A generalization of Chu's identity

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Abstract

By means of partial fraction decomposition, the purpose of this paper is to obtain a generalization of an algebraic identity which was given by Chu in *The Electronic J. Comb.*, 11(2004), #N15.

AMS classification: 05A10, 05A19, 33D15

Key Words: Partial fraction decomposition, Identity, q-analogue

1 Natation and Introduction

Let H_n be the Harmonic number which is defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

In [2], by the WZ method, the following identity was confirmed successfully.

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \{1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k\} = 0. \qquad (1)$$

Ahlgren and Ono [1] have shown that Beukers' conjecture is implied by this beautiful binomial identity. See [3] or [1, Theorem 7]) for Beukers' conjecture.

Recently, by means of partial fraction decomposition, W.-C. Chu [4] gave the following beautiful algebraic identity which is a generalization of identity (1): Let x be an indeterminate. There holds

$$\begin{aligned}\frac{x(1-x)_n^2}{(x)_{n+1}^2} &= \frac{1}{x}+\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\left\{\frac{-k}{(x+k)^2}\right.\\ &\quad\left.+\frac{1+2kH_{n+k}+2kH_{n-k}-4kH_k}{x+k}\right\},\end{aligned} \tag{2}$$

where $(x)_0=1$ and $(x)_n=\prod_{k=0}^{n-1}(x+k)$.

The purpose of this paper is to give a generalization of (2). As applications, we obtain a number of interesting identities.

Throughout the paper, let $f(x)$ be any polynomial of degree $\leq 2n$.

2 Main Results

Theorem 2.1 *Let a_0, a_1, a_2, $\ldots$, a_n be a real sequence, and $a_i \neq a_j$, when $i \neq j$, $i, j = 1, 2, \ldots, n$, and $f(a_i) \neq 0$, $(i = 0, 1, \ldots, n)$. Then*

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(a_0-a_k)f(-a_k)}{(x+a_k)\prod\limits_{\substack{j=0\\ j\neq k}}^{n}(a_j-a_k)^2}\left\{\frac{1}{x+a_k}+\frac{1}{(a_0-a_k)}\right.\\ &\qquad\left.+\frac{f'(-a_k)}{f(-a_k)}-\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{a_j-a_k}\right\}\\ &= \frac{f(x)}{(x+a_0)(x+a_1)^2\ldots(x+a_n)^2}.\end{aligned} \tag{3}$$

Proof. Applying the partial fraction decomposition, we have

$$g(x) := \frac{f(x)}{(x+a_0)(x+a_1)^2(x+a_2)^2\ldots(x+a_n)^2}$$

$$= \frac{A}{x+a_0} + \sum_{k=1}^{n} \left\{ \frac{B_k}{(x+a_k)^2} + \frac{C_k}{x+a_k} \right\}.$$

In order to determine the coefficients A and $\{B_k,\ C_k\}$, we consider the way of taking limit. First, we can easily compute the coefficients A and $\{B_k\}$ as follows:

$$A = \lim_{x \to -a_0} (x+a_0)g(x) = \lim_{x \to -a_0} \frac{f(x)}{\prod_{j=1}^{n} (x+a_j)^2} = \frac{f(-a_0)}{\prod_{j=1}^{n} (a_j - a_0)^2};$$

$$\begin{aligned} B_k &= \lim_{x \to -a_k} (x+a_k)^2 g(x) = \lim_{x \to -a_k} \frac{(x+a_k)^2 (x+a_0) f(x)}{\prod_{j=0}^{n} (x+a_j)^2} \\ &= \frac{(a_0 - a_k) f(-a_k)}{\prod_{\substack{j=0 \\ j \neq k}}^{n} (a_j - a_k)^2}. \end{aligned}$$

According to the L'Hôspital rule, the coefficients $\{C_k\}$ can be obtained:

$$C_k = \lim_{x \to -a_k} (x+a_k) \left\{ g(x) - \frac{B_k}{(x+a_k)^2} \right\}$$

$$= \lim_{x \to -a_k} \frac{(x+a_k)^2 g(x) - B_k}{x+a_k}$$

$$= \lim_{x \to -a_k} \frac{d}{dx} \left\{ (x+a_k)^2 g(x) - B_k \right\}$$

$$= \lim_{x \to -a_k} \frac{d}{dx} \frac{(x+a_k)^2 (x+a_0) f(x)}{\prod_{j=0}^{n} (x+a_j)^2}$$

$$= \lim_{x \to -a_k} \frac{d}{dx} \frac{(x+a_0) f(x)}{\prod_{\substack{j=0 \\ j \neq k}}^{n} (x+a_j)^2}$$

$$= \frac{f(-a_k)}{\prod_{\substack{j=0 \\ j \neq k}}^{n} (a_j - a_k)^2} \left[1 + \frac{(a_0 - a_k) f'(-a_k)}{f(-a_k)} - \sum_{\substack{j=0 \\ j \neq k}}^{n} \frac{2(a_0 - a_k)}{a_j - a_k} \right].$$

Hence we have

$$\frac{f(x)}{(x+a_0)(x+a_1)^2(x+a_2)^2\dots(x+a_n)^2}$$

$$= \frac{f(-a_0)}{(a_1-a_0)^2(a_2-a_0)^2\dots(a_n-a_0)^2}\cdot\frac{1}{x+a_0}+\sum_{k=1}^{n}\frac{f(-a_k)}{\prod\limits_{\substack{j=0\\ j\neq k}}^{n}(a_j-a_k)^2}$$

$$\times\left\{\frac{a_0-a_k}{(x+a_k)^2}+\frac{1+(a_0-a_k)\frac{f'(-a_k)}{f(-a_k)}-(a_0-a_k)\sum\limits_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{a_j-a_k}}{x+a_k}\right\},$$

that is

$$\sum_{k=0}^{n}\frac{(a_0-a_k)f(-a_k)}{(x+a_k)\prod\limits_{\substack{j=0\\ j\neq k}}^{n}(a_j-a_k)^2}\left\{\frac{1}{x+a_k}+\frac{1}{(a_0-a_k)}+\frac{f'(-a_k)}{f(-a_k)}-\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{a_j-a_k}\right\}$$

$$=\frac{f(x)}{(x+a_0)(x+a_1)^2\dots(x+a_n)^2}.$$

the proof of the theorem is completed. □

Differentiating both sides of equation (3) with respect to x, then we have

Theorem 2.2

$$\sum_{k=0}^{n}\frac{(a_0-a_k)f(-a_k)}{(x+a_k)^2\prod\limits_{\substack{j=0\\ j\neq k}}^{n}(a_j-a_k)^2}\left\{\frac{2}{x+a_k}+\frac{1}{(a_0-a_k)}+\frac{f'(-a_k)}{f(-a_k)}-\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{a_j-a_k}\right\}$$

$$= \frac{-1}{(x+a_0)^2\dots(x+a_n)^2}\left\{(x+a_0)f'(x)+f(x)-(x+a_0)f(x)\right.$$
$$\left.\times\sum_{i=0}^{n}\frac{2}{x+a_i}\right\} \tag{4}$$

Differentiating both sides of equation (3) twice with respect to x, then we have

Theorem 2.3

$$\sum_{k=0}^{n}\frac{(a_0-a_k)f(-a_k)}{(x+a_k)^3\prod\limits_{\substack{j=0\\j\neq k}}^{n}(a_j-a_k)^2}\left\{\frac{3}{x+a_k}+\frac{1}{(a_0-a_k)}+\frac{f'(-a_k)}{f(-a_k)}\right.$$
$$\left.-\sum_{\substack{j=0\\j\neq k}}^{n}\frac{2}{a_j-a_k}\right\}$$
$$= \frac{1}{2(x+a_0)^2\dots(x+a_n)^2}\left\{(x+a_0)f''(x)+2f'(x)-2\left((x+a_0)\right.\right.$$
$$\times f'(x)+f(x)\Big)\sum_{i=0}^{n}\frac{2}{x+a_i}+(x+a_0)f(x)\left(\left(\sum_{i=0}^{n}\frac{2}{x+a_i}\right)^2\right.$$
$$\left.\left.+\sum_{i=0}^{n}\frac{2}{(x+a_i)^2}\right)\right\}. \tag{5}$$

3 Some Applications

As a direct applications of Theorems 2.1, and 2.2 and 2.3, according to different choices of a_k, $f(x)$, we can get a number of interesting identities.

3.1 The case $a_k = k$

Taking $a_k = k$ and applying $\sum\limits_{\substack{j=0\\j\neq k}}^{n}\frac{1}{j-k} = H_{n-k} - H_k$, we have

Corollary 3.1

$$\sum_{k=0}^{n} \frac{f(-a_k)}{x+k}\binom{n}{k}^2 \left\{\frac{-k}{x+k} + 1 - k\frac{f'(-k)}{f(-k)} + 2kH_{n-k} - 2kH_k\right\}$$
$$= \frac{n!^2}{x(x+1)^2\dots(x+n)^2} f(x),$$

$$\sum_{k=0}^{n} \frac{f(-a_k)}{(x+k)^2}\binom{n}{k}^2 \left\{\frac{-2k}{x+k} + 1 - k\frac{f'(-k)}{f(-k)} + 2kH_{n-k} - 2kH_k\right\}$$
$$= \frac{-n!^2}{x^2(x+1)^2\dots(x+n)^2}\left\{xf'(x) + f(x) - xf(x)\sum_{i=0}^{n}\frac{2}{x+i}\right\}$$

and

$$\sum_{k=0}^{n} \frac{f(-a_k)}{(x+k)^3}\binom{n}{k}^2 \left\{\frac{-3k}{x+k} + 1 - k\frac{f'(-k)}{f(-k)} + 2kH_{n-k} - 2kH_k\right\}$$
$$= \frac{n!^2}{2x^2(x+1)^2\dots(x+n)^2}\{xf''(x) + 2f'(x) - 2\left(xf'(x) + f(x)\right)$$
$$\times \sum_{i=0}^{n}\frac{2}{x+i} + xf(x)\left(\left(\sum_{i=0}^{n}\frac{2}{x+i}\right)^2 + \sum_{i=0}^{n}\frac{2}{(x+i)^2}\right)\Bigg\}.$$

Example 3.2 *Taking $f(x) = 1$ in Corollary 3.1, then we have*

$$\sum_{k=0}^{n} \frac{1}{x+k}\binom{n}{k}^2 \left\{\frac{x}{x+k} + 2kH_{n-k} - 2kH_k\right\}$$
$$= \frac{n!^2}{x(x+1)^2\dots(x+n)^2}, \tag{6}$$

$$\sum_{k=0}^{n} \frac{1}{(x+k)^2}\binom{n}{k}^2 \left\{\frac{x-k}{x+k} + 2kH_{n-k} - 2kH_k\right\}$$
$$= \frac{-n!^2}{x^2(x+1)^2\dots(x+n)^2}\left\{1 - x\sum_{i=0}^{n}\frac{2}{x+i}\right\} \tag{7}$$

and

$$\sum_{k=0}^{n} \frac{1}{(x+k)^3} \binom{n}{k}^2 \left\{ \frac{x-2k}{x+k} + 2kH_{n-k} - 2kH_k \right\}$$
$$= \frac{n!^2}{2x^2(x+1)^2 \dots (x+n)^2} \left\{ -2 \sum_{i=0}^{n} \frac{2}{x+i} \right.$$
$$\left. +x \left(\sum_{i=0}^{n} \frac{2}{x+i} \right)^2 + x \sum_{i=0}^{n} \frac{2}{(x+i)^2} \right\}.$$

Taking $x = 1$ in (6) and (7), we obtain the following results.

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}^2 \left\{ \frac{1}{k+1} + 2kH_{n-k} - 2kH_k \right\}$$
$$= \frac{1}{(n+1)^2}, \tag{8}$$

and

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2} \binom{n}{k}^2 \left\{ \frac{1-k}{k+1} + 2kH_{n-k} - 2kH_k \right\}$$
$$= \frac{1}{(n+1)^2} \left\{ 2H_n - 1 \right\}. \tag{9}$$

Example 3.3 *Taking $f(x) = (1-x)_n^2 = (1-x)^2(2-x)^2 \dots (n-x)^2$, then*

$$f(-k) = (k+1)^2(k+2)^2 \dots (k+n)^2 = \left(\frac{(n+k)!}{k!} \right)^2,$$

$$f'(x) = -2(1-x)^2(2-x)^2 \dots (n-x)^2 \sum_{j=1}^{n} \frac{1}{j-x},$$

$$f''(x) = (1-x)^2(2-x)^2 \dots (n-x)^2 \left\{ \left(\sum_{j=1}^{n} \frac{2}{j-x} \right)^2 - \sum_{j=1}^{n} \frac{2}{(j-x)^2} \right\}.$$

Hence Theorem 2.1 reduces to Chu's identity. Theorems 2.2 and 2.3 reduce to the following identities:

$$\sum_{k=0}^{n} \frac{1}{(x+k)^2} \binom{n}{k}^2 \binom{n+k}{k}^2 \Big\{ -\frac{2k}{x+k} + 1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k \Big\}$$
$$= \frac{-(1-x)^2(2-x)^2\dots(n-x)^2}{x^2(x+1)^2\dots(x+k)^2} \left\{ -2x\sum_{j=1}^{n}\frac{1}{j-x} + 1 - x\sum_{i=0}^{n}\frac{2}{x+i} \right\} \quad (10$$

and

$$\sum_{k=0}^{n} \frac{1}{(x+k)^3} \binom{n}{k}^2 \binom{n+k}{k}^2 \Big\{ -\frac{3k}{x+k} + 1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k \Big\}$$
$$= \frac{(1-x)^2(2-x)^2\dots(n-x)^2}{2x^2(x+1)^2\dots(x+k)^2} \left\{ x\left(\sum_{j=1}^{n}\frac{2}{j-x}\right)^2 - x\sum_{j=1}^{n}\frac{2}{(j-x)^2} \right.$$
$$-2\sum_{j=1}^{n}\frac{2}{j-x} + 2x\left(\sum_{j=1}^{n}\frac{2}{j-x}\right)\left(\sum_{i=0}^{n}\frac{2}{x+i}\right) - 2\sum_{i=0}^{n}\frac{2}{x+i}$$
$$\left. +x\left(\sum_{i=0}^{n}\frac{2}{x+i}\right)^2 - x\left(\sum_{i=0}^{n}\frac{2}{x+i}\right) \right\}, \quad (11)$$

respectively.

3.2 The case $a_k = \frac{1-q^k}{1-q}$

Suppose $[k] = \frac{1-q^k}{1-q}$ and take $a_k = \frac{1-q^k}{1-q}$ in Theorems 2.1, and 2.2 and 2.3. Then we have

Corollary 3.4

$$\sum_{k=0}^{n} \frac{f(-[k])}{q^{2nk-k^2-k}(x+[k])} \begin{bmatrix} n \\ k \end{bmatrix}^2 \Big\{ -\frac{[k]}{x+[k]} + 1 - [k]\frac{f'(-[k])}{f(-[k])}$$

$$+[k]\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{[j]-[k]}\Bigg\}$$

$$=\frac{[1]^2[2]^2\dots[n]^2}{x^2(x+[1])^2\dots(x+[n])^2}f(x). \tag{12}$$

$$\sum_{k=0}^{n}\frac{f(-[k])}{q^{2nk-k^2-k}(x+[k])^2}\begin{bmatrix} n\\ k\end{bmatrix}^2\Bigg\{-\frac{2[k]}{x+[k]}+1-[k]\frac{f'(-[k])}{f(-[k])}$$

$$+[k]\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{[j]-[k]}\Bigg\}$$

$$=\frac{-[1]^2[2]^2\dots[n]^2}{x^2(x+[1])^2\dots(x+[n])^2}\left\{xf'(x)+f(x)-xf(x)\sum_{i=0}^{n}\frac{2}{x+[i]}\right\} \tag{13}$$

and

$$\sum_{k=0}^{n}\frac{f(-[k])}{q^{2nk-k^2-k}(x+[k])^3}\begin{bmatrix} n\\ k\end{bmatrix}^2\Bigg\{-\frac{3[k]}{x+[k]}+1-[k]\frac{f'(-[k])}{f(-[k])}$$

$$+[k]\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2}{[j]-[k]}\Bigg\}$$

$$=\frac{[1]^2[2]^2\dots[n]^2}{2x^2(x+[1])^2\dots(x+[n])^2}\{xf''(x)+2f'(x)-2(xf'(x)+f(x))$$

$$\times\sum_{i=0}^{n}\frac{2}{x+[i]}+xf(x)\left(\left(\sum_{i=0}^{n}\frac{2}{x+[i]}\right)^2+\sum_{i=0}^{n}\frac{2}{(x+[i])^2}\right)\Bigg\}. \tag{14}$$

If we take $f(x)=([1]-qx)^2([2]-q^2x)^2\dots([n]-q^nx)^2$ in (12), then this reduces the q-analog of Chu's identity (2):

$$\frac{([1]-qx)^2([2]-q^2x)^2\dots([n]-q^nx)^2}{x(x+[1])^2(x+[2])^2\dots(x+[n])^2}$$

$$= \frac{1}{x} + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 q^{k(k+1-2n)} \left(\frac{-[k]}{(x+[k])^2} + \frac{1 + 2[k] \sum_{j=1}^{n} \frac{q^j}{[k+j]} + 2[k] \sum_{j=0\ j\neq k}^{n} \frac{1-q}{q^k - q^j}}{x+[k]} \right), \quad (15)$$

where $[k] = \frac{1-q^k}{1-q}$.

When $x = 0$, we obtain the q-analog of identity (1):

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \frac{1-q^k}{q^{k(2n-k)}} \left(\frac{q^k}{1-q^k} + 2\sum_{j=1}^{n+k} \frac{q^j}{1-q^j} + 2\sum_{j=1}^{n-k} \frac{1}{1-q^j} - 4\sum_{j=1}^{k} \frac{q^j}{1-q^j} \right)$$
$$= q^{2\binom{n+1}{2}} - 1.$$

3.3 The case: $a_k = -(y+k)^2$

In Theorem 2.1, taking $x \to x^2$, $a_k = -(y+k)^2$ and $f(x) = 1$, we get the following identity.

$$\sum_{k=0}^{n} \frac{(x^2 - (y+k)^2)}{x^2 - y^2} \frac{\binom{x+y+n}{n-k}^2 \binom{x-y-k-1}{n-k}^2 \binom{x+y+k-1}{k}^2 \binom{x-y}{k}^2}{\binom{2y+2k-1}{k}^2 \binom{2y+n+k}{n-k}^2}$$
$$\times \left\{ \frac{k(2y+k)}{x^2 - (y+k)^2} + 1 - k(2y+k) \sum_{\substack{j=0 \\ j \neq k}}^{n} \frac{2}{(k-j)(2y+k+j)} \right\}$$
$$= 1 \quad (16)$$

Acknowledgements

We would like to thank the anonymous referee for his valuable suggestions. This research is supported by the National Natural Science Foundation of China (Grant No. 10471016), the Natural Science Foundation of Henan Province (Grant No. 0511010300) and the

Natural Science Foundation of the Education Department of Henan Province (Grant No.200510482001).

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