Semi-regular harmonic graph and equi-bipartite harmonic graph *

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Abstract

Let G be a graph on n vertices $v_1, v_2, \cdots, v_n$ and let $d(v_i)$ be the degree of the vertex v_i. If $(d(v_1), d(v_2), \cdots, d(v_n))^t$ is an eigenvector of the (0,1)-adjacency matrix of G, then G is said to be harmonic. Semi-regular harmonic graph is the harmonic graph which has exactly two different degrees. Equi-bipartite harmonic graph is the bipartite graph $H = (X, Y; E)$ with $|X| = |Y|$. In this paper, we characterize the semi-regular harmonic graph and equi-bipartite harmonic graph, and the degree sequence of equi-bipartite 3-harmonic graphs.

Key words harmonic graph , bipartite graph , degree sequence.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph, $|V(G)| = n, |E(G)| = m$, whose vertices are $v_1, v_2, \cdots, v_n$. The number of first neighbors of the vertex v_i is the degree of this vertex and is denoted by $d(v_i)$. A vertex of degree k will be referred to as a k-vertex. In addition, vertex of degree zero is called isolated vertex. The maximum and minimum degree of G is denoted by Δ and δ, respectively. $D(G) = \{d(v)|v \in V(G)\}$ is referred to as the degree set of G. The column-vector $(d(v_1), d(v_2), \cdots, d(v_n))^t$ is denoted by $d(G)$. The (0,1)-adjacency matrix $A(G)$ of the graph G is the square matrix of order n whose (i, j)-entry is equal to unity if $(v_i, v_j) \in E(G)$ and is zero if $(v_i, v_j) \notin E(G)$. The eigenvalues and eigenvectors of $A(G)$ are said to be the eigenvalues and eigenvectors of the graph G, respectively. The number of k-vertices is denoted by n_k. Evidently,

$$\sum_{k \geq 0} n_k = n \tag{1}$$

The graph G is said to be harmonic if there exists a constant λ, such that the equation

$$\lambda d(v_i) = \sum_{(v_i, v_j) \in E(G)} d(v_j) \tag{2}$$

holds for all $i = 1, 2, \cdots, n$.

It is easy to see that Equation (2) for $i = 1, 2, \cdots, n$ are equivalent to

$$A(G)d(G) = \lambda d(G) \tag{3}$$

i.e., the graph G is harmonic if and only if $d(G)$ is one of its eigenvectors. A graph satisfying Equation (2) or (3) will be referred to as a λ -harmonic graph. Clearly, λ is the eigenvalue corresponding to the eigenvector $d(G)$.

The harmonic graph G with $|D(G)| = 2$ is said to be semi-regular harmonic graph. Equi-bipartite harmonic graph is the bipartite graph $H = (X, Y; E)$ with $|X| = |Y|$.

Lemma 1.1 ([2]) *(a) Let the graph G' be obtained from the graph G by adding to it an arbitrary number of isolated vertices. Then G' is harmonic if and only if G is harmonic.*

(b) If G is a graph without isolated vertices, then G is λ-harmonic if and only if all its components are λ-harmonic.

(c) Every regular graph is λ-harmonic, with λ equal to the degree of the graph.

Lemma 1.2 ([2]) *Let G be a connected λ-harmonic graph.*

(a) λ is the greatest eigenvalue of G and its multiplicity is one.

(b) If $m > 0$ then $\lambda \geq 1$.

(c) $\lambda = 1$ if and only if $G = K_2$.

Let λ be a positive integer. Construct the tree T_λ in the following manner. T_λ has a total of $\lambda^3 - \lambda^2 + \lambda + 1$ vertices, of which one vertex is a $\lambda^2 - \lambda + 1$-vertex, $\lambda^2 - \lambda + 1$ vertices are λ-vertices and $(\lambda - 1)(\lambda^2 - \lambda + 1)$ vertices are 1-vertices, i.e., each λ-vertex is connected to (λ-1) 1-vertices and to the $\lambda^2 - \lambda + 1$-vertex.

Theorem 1.3 ([2]) *The tree T_2 is the unique connected non-regular 2-harmonic graph.*

In view of Lemma 1.1, it is reasonable to restrict our considerations to connected graphs. And bearing in mind Lemma 1.2 and

Theorem 1.3, in what follows we may assume that $\lambda \geq 3$. In this paper, we characterize the semi-regular harmonic graph and equi-bipartite harmonic graph, and the degree sequence of equi-bipartite 3-harmonic graph with $|D(G)| \leq 3$.

2 Semi-regular harmonic graph

The following lemmas are needed.

Lemma 2.1 *For a λ-harmonic graph G, $\delta \leq \lambda \leq \Delta$.*

Proof. By Lemma 1.2 and reference [3],

$$min\{r_1, r_2, \cdots, r_n\} \leq \lambda \leq max\{r_1, r_2, \cdots, r_n\},$$

where r_i is the sum of the entries of the i-th row of $A(G)$. And it is easy to see that $r_i = d(v_i), 1 \leq i \leq n$. Thus $\delta = min\{r_1, r_2, \cdots, r_n\} \leq \lambda \leq max\{r_1, r_2, \cdots, r_n\} = \Delta$. ∎

Lemma 2.2 *For a λ-harmonic graph G, G is regular if and only if $\delta = \lambda$.*

Proof. It clearly just needs to prove the sufficiency.

If $\delta = \lambda$, then there exist a vertex $v \in V(G)$ such that $d(v) = \delta = \lambda$. Let $v_1, v_2, \cdots, v_\lambda$ be the neighbors of v. By the definition of harmonic graph, $\lambda^2 = \lambda d(v) = d(v_1) + d(v_2) + \cdots + d(v_\lambda)$, and $d(v_i) \geq \lambda, 1 \leq i \leq \lambda$. Thus it is easy to see that $d(v_i) = \lambda \quad (1 \leq i \leq \lambda)$. For any $u \in V(G)$, there exists a path $t_1 t_2 \cdots t_s$ in G with $v = t_1, u = t_s$. By the above discussion, t_2 is a λ-vertex. Similarly, $t_3, \cdots, t_s = u$ are λ-vertices. Thus, G is a λ-regular graph. ∎

Lemma 2.3 *If x is a vertex of a λ-harmonic graph G, then $d(x) \le \lambda^2 - \lambda\delta + \delta$.*

Proof. Let $y_i, i = 1, 2, \cdots, d(x)$, be the vertices adjacent to x and let $z_{ij}, j = 1, 2, \cdots, d(y_i) - 1$ be the vertices adjacent to y_i other than x. Then by Equation (2),

$$\lambda d(y_i) = d(x) + \sum_{j=1}^{d(y_i)-1} d(z_{ij}) \ge d(x) + \delta(d(y_i) - 1),$$

from which $d(y_i) \ge [d(x) - \delta]/(\lambda - \delta)$. On the other hand,

$$\lambda d(x) = \sum_{i=1}^{d(x)} d(y_i) \ge d(x)(d(x) - \delta)/(\lambda - \delta),$$

from which $d(x) \le \lambda^2 - \lambda\delta + \delta$. ∎

Remark 1 *By the prove of Lemma 2.3, we know that $\Delta \le \lambda^2 - \lambda\delta + \delta$, and the equation holds if and only if $d(y_i) = [\Delta - \delta]/(\lambda - \delta)$ and $d(z_{ij}) = \delta$, $i = 1, 2, \cdots, \Delta, j = 1, 2, \cdots, d(y_i) - 1$.*

Lemma 2.4 *For all $d_1, d_2, \lambda \in Z^+, d_1 < d_2$, if d_1, d_2, λ satisfy:*
(i) $1 < d_1 < \lambda < d_2$,
(ii) $d_1(d_2 - \lambda) \mid {}_{(d_2 - d_1)}$,
(iii) $d_2(\lambda - d_1) \mid {}_{(d_2 - d_1)}$ *and*
(iv) $d_1(\lambda - 1)/(d_1 - 1) \le d_2 \le d_1(\lambda + 1 - d_1)$,
then there exists a λ-harmonic graph G, such that $D(G) = \{d_1, d_2\}$.

Proof. Clearly,

$$\lambda d_1 = \frac{d_1(d_2 - \lambda)d_1 + d_1(\lambda - d_1)d_2}{d_2 - d_1} = a_1 d_1 + a_2 d_2$$

$$\lambda d_2 = \frac{d_2(\lambda - d_1)d_2 + d_2(d_2 - \lambda)d_1}{d_2 - d_1} = b_1 d_1 + b_2 d_2$$

where,

$$a_1 = \frac{d_1(d_2 - \lambda)}{d_2 - d_1} \quad , \quad a_2 = \frac{d_1(\lambda - d_1)}{d_2 - d_1}$$

$$b_1 = \frac{d_2(d_2 - \lambda)}{d_2 - d_1} \quad , \quad b_2 = \frac{d_2(\lambda - d_1)}{d_2 - d_1}$$

By (ii) and (iii), we know $a_1, b_2 \in Z$. And $a_2 = d_1 - a_1, b_1 = d_2 - b_2$, therefore, $a_2, b_1 \in Z$.

It's easy to get $1 \leq b_1, b_2 \leq d_2 - 1$ and by (iv), we know $1 \leq a_1, a_2 \leq d_1 - 1$.

Let $g = lcm[a_2, b_1] = a_2 g_2, t = b_1 g_2 - 1$, construct bipartite graph $G = (V_1, V_2; E)$ as follow:

$V_1 = \{x_0, x_1, \cdots, x_t, u_0, u_1, \cdots, u_{g-1}\}$

$V_2 = \{y_0, y_1, \cdots, y_t, v_0, v_1, \cdots, v_{g-1}\}$

$E = \{(x_i, y_r) | i = 0, 1, \cdots, t; j = 0, 1, \cdots, a_1 - 1; r \equiv i + j (mod\ t + 1); 0 \leq r \leq t\} \cup \{(u_i, v_r) | i = 0, 1, \cdots, g - 1; j = 0, 1, \cdots, b_2 - 1; r \equiv i + j (mod\ g); 0 \leq r \leq g - 1\} \cup \{(x_p, v_q) | q = 0, 1, \cdots, g - 1; q \equiv r (mod\ g_2); 0 \leq r \leq g_2 - 1; p = b_1 r, \cdots, b_1(r + 1) - 1\} \cup \{(y_p, u_q) | q = 0, 1, \cdots, g - 1; q \equiv r (mod\ g_2); 0 \leq r \leq g_2 - 1; p = b_1 r, \cdots, b_1(r + 1) - 1\}$

In this case, for all $x_i, y_i, u_j, v_j (0 \leq i \leq t, 0 \leq j \leq g - 1)$, $d(x_i) = d(y_i) = d_1, d(u_j) = d(v_j) = d_2$, and x_i has a_1, a_2 neighbors in Y and V, respectively, y_i has a_1, a_2 neighbors in X and U, respectively, u_j has b_1, b_2 neighbors in Y and V, respectively, v_j has b_1, b_2 neighbors in X and U, respectively, where $X = \{x_0, x_1, \cdots, x_t\}, U = \{u_0, u_1, \cdots, u_{g-1}\}, Y = \{y_0, y_1, \cdots, y_t\}, V = \{v_0, v_1, \cdots, v_{g-1}\}$. It follows that for all $x \in V(G)$, $\lambda d(x) = \sum_{(x,y) \in E} d(y)$. Thus G is a λ-harmonic graph, and $D(G) = \{d_1, d_2\}$. ∎

In fact, the λ-harmonic graph G with $D(G) = \{d_1, d_2\}$ is not necessary a bipartite graph. The construction of non-bipartite graph is not difficult, too.

Lemma 2.5 *G is a λ-harmonic graph with $D(G) = \{d_1, d_2\}$, $d_1, d_2, \lambda \in Z^+, d_1 < d_2$,, then d_1, d_2, λ satisfy above* (i) *to* (iv).

Proof. By Equation (2), 1-vertices must be adjacent to λ-vertices, but clearly, $d_1 < \lambda < d_2$. Therefore (i) holds. And there exists vertices u and v such that $d(v) = d_1, d(u) = d_2$. Suppose vertex v has x d_1-vertices as its neighbors, and other $(d_1 - x)$ neighbors are d_2-vertices. Suppose vertex u has y d_2-vertices as its neighbors, and other $(d_2 - y)$ neighbors are d_1-vertices. By the definition of λ-harmonic graph,

$$\lambda d_1 = x d_1 + (d_1 - x) d_2 \quad , \quad \lambda d_2 = y d_2 + (d_2 - y) d_1$$

from which

$$x = \frac{d_1(d_2 - \lambda)}{d_2 - d_1} \quad , \quad y = \frac{d_2(\lambda - d_1)}{d_2 - d_1}$$

Because the number of neighbors are integers, (ii) and (iii) holds. Moreover G is connected and x, y are constants, then $1 \leq x = d_1(d_2 - \lambda)/(d_2 - d_1) \leq d_1 - 1$ from which (iv) holds. ∎

Combining Lemmas 2.4 and 2.5, we get:

Theorem 2.6 *For all $d_1, d_2, \lambda \in Z^+, d_1 < d_2$, if d_1, d_2, λ satisfy* (i) *to* (iv) *then there exists λ-harmonic graph G, such that $D(G) = \{d_1, d_2\}$. On the other hand, if G is a λ-harmonic graph with $D(G) = \{d_1, d_2\}$, $d_1, d_2, \lambda \in Z^+, d_1 < d_2$,, then d_1, d_2, λ satisfy* (i) *to* (iv).

3 Equi-bipartite harmonic graph

Let $G = (X, Y; E)$ be a bipartite graph with $|X| = |Y| = n, |E| = m$. The number of k-vertices in X and Y are denoted by n_{1k} and n_{2k}, respectively. The degree sequence of X is the row-vector $(d(v_1), d(v_2), \cdots, d(v_n))$ such that $X = \{v_1, v_2, \cdots, v_n\}$ and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$, denoted by $d(X)$. The definition of degree sequence of Y is similar.

Lemma 3.1 *$G = (X, Y; E)$ is a equi-bipartite harmonic graph, then following equations hold:*

$$\begin{cases} \sum_{k\geq 0} n_{1k} = n & (4) \\ \sum_{k\geq 0} n_{2k} = n & (5) \\ \sum_{k\geq 0} kn_{1k} = \sum_{k\geq 0} kn_{2k} & (6) \\ \sum_{k\geq 0}(\lambda k - k^2)n_{1k} = 0 & (7) \\ \sum_{k\geq 0}(\lambda k - k^2)n_{2k} = 0 & (8) \end{cases}$$

Proof. By $|X| = |Y| = n$, (4) and (5) obviously hold. And

$$m = \sum_{x\in X} d(x) = \frac{1}{\lambda} \sum_{x\in X} \lambda d(x) = \frac{1}{\lambda} \sum_{x\in X} \sum_{(x,y)\in E} d(y).$$

Note that $d(y)$ appears exactly $d(y)$ times, therefore

$$m = \sum_{x\in X} d(x) = \frac{1}{\lambda} \sum_{y\in Y} d^2(y). \tag{9}$$

Similarly,

$$m = \sum_{y\in Y} d(y) = \frac{1}{\lambda} \sum_{x\in X} d^2(x). \tag{10}$$

And

$$\sum_{x\in X} d(x) = \sum_{k\geq 0} kn_{1k} \quad , \quad \sum_{x\in X} d^2(x) = \sum_{k\geq 0} k^2 n_{1k},$$

$$\sum_{y\in Y} d(y) = \sum_{k\geq 0} kn_{2k} \quad , \quad \sum_{y\in Y} d^2(y) = \sum_{k\geq 0} k^2 n_{2k}.$$

Thus by Equations (9) and (10),

$$\lambda \sum_{k\geq 0} kn_{1k} = \lambda \sum_{k\geq 0} kn_{2k} = \sum_{k\geq 0} k^2 n_{1k} = \sum_{k\geq 0} k^2 n_{2k},$$

from which Equations (6), (7) and (8) are obtained. ∎

Lemma 3.2 *Let $G = (X, Y; E)$ be a equi-bipartite harmonic graph with $|D(G)| = 2$, then $d(X) = d(Y)$ and $\lambda \notin D(G)$, therefore $\delta \geq 2$.*

Proof. Let $D(G) = \{j, k\}$.

If G has λ-vertices, we may assume $j = \lambda$. Then by Equations (7) and (8),

$$(\lambda k - k^2)n_{tk} + (\lambda\lambda - \lambda^2)n_{t\lambda} = 0 \quad , \quad (t = 1, 2)$$

from which $n_{tk} = 0, t = 1, 2$, contrary to the fact $|D(G)| = 2$. By Equation (2), 1-vertices must be adjacent to λ-vertices, therefore $\delta \geq 2$.

We may assume $j < \lambda < k$, combining Equations (4) and (7), (5) and (8), we get

$$n_{tk} = \frac{j(\lambda - j)n}{j(\lambda - j) - k(\lambda - k)} \tag{11}$$

$$n_{tj} = \frac{k(\lambda - k)n}{k(\lambda - k) - j(\lambda - j)} \quad , \quad t = 1, 2 \tag{12}$$

Obviously, $n_{1k} = n_{2k}, n_{1j} = n_{2j}$. Thus, the degree sequences of X and Y are the same.

∎

In fact, by the prove of Lemma 3.2, if G is simple graph with $|D(G)| = 2$, then G has no λ-vertices, therefore $\delta \geq 2$.

Lemma 3.3 *Let $G = (X, Y; E)$ be a equi-bipartite harmonic graph with $|D(G)| = \{i, j, k\}$ and $\frac{i}{j} + \frac{j}{k} + \frac{k}{i} \neq \frac{k}{j} + \frac{i}{k} + \frac{j}{i}$, then $d(X) = d(Y)$.*

Proof. We may assume $i < j < k$. Combining Equations (4) and (7), (5) and (8), we get

$$n_{tk} = \frac{xn - (x - y)n_{tj}}{x - z}, \tag{13}$$

$$n_{ti} = \frac{zn - (z - y)n_{tj}}{z - x} \quad , \quad t = 1, 2 \tag{14}$$

where $x = i(\lambda - i), y = j(\lambda - j), z = k(\lambda - k)$. Substituting Equation (6) for (13) and (14), we get $n_{1j} = n_{2j}$, therefore $n_{1i} = n_{2i}, n_{1k} = n_{2k}$. Thus, $d(X) = d(Y)$. ∎

Theorem 3.4 *For a equi-bipartite 3-harmonic graph $G = (X, Y; E)$ with $|D(G)| \leq 3$, one of the following must hold:*

(i) *If $|D(G)| = 1$, then G is a 3-regular graph with $d(X) = d(Y) = (3, 3, \cdots, 3), n_{13} = n_{23} \geq 3$*

(ii) *If $|D(G)| = 2$, then $d(X) = d(Y) = (2, \cdots, 2, 4, \cdots, 4)$ where $n_{12} = n_{22} = \frac{2n}{3}, n_{14} = n_{24} = \frac{n}{3}, n \geq 6$*

(iii) *If $|D(G)| = 3$, then*

1° *$d(X) = d(Y) = (2, \cdots, 2, 3, \cdots, 3, 4, \cdots, 4)$ where $n_{i4} = t, n_{i2} = 2t, n_{i3} = n - 3t, i = 1, 2; t \in Z^+, n \geq 5$*

2° *$d(X) = d(Y) = (1, \cdots, 1, 3, \cdots, 3, 5, \cdots, 5)$ where $n_{i5} = t, n_{i1} = 5t, n_{i3} = n - 6t, i = 1, 2; t \in Z^+, n \geq 11$*

Proof. By Lemmas 2.2, 3.2, and 2.4, (i) and (ii) obviously hold. The smallest graphs with $|D(G)| = 1$ and $|D(G)| = 2$ are shown in Fig. 3(1), (2) and (3), respectively.

We now suppose $|D(G)| = 3$:

Case 1: If $\delta = 2$, then $\Delta \leq 5$.

Case 1.1: If $D(G) = \{2,3,4\}$, then $n_{i4} = t, n_{i2} = 2t, n_{i3} = n - 3t, i = 1,2; t \in Z^+$ by Lemma 3.1. By

$$3 \times 2 = 2 + 4 = 3 + 3 \quad , \quad 3 \times 3 = 3 + 3 + 3 = 2 + 3 + 4,$$

$$3 \times 4 = 3 + 3 + 3 + 3 = 2 + 2 + 4 + 4 = 2 + 3 + 3 + 4,$$

there must exists 3-vertex adjacent to 2-vertex, otherwise G is not connected. Therefore $n_{23} = n_{13} \geq 2, n \geq 5$. The smallest graph is shown in Fig. 3(4).

Case 1.2: If $D(G) = \{2,3,5\}$, then $n_{i5} = t, n_{i1} = 5t, n_{i3} = n - 6t, i = 1,2; t \in Z^+$ by Lemma 3.1. By

$$3{\times}2 = 3{+}3 \quad , \quad 3{\times}3 = 3{+}3{+}3 = 2{+}2{+}5 \quad , \quad 3{\times}5 = 3{+}3{+}3{+}3{+}3,$$

each 5-vertices of X adjacent to five 3-vertices of Y, and these 3-vertices are adjacent to 2-vertices of X. These 2-vertices are adjacent to 3-vertices of Y, by the partition these 3-vertices are adjacent to 5-vertices of X (see Figure 1).

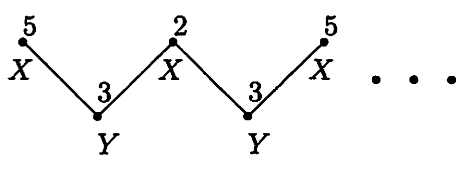

Figure 1

Thus any 5-vertices of X and Y are not in the same component of G, a contradiction.

Case 1.3: If $D(G) = \{2, 4, 5\}$, then $\delta = 2, \Delta = 5$. By Remark 1, G must have 3-vertices. Contradiction.

Case 2: If $\delta = 1$, then $\Delta \leq 6, 3 \in D(G)$.

Case 2.1: If $D(G) = \{1, 3, 4\}$, then $n_{i4} = t, n_{i1} = 2t, n_{i3} = n - 3t, i = 1, 2; t \in Z^+$ by Lemma 3.1. By

$$3 \times 3 = 3 + 3 + 3 = 1 + 4 + 4 \quad , \quad 3 \times 4 = 3 + 3 + 3 + 3$$

each 4-vertices of X adjacent to four 3-vertices of Y, and these 3-vertices are adjacent to 1-vertex and other 4-vertex of X (see Figure 1).

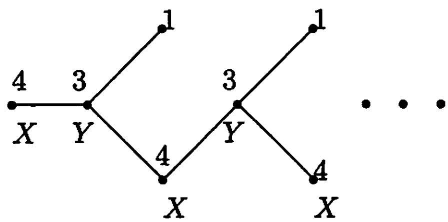

Figure 2

Thus any 4-vertices of X and Y are not in the same component of G, a contradiction.

Case 2.2: If $D(G) = \{1, 3, 5\}$, then $n_{i5} = t, n_{i1} = 5t, n_{i3} = n - 6t, i = 1, 2; t \in Z^+$ by Lemma 3.1. And $n_{i1} \leq n_{i3}$ because a 3-vertex is at most adjacent to one 1-vertex. Thus $n \geq 11t \geq 11$. The smallest graph is shown in Fig. 3(5).

Case 2.3: G with $D(G) = \{1, 3, 6\}$ does not exist, because $3 \times 3 = 6 + x_1 + x_2$ has no solution in $D(G)$.

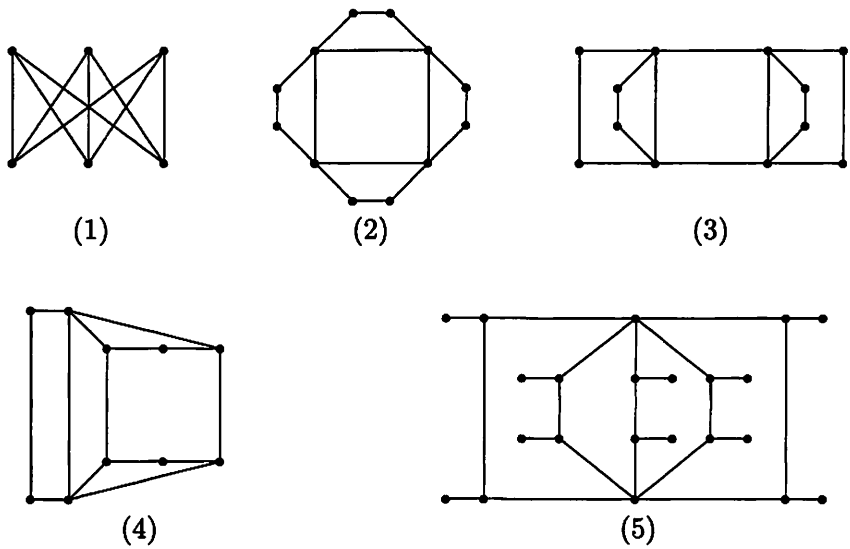

Figure 3

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