MORE GENERAL IDENTITIES INVOLVING THE TERMS OF $\{W_n(a,b;p,q)\}$

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ABSTRACT. In this paper, we obtain new general results containing sums of binomial and multinomial with coefficients satisfying a general third order linear recursive relations with indices in arithmetic progression.

1. INTRODUCTION

The second order linear recurrence $\{W_n(a,b;p,q)\}$ is defined as for $n \geq 2$

$$W_n = pW_{n-1} - qW_{n-2} \tag{1.1}$$

where $W_0 = a, W_1 = b$. (Denote $W_n(a,b;p,q)$ by W_n).

Let α and β be the distinct roots of $\lambda^2 - p\lambda + q = 0$. The Binet formula of $\{W_n\}$ is

$$W_n = A\alpha^n + B\beta^n \tag{1.2}$$

where $A = (b - a\beta)/(\alpha - \beta)$ and $B = (a\alpha - b)/(\alpha - \beta)$.

Define $V_n = W_n(2,p;p,q)$. Thus the Binet form of V_n is given by $V_n = \alpha^n + \beta^n$.

Using the method of Carlitz and Ferns [1], some identities involving the terms of $\{W_n\}$ were given in [3, 4].

From [2], we have for $k > 0$ and $n > 1$,

$$W_{kn} = V_k W_{k(n-1)} - q^k W_{k(n-2)} \tag{1.3}$$

where V_n is defined as before. The characteristic polynomial of $\{W_{kn}\}$ is $\lambda^2 - V_k\lambda + q^k$ and its roots are α^k and β^k.

In this paper, we derive more general cases of the results of [3, 4] concerning the sequence $\{W_{kn}\}$, for a fixed positive $k > 0$.

2. THE MAIN RESULTS

Firstly we give the following result to generalize the results of [3, 4].

2000 *Mathematics Subject Classification.* 11B37.
Key words and phrases. Binomial sums, recurrences.

Theorem 1. *For $n > 0$ and $c \geq 0$,*

$$W_{2kn+c} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} V_k^j q^{k(n-j)} W_{kj+c}.$$

Proof. Since α^k and β^k are the roots of the equation $\lambda^2 - V_k\lambda + q^k = 0$,

$$\alpha^{2k} = V_k\alpha^k - q^k \text{ and } \beta^{2k} = V_k\beta^k - q^k.$$

Here, by the binomial theorem, we have

$$\alpha^{2kn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} V_k^j q^{k(n-j)} \alpha^{kj} \quad (2.1)$$

$$\beta^{2kn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} V_k^j q^{k(n-j)} \beta^{kj}. \quad (2.2)$$

By multiplying both sides of (2.1) and (2.2) by $A\alpha^c$ and $B\beta^c$, respectively, and adding, the proof follows from the Binet form (1.2). □

Lemma 1. *For $k, m, r > 0$,*

$$-q^{k(m+r)} + V_{kr} q^{km} u^{kr} + u^{2k(m+r)} = V_{km} u^{k(m+2r)}$$

where u is either α or β.

Proof. Since $\alpha^{2k} = V_k\alpha^k - q^k$ and $\beta^{2k} = V_k\beta^k - q^k$,

$$\begin{aligned} -q^{k(m+r)} + V_{kr} q^{km} u^{kr} + u^{2k(m+r)} &= q^{km}\left(V_{kr}u^{kr} - q^{kr}\right) + u^{2k(m+r)} \\ &= u^{k(m+2r)}\left(q^{km}u^{-km} + u^{km}\right) \\ &= u^{k(m+2r)}\left(\beta^{km} + \alpha^{km}\right) \\ &= V_{km} u^{k(m+2r)}. \end{aligned}$$

□

Theorem 2. *For $n, k, m, r > 0$,*

$$-q^{k(m+r)} W_n + V_k q^{mk} W_{kr+n} + W_{2k(m+r)+n} = V_{km} W_{k(m+2r)+n}$$

Proof. By Lemma 1, we have

$$\begin{aligned} -q^{k(m+r)} + V_{kr} q^{km} \alpha^{kr} + \alpha^{2k(m+r)} &= V_{km}\alpha^{k(m+2r)}, \\ -q^{k(m+r)} + V_{kr} q^{km} \beta^{kr} + \beta^{2k(m+r)} &= V_{km}\beta^{k(m+2r)}. \end{aligned}$$

The proof follows if we multiply both sides of the previous two identities by $A\alpha^n$ and $B\beta^n$, respectively, add, and then use the Binet form (1.2). □

Theorem 3. *For $n, k, m, r > 0$,*

$$W_{krn+c} = \left(V_{kr} q^{km}\right)^{-n} \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j q^{k(m+r)s} V_{km}^i W_{k(m+2r)i+2k(m+r)j+c},$$

$$W_{kn(m+2r)+c} = V_{km}^{-n} \sum_{i+j+s=n} \binom{n}{i,j} (-1)^s q^{kmj+k(m+r)s} V_{kr}^j W_{2k(m+r)i+krj+c},$$

$$W_{2kn(m+r)+c} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j V_{kr}^j q^{kmj+k(m+r)s} V_{km}^i W_{k(m+2r)i+krj+c}.$$

where the symbol $\binom{n}{i,j}$ is defined by $\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}$.

Proof. By Lemma 1 and the multinomial theorem, we have

$$\begin{aligned}
(V_{kr}q^{km})^n u^{krn} &= \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j q^{k(m+r)s} V_{km}^i u^{k(m+2r)i+2k(m+r)j} \\
V_{km}^n u^{kn(m+2r)} &= \sum_{i+j+s=n} \binom{n}{i,j} (-1)^s q^{kmj+k(m+r)s} V_k^j u^{2k(m+r)i+krj} \\
u^{2kn(m+r)} &= \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j V_{kr}^j q^{kmj+k(m+r)s} V_{km}^i u^{k(m+2r)i+krj}
\end{aligned}$$

Multiplying both sides in the preceding identities by u^c, adding, and using (1.2), we get the conclusion. □

Theorem 4. *For* $n, k, m, r, c > 0$,

$$\begin{aligned}
V_{kr}^n q^{kmn} W_{krn+c} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{k(m+r)(n-j)} W_{2k(m+r)j+c} &\equiv 0 \pmod{V_{km}}, \\
W_{2k(m+r)n+c} - (-1)^n q^{kmn} W_{2krn+c} &\equiv 0 \pmod{V_{km}}.
\end{aligned}$$

Proof. The proof follows from the first two results of Theorem 3 and the decomposition

$$\sum_{i+j+s=n} = \sum_{i+j+s=n,\ i=0} + \sum_{i+j+s=n,\ i\neq 0}.$$

□

When $k = r = 1$ in Theorems 2-4, the results are reduced to the results of [3, 4].

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