M)$. So, $T \in \mathcal{B}_1$ (Type 2). □

Now we study trees T with $\gamma'_s(T) = 3$ for which every γ'_s-function of T assigns -1 to at least a non-pendant edge of T. Let $\mathcal{A}$ be the collection of trees, T, in which $\gamma'_s(T) = 2$ and $\ell(v) \geq \lfloor\deg(v)/2\rfloor$ for each $v \in V(T)$. Obviously, $\mathcal{A} \subset \mathcal{T}_2$. The proof of the following lemma is straightforward.

Lemma 4. *Let T be a tree. Then $T \in \mathcal{A}$ if and only if $\gamma'_s(T) = 2$ and each γ'_s-function of T assigns 1 to at least one pendant edge at the unique vertex of even degree.*

Let $\mathcal{B}_2$ be the collection of trees, T, which satisfy one of the following properties:
Type 1: $T = T_1 \cup T_2 + \{w_1w_2\}$, where $T_1, T_2 \in \mathcal{A}$, u_1, u_2 are the unique vertices of even degree in T_1, T_2 and u_1w_1, u_2w_2 are pendant edges in T_1, T_2, respectively.
Type 2: $T = T_1 \cup T_2 + \{w_1u_2\}$, where $T_1 \in \mathcal{A}$, $T_2 \in (\mathcal{T}_2 \setminus \mathcal{A})$, u_1, u_2 are the unique vertices of even degree in T_1, T_2, respectively, and u_1w_1 is a pendant edge in T_1.
Type 3: $T = T_1 \cup T_2 + \{u_1u_2\}$, where $T_i \in (\mathcal{T}_2 \setminus \mathcal{A})$ and u_i is the unique vertex of even degree in T_i, $i = 1, 2$.

We leave for the reader to check that $\gamma'_s(T) = 3$ for every $T \in \mathcal{B}_2$. The following lemma shows that, under a certain condition, the inverse is also true.

Lemma 5. *Let T be a tree with $\gamma'_s(T) = 3$. If every γ'_s-function of T assigns -1 to a non-pendant edge of T, then $T \in \mathcal{B}_2$.*

Proof. Let T be a tree with $\gamma'_s(T) = 3$ and let f be a γ'_s-function of T. Then $f(e) = -1$ for a non-pendant edge $e = uv$, by assumption. Let T_1 and T_2 be the connected components of $T - e$ with $u \in T_1$. We have $f(E(T_1)) + f(E(T_2)) = 4$. Obviously, f restricted to T_i is an SEDF of T_i for $i = 1, 2$. If $f(E(T_1)) = 1$ (the case $f(E(T_2)) = 1$ is similar) then f restricted to T_1 is a γ'_s-function of T_1 by Theorem 1. Let e' be any edge of T_1 at u. Then, by Theorem 1, we have $f[e'] = 1$ in T_1. So $f[e'] = 0$ in T, which is a contradiction. Therefore, $f(E(T_1)) = f(E(T_2)) = 2$, and hence, $\gamma'_s(T_i) \leq 2$ for $i = 1, 2$. Now if $\gamma'_s(T_i) = 1$ for $i = 1$ or 2 then, since $f(e) = -1$, there exists a $\gamma'_s(T)$-function such that it assigns 1 to every non-pendant edge of T, which is a contradiction. Therefore, the function f, restricted to T_i, is a γ'_s-function of T_i for $i = 1, 2$. Hence, $T_i \in \mathcal{T}_2$. So by Theorem 1, the number of vertices of even degree in T is 0, 2 or 4. Now we consider three cases.

Case 1. T has four vertices of even degree.
This forces $\deg_{T_1}(u)$ and $\deg_{T_2}(v)$ to be odd. Let $\deg_{T_1}(u) \geq 3$ (the case $\deg_{T_2}(v) \geq 3$ is similar). Then there exists a pendant edge, say e', at u by Theorem 1. Now we have $f[e'] = f(u) = 1$ in T_1, by Corollary 2. This implies $f[e'] = 0$ in T, which is a contradiction. Therefore $\deg_{T_1}(u) = \deg_{T_2}(v) = 1$. Let $uu_1 \in E(T_1)$ and $vv_1 \in E(T_2)$. Since $f[uv] \geq 1$ in T we must have $f(uu_1) = f(vv_1) = 1$. Obviously, $\deg(u_1), \deg(v_1) > 1$. We claim that $\deg(u_1)$ and $\deg(v_1)$ are even. Let $\deg(u_1) \geq 3$ be odd. Then there is a pendant edge, say e', at u_1 by Theorem 1. Now we have

$f[e'] = f(u_1) = 1$ in T_1, which implies $f[uu_1] = f(u_1) = 1$ in T_1, by Corollary 2. Hence, $f[uu_1] = 0$ in T, which is a contradiction. So $\deg(u_1)$ is even. Similarly, $\deg(v_1)$ is also even. In order to show that $T_1, T_2 \in \mathcal{A}$ it is sufficient to prove that $\ell(u_1) \geq \deg(u_1)/2$ in T_1 and $\ell(u_2) \geq \deg(u_2)/2$ in T_2. By Theorem 1 and Corollary 2 we have $2 = f(u_1) = \deg(u_1) - 2\ell^-(u_1)$ in T_1, where $\ell^-(u_1)$ is the number of pendant edges e' at u_1 for which $f(e') = -1$. So $\ell^-(u_1) = (\deg(u_1) - 2)/2$ in T_1. Now since $f(uu_1) = 1$ it follows that $\ell(u_1) \geq ((\deg(u_1) - 2)/2) + 1 = \deg(u_1)/2$ in T_1. Similarly $\ell(u_2) \geq \deg(u_2)/2$ in T_2. Hence, $T \in \mathcal{B}_2$ (Type 1).

Case 2. T has exactly two vertices of even degree.
Without loss of generality we may assume $\deg(u)$ is even and $\deg(v)$ is odd. An arguments similar to that described above shows that $\deg_{T_1}(u) = 1$ and $T_1 \in \mathcal{A}$. As in Case 1 one can also see that if $uu_1 \in T_1$ then $\deg_{T_1}(u_1)$ is even and $f(uu_1) = 1$. Let $T_2 \in \mathcal{A}$. Then $\ell(v) \geq \deg(v)/2$ in T_2. This forces that f assigns 1 to a pendant edge at v, say e', in T_2, by Lemma 4. Now define $g : E(T) \longrightarrow \{-1, +1\}$ by

$$g(e') = -1, \quad g(uv) = 1 \text{ and } g(e) = f(e) \text{ if } e \neq e', uv.$$

Obviously g is a γ'_s-function of T. In addition, g assigns 1 to every non-pendant edges of T, which is a contradiction by assumption. So $T_2 \notin \mathcal{A}$. Hence, $T \in \mathcal{B}_2$ (Type 2).

Case 3. T has no vertex of even degree.
Then obviously $\deg_{T_1}(u)$ and $\deg_{T_2}(v)$ are even. An argument similar to that presented in Case 2 shows that $T_1, T_2 \notin \mathcal{A}$. Hence, $T \in \mathcal{B}_2$ (Type 3). □

Now we are ready to state the main theorem of this section.

Theorem 6. *Let T be a tree. Then $\gamma'_s(T) = 3$ if and only if $T \in \mathcal{B}_1 \cup \mathcal{B}_2$.*

Acknowledgements. The authors would like to thank the referee whose suggestions were most helpful in writing the final version of this paper.

REFERENCES

[1] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.
[2] B. Xu, *On signed edge domination numbers of graphs*, Discrete Mathematics **239** (2001), 179–189.
[3] B. Zelinca and Liberec, *On signed edge domination numbers of trees*, Mathematica Bohemica **127** (2002), 49–55.