The Product Summation over all Ordered Partitions *

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Abstract

In [5], a product summation of ordered partition $f(n,m,r) = \sum c_1^r c_2^r \cdots c_m^r$ was defined, where for two given positive integers m, r, the sum is over all positive integers $c_1, c_2, \cdots, c_m$ with $c_1 + c_2 + \ldots + c_m = n$. $f(n,r) = \sum_{i=1}^{n} f(n,m,r)$ was also defined. Many results on $f(n,m,r)$ were found. However, few things have been known about $f(n,r)$. In this paper, we give more details for $f(n,r)$, including its two recurrences, its explicit formula via an entry of a matrix and its generating function. Unexpectedly, we obtain some interesting combinatorial identities, too.

Key words: product summation, Fibonacci sequence, k–generalized Fibonacci sequence, ordered partition, recurrence, identity, matrix

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*Supported by NNSF of China(10771080)and SRFDP of China (No. 20070574006).

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1 Introduction

The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and it has been studied in many articles and books (see [1,2,3]). Here we introduce a generalization of the Fibonacci sequence $\{F_n^{(k)}\}$ which is called the $k-$generalized Fibonacci sequence for integer $k \geq 2$ [4]. It is defined as

$$F_n^{(k)} = a_1 F_{n-1}^{(k)} - a_2 F_{n-2}^{(k)} + a_3 F_{n-3}^{(k)} - \cdots + (-1)^{k-1} a_k F_{n-k}^{(k)}, \quad for \quad n \geq 1,$$

where $F_0^{(k)} = 1$, and for each negative i, $F_i^{(k)} = 0$, and $a_1, a_2, \cdots, a_k$ are arbitrary real numbers. Setting $k = 2$, $a_1 = 1$ and $a_2 = -1$, we obtain the Fibonacci sequence.

The generating function for $\{F_n^{(k)}\}$ is

$$F^{(k)}(t) = \sum_{n \geq 0} F_n^{(k)} t^n = \frac{1}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k}.$$

Suppose that $a_1, a_2, \cdots, a_k$ are defined as above. Consider the general order-k linear homogeneous recurrence relation

$$g_n = a_1 g_{n-1} - a_2 g_{n-2} + a_3 g_{n-3} - \cdots + (-1)^{k-1} a_k g_{n-k}, \quad for \quad n \geq k,$$

and it begins with arbitrary initial values $g_0, g_1, \cdots, g_{k-1}$.

Theorem A([4]) *The generating function for* $\{g_n\}$ *is*

$$G(t) = \sum_{n \geq 0} g_n t^n = \frac{\sum_{s=0}^{k-1} \left(\sum_{i=0}^{s} (-1)^i a_i g_{s-i} \right) t^s}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k},$$

and $\{g_n\}$ *satisfies that*

$$g_n = \sum_{s=0}^{k} \left(\sum_{i=0}^{s} (-1)^i a_i g_{s-i} \right) F_{n-s}^{(k)},$$

where $a_0 = 1$.

In [1], Louis Comtet gives an exercise to show that $\sum\limits_{c_1+c_2+\cdots+c_k=n} c_1c_2\cdots c_k = \binom{n+k-1}{2k-1}$. Generally, we consider

$$f(n,m,r) := \sum_{c_1+c_2+\cdots+c_m=n} c_1^r c_2^r \cdots c_m^r,$$

where all integers $c_i > 0$, and m, r are two given positive integers. For a given integer r, define $f(n,r) = \sum\limits_{m=1}^{n} f(n,m,r)$. Many results on $f(n,m,r)$ were found, such as its generating function, the explicit formulas for $f(n,m,2)$, $f(n,m,3)$ and $f(n,m,4)$ and so on [5]. We also prove that $f(n,1) = F_{2n-1}$. However, few results have been known about $f(n,r)$ for $r \geq 2$. In this paper, we concentrate on $f(n,r)$ and obtain more results about it, including its two recurrences, its explicit formula via an entry of a matrix and its generating function. Additionally, we gain some unexpected and interesting combinatorial identities.

2 One recurrence and the explicit formula for $f(n,r)$

For an integer $n > 0$, it has 2^{n-1} ordered partitions [3]. Take $n = 4$ for example. All its ordered partitions are 4, 1+3, 3+1, 2+2, 1+1+2, 1+2+1, 2+1+1 and 1+1+1+1. From the definition of $f(n,r)$, one can write $f(n,r) = \sum a_1^r a_2^r \cdots a_m^r$ where the sum is over all 2^{n-1} ordered partitions $a_1 + a_2 + \cdots + a_m = n$. According to the value of a_1, one can easily get the recurrence of $f(n,r)$ as follows.

$$\begin{aligned} f(n,r) = f(n-1,r) + 2^r f(n-2,r) + 3^r f(n-3,r) + \cdots \\ +(n-1)^r f(1,r) + n^r \quad for \quad n \geq 1. \end{aligned} \tag{1}$$

Since $f(n,1) = F_{2n-1}$, we have a recurrence of Fibonacci number as

$$F_{2n-1} = F_{2n-3} + 2F_{2n-5} + 3F_{2n-7} + \cdots + (n-1)F_1 + n \quad for \quad n \geq 1.$$

Let C_n be an $n \times n$ matrix which is defined as

$$C_n = \begin{pmatrix} 1 & 2^r & 3^r & 4^r \cdots & (n-2)^r & (n-1)^r & n^r \\ 1 & 0 & 0 & 0 \quad \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \quad \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \quad \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \quad \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \quad \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

It follows that the recurrence relation (1) can be rewritten as

$$\begin{pmatrix} f(n,r) \\ f(n-1,r) \\ f(n-2,r) \\ \vdots \\ f(2,r) \\ f(1,r) \end{pmatrix} = C_n \begin{pmatrix} f(n-1,r) \\ f(n-2,r) \\ f(n-3,r) \\ \vdots \\ f(1,r) \\ 1 \end{pmatrix}, \quad for \quad n \geq 1.$$

By iteration one obtains

$$\begin{pmatrix} f(n,r) \\ f(n-1,r) \\ f(n-2,r) \\ \vdots \\ f(2,r) \\ f(1,r) \end{pmatrix} = C_n^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad for \quad n \geq 1. \tag{3}$$

From (3), it follows that $f(n-j+1,r) = [C_n^n]_{j1}$ for $1 \leq j \leq n$ and $[C_n^n]_{j1}$ denotes the $(j,1)$-entry of matrix C_n^n. Then the following result holds.

Theorem 2.1 *For $n \geq 1$, we have $f(n,r) = [C_n^n]_{11}$ where the matrix C_n is defined as (2).*

Applying this theorem and using some mathematical software, one is able to compute $f(n,r)$ more easily and faster, such as $f(1,r) = 1$, $f(2,r) = 1 + 2^r$, $f(3,r) = 1 + 2 \cdot 2^r + 3^r$, $f(4,r) = 1 + 3 \cdot 2^r + 2 \cdot 3^r + 2 \cdot 4^r$, $f(5,r) = 1 + 4 \cdot 2^r + 3 \cdot 3^r + 5 \cdot 4^r + 5^r + 2 \cdot 6^r$, $f(6,r) = 1 + 5 \cdot 2^r + 4 \cdot 3^r + 9 \cdot 4^r + 2 \cdot 5^r + 7 \cdot 6^r + 3 \cdot 8^r + 9^r$ and so on.

3 Another recurrence and the generating function for $f(n,r)$

Before presenting the other recurrence for $f(n,r)$, we must prove some lemmas. Firstly, some terminologies are needed.

Suppose that M and N are two sets. Use M^N to denote the set which contains all the maps from N to M. Then it follows that $|M^N| = m^n$ if $|M| = m$ and $|N| = n$. It is known that the number of one-to-one maps of N with $|N| = n$ is $n!$.

Lemma 3.1 *Let r be a positive integer and k be any nonnegative integer. Then*

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j (r + k - j)^r = r!. \tag{4}$$

Proof. Noting that the right hand of Eq.(4) enumerates the one-to-one maps of an $r-$*entries* set, say R, i.e., $|R| = r$. On the left hand, we construct some maps and then we can obtain the one-to-one maps of R.

Suppose that K is a set with $|K| = k$. Let $M_j = (R \cup K) - R_j$, where R_j is a subset of R with $|R_j| = j$ and integer $0 \leq j \leq r$.

Given j, there are $\binom{r}{j}$ ways to choose R_j and $|M_j^R| = (r+k-j)^r$ for each R_j. By the Principle of Inclusion-Exclusion, it follows that $\sum_{j=0}^{r} \binom{r}{j}(-1)^j(r+k-j)^r$ calculate the number of the one-to-one maps of R, i.e., Eq(4) holds. ∎

Take $R = \{a, b\}$ and $K = \{c, d\}$ for instance. $|M_0^R| = 4^2$, $|M_1^R| = 3^2$ and $|M_2^R| = 2^2$. Then $4^2 - \binom{2}{1} \cdot 3^2 + 2^2 = 2!$. If we take $R = \{a, b\}$ and $K^* = \{c, d, e, f, g\}$, we have $7^2 - \binom{2}{1} \cdot 6^2 + 5^2 = 2!$.

Applying identity (4), we can obtain several interesting identities.

Corollary 1 *Let r be a positive integer and k be any nonnegative integer. Then*

$$\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j(r+k-j)^{r-1-i}(r+k-j-1)^i = r!. \qquad (5)$$

Proof. Since $\binom{r}{j} = \binom{r-1}{j} + \binom{r-1}{j-1}$ for $j \geq 1$, we have

$$\begin{aligned}
& \sum_{j=0}^{r} \binom{r}{j}(-1)^j(r+k-j)^r \\
= \; & (r+k)^r + \sum_{j=1}^{r-1}\left(\binom{r-1}{j} + \binom{r-1}{j-1}\right)(-1)^j(r+k-j)^r + (-1)^r k^r \\
= \; & \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j(r+k-j)^r + \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{j+1}(r+k-j-1)^r \\
= \; & \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j[(r+k-j)^r - (r+k-j-1)^r] \\
= \; & \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\sum_{i=0}^{r-1}(r+k-j)^{r-1-i}(r+k-j-1)^i
\end{aligned}$$

By applying Eq.(4), we obtain (5). ∎

Corollary 2 *Let r be a positive integer and k be any nonnegative integer. Then*

$$\sum_{j=0}^{r+1}\binom{r+1}{j}(-1)^j(r+k+1-j)^r = 0. \qquad (6)$$

Proof. Note that Eq.(4) holds for any integer $k \geq 0$. It follows that

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j (r+k-j)^r = r! \tag{7}$$

and

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j (r+k+1-j)^r = r! \tag{8}$$

Eq.(7) subtracted from Eq.(8) gives that

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j (r+k+1-j)^r - \sum_{j=0}^{r} \binom{r}{j} (-1)^j (r+k-j)^r = 0 \tag{9}$$

The left hand of (9) becomes

$$(r+k+1)^r + \sum_{j=1}^{r} \tbinom{r}{j} (-1)^j (r+k+1-j)^r - \sum_{j=0}^{r} \tbinom{r}{j} (-1)^j (r+k-j)^r$$
$$= (r+k+1)^r + \sum_{j=0}^{r-1} \tbinom{r}{j+1} (-1)^{j+1} (r+k-j)^r - \sum_{j=0}^{r} \tbinom{r}{j} (-1)^j (r+k-j)^r$$
$$= (r+k+1)^r + \sum_{j=0}^{r-1} [\tbinom{r}{j+1} + \tbinom{r}{j}] (-1)^{j+1} (r+k-j)^r + (-1)^{r+1} k^r$$
$$= (r+k+1)^r + \sum_{j=1}^{r-1} \tbinom{r+1}{j+1} (-1)^{j+1} (r+k-j)^r + (-1)^{r+1} k^r$$
$$= \sum_{j=0}^{r+1} \tbinom{r+1}{j} (-1)^j (r+k+1-j)^r \qquad \blacksquare$$

Using the method of proving (6) and from (5), we deduce the following result.

Corollary 3 *Let r be a positive integer and k be any nonnegative integer. Then*

$$\sum_{i=0}^{r-1} \sum_{j=0}^{r} \binom{r}{j} (-1)^j (r+k+1-j)^{r-1-i} (r+k-j)^i = 0. \tag{10}$$

Now we turn to $f(n,r)$. From Eq.(1), if n is sufficiently large, we have

$$f(n-1,r) = f(n-2,r) + 2^r f(n-3,r) + 3^r f(n-4,r) + \cdots + (n-2)^r f(1,r) + (n-2)^r. \quad (11)$$

(11) subtracted from (1) yields that
$f(n,r) - f(n-1,r) = f(n-1,r) + (2^r - 1)f(n-2,r) + (3^r - 2^r)f(n-3,r) + \cdots + [(n-1)^r - (n-2)^r]f(1,r) + n^r - (n-1)^r$ that is

$$f(n,r) = 2f(n-1,r) + \sum_{i=1}^{n-2}[(i+1)^r - i^r]f(n-i-1,r) + n^r - (n-1)^r. \quad (12)$$

Thus we have

$$f(n-1,r) = 2f(n-2,r) + \sum_{i=1}^{n-3}[(i+1)^r - i^r]f(n-i-2,r) + (n-1)^r - (n-2)^r. \quad (13)$$

(13) subtracted from (12) gives that

$$f(n,r) = 3f(n-1,r) + (2^r - 1 - 2)f(n-2) + n^r - 2(n-1)^r + (n-2)^r + \sum_{i=1}^{n-3}[(i+2)^r - 2(i+1)^r + i^r]f(n-i-2,r). \quad (14)$$

For the convenience of the discussion, we call the operation from Eq.(1) to Eq.(12) an $1-$*subtract operation* (1-S.O for short). Then the operation from Eq.(12) to Eq.(14) is called the 2-S.O and so on, i.e., if the subtraction operation is operated for l times from Eq.(1), we get the $l-$S.O, where integer $l \geq 0$. After $l-$S.O, we define that $f(n,r) = b_1^{(l)}f(n-1,r) + b_2^{(l)}f(n-2,r) + b_3^{(l)}f(n-3,r) + \cdots + b_{n-1}^{(l)}f(1,r) + b_n^{(l)}$, for integer $l \geq 0$. For example, $b_i^{(0)} = i^r$ for $i = 1, 2, \cdots, n$ and $b_1^{(2)} = 2$, $b_2^{(2)} = 2^r - 1 - 2$, $b_i^{(2)} = i^r - 2(i-1)^r + (i-2)^r$ for $i = 3, 4, \cdots, n$. It should be noted that $b_{i+1}^{(l+1)} = b_{i+1}^{(l)} - b_i^{(l)}$, for $1 \leq i \leq n-1$.

Proposition After l–S.O, $b_i^{(l)} = \sum_{j=0}^{l} \binom{l}{j}(-1)^j(i-j)^r$, for $i \geq l+1$.

Proof. The result is proved by the induction of l. If $l=0$, it is easy to see that $b_i^{(l)} = i^r$ for $i \geq 1$. Suppose that $b_i^{(l)} = \sum_{j=0}^{l} \binom{l}{j}(-1)^j(i-j)^r$, for $i \geq l+1$. From the definition of l–S.O, we have

$$\begin{aligned}
b_{i+1}^{(l+1)} &= b_{i+1}^{(l)} - b_i^{(l)} \\
&= \sum_{j=0}^{l} \binom{l}{j}(-1)^j(i+1-j)^r - \sum_{j=0}^{l} \binom{l}{j}(-1)^j(i-j)^r \\
&= (i+1)^r + \sum_{j=1}^{l} \binom{l}{j}(-1)^j(i+1-j)^r + \sum_{j=0}^{l-1} \binom{l}{j}(-1)^{j+1}(i-j)^r \\
&\qquad + (-1)^{l+1}(i-l)^r \\
&= (i+1)^r + \sum_{j=0}^{l-1}[\binom{l}{j+1} + \binom{l}{j}](-1)^{j+1}(i-j)^r + (-1)^{l+1}(i-l)^r \\
&= \sum_{j=0}^{l+1} \binom{l+1}{j}(-1)^j(i+1-j)^r,
\end{aligned}$$

for $i \geq l+1$, which implies that

$$b_i^{(l+1)} = \sum_{j=0}^{l+1} \binom{l+1}{j}(-1)^j(i-j)^r,$$

for $i \geq (l+1)+1$. ∎

Now we give another recurrence for $f(n,r)$.

Theorem 3.2

$$f(n,r) = b_1^{(r+1)} f(n-1,r) + b_2^{(r+1)} f(n-2,r) + b_3^{(r+1)} f(n-3,r) + \cdots + b_{r+1}^{(r+1)} f(n-r-1,r), \quad for \quad n \geq r+2,$$

where $b_1^{(r+1)} = r+2$, $b_2^{(r+1)} = 2^r - \binom{r+2}{2}$ *and* $b_i^{(r+1)} = b_i^{(0)} - b_{i-1}^{(0)} - b_{i-1}^{(1)} - \cdots - b_{i-1}^{(r)}$ *for* $2 \leq i \leq r+1$.

Proof. From Proposition and Lemma 3.1, we know that $b_i^{(r)} = \sum_{j=0}^{r} \binom{r}{j}(-1)^j(i-j)^r = r!$, for $i \geq r+1$. Thus $b_{i+1}^{(r+1)} = b_{i+1}^{(r)} - b_i^{(r)} = r! - r! = 0$, for $i \geq r+1$, implying that

$$f(n,r) = b_1^{(r+1)} f(n-1,r) + b_2^{(r+1)} f(n-2,r) + b_3^{(r+1)} f(n-3,r) + \cdots + b_{r+1}^{(r+1)} f(n-r-1,r).$$

On the other hand, it is clear that $b_1 = b_1^{(r+1)} = r+2$ and $b_i = b_i^{(r+1)} = b_i^{(r)} - b_{i-1}^{(r)} = (b_i^{(r-1)} - b_{i-1}^{(r-1)}) - b_{i-1}^{(r)} = b_i^{(0)} - b_{i-1}^{(0)} - b_{i-1}^{(1)} - \cdots - b_{i-1}^{(r)}$ for $i \geq 2$. And particularly, $b_2 = b_2^{(r+1)} = 2^r - 1 - 2 - \cdots - (r+1) = 2^r - \binom{r+2}{2}$. ∎

This theorem gives an easy way to enumerate $f(n,r)$ if r is smaller enough, such as $f(n,1) = 3f(n-1,1) - f(n-2,1)$, $f(n,2) = 4f(n-1,2) - 2f(n-2,2) + f(n-3,2)$, $f(n,3) = 5f(n-1,3) - 2f(n-2,3) + 5f(n-3,3) - f(n-4,3)$, $f(n,4) = 6f(n-1,4) + f(n-2,4) + 21f(n-3,4) - 4f(n-4,4) + f(n-5,4)$, and so on.

This theorem means that the recurrence of $f(n,r)$ is a general order-$(r+1)$ linear homogeneous recurrence relation, which reminds us that we can obtain its generating function via the generating function of some $(r+1)$-generalized Fibonacci sequence.

Define the $(r+1)$-generalized Fibonacci sequence $\{F_n^{(r+1)}\}$ as

$$F_n^{(r+1)} = b_1^{(r+1)} F_{n-1}^{(r+1)} + b_2^{(r+1)} F_{n-2}^{(r+1)} + \cdots + b_{r+1}^{(r+1)} F_{n-r-1}^{(r+1)},$$

which satisfies the same recursive pattern as $\{f(n,r)\}$. From Theorem A, we have the relation between $\{f(n,r)\}$ and $F_n^{(r+1)}$ as

$$f(n,r) = \sum_{s=1}^{r+1} \left(b_0^{(r+1)} f(s,r) - \sum_{i=1}^{s-1} b_i^{(r+1)} f(s-i,r) \right) F_{n-s}^{(r+1)},$$

where we define $b_0^{(r+1)} = 1$ and note that $f(0,r) = 0$. Base on Theorem A, the following result holds.

Theorem 3.3 *The generating function of* $\{f(n,r)\}$ *is*

$\Psi_r(t) = \sum_{n\geq 0} f(n,r) t^n$

$$= \frac{\sum_{s=1}^{r}\left(b_0^{(r+1)}f(s,r) - \sum_{i=1}^{s-1} b_i^{(r+1)} f(s-i,r)\right)t^s}{1 - b_1^{(r+1)}t - b_2^{(r+1)}t^2 - \cdots - b_{r+1}^{(r+1)}t^{r+1}}.$$

For example, $\Psi_1(t) = \frac{t}{1-3t+t^2}$, $\Psi_2(t) = \frac{t+t^2}{1-4t+2t^2-t^3}$ and $\Psi_3(t) = \frac{t+4t^2+t^3}{1-5t+2t^2-5t^3+t^4}$. It is equivalent to mean that if one sets $F_n^{(2)} = 3F_{n-1}^{(2)} - F_{n-2}^{(2)}$, $F_n^{(3)} = 4F_{n-1}^{(3)} - 2F_{n-2}^{(3)} + F_{n-3}^{(3)}$ and $F_n^{(4)} = 5F_{n-1}^{(4)} - 2F_{n-2}^{(4)} + 5F_{n-3}^{(4)} - F_{n-4}^{(4)}$, respectively, it follows that $f(n,1) = F_{n-1}^{(2)}$, $f(n,2) = F_{n-1}^{(3)} + F_{n-2}^{(3)}$ and $f(n,3) = F_{n-1}^{(4)} + 4F_{n-2}^{(4)} + F_{n-3}^{(4)}$, respectively.

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