Identities Involving the Fibonacci Polynomials

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bstract

In this paper, by using the generating functions of Fibonacci polynomial sequences nd their partial derivatives, we work out some identities involving the Fibonacci olynomials. As their primary applications, we obtain several identities involving the ibonacci numbers and Lucas numbers.

eywords: Fibonacci number; Lucas number; Generating Function.

. Introduction

Define the generalized Fibonacci polynomial sequence $\{F_n(x,y)\}$ by the econd-order linear recurrence relation

$$F_{n+2}(x,y) = xF_{n+1}(x,y) + yF_n(x,y), \tag{1}$$

r $n \geq 0$ with $F_0(x,y) = 0$, $F_1(x,y) = 1$. Let

$$\alpha = \frac{x + \sqrt{x^2+4y}}{2} \; and \; \beta = \frac{x - \sqrt{x^2+4y}}{2}$$

enote the roots of the characteristic polynomials of the sequence $\{F_n(x,y)\}$. this case, the terms of the sequence $\{F_n(x,y)\}$ can be expressed as

$$F_n(x,y) = \frac{1}{\alpha - \beta}\{\alpha^n - \beta^n\},$$

r $n \geq 0$. Considering the generating function of $\{F_n(x,y)\}$: $W(t,x,y) = \sum_{n=0}^{\infty} F_n(x,y)t^n$ and $G(t,x,y) = \frac{W(t,x,y)}{t}$, we can deduce from (1) that

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$$W(t,x,y)=\sum_{n=0}^{\infty}F_n(x,y)t^n=\frac{t}{1-xt-yt^2},$$

$$G(t,x,y)=\sum_{n=0}^{\infty}F_{n+1}(x,y)t^n=\frac{1}{1-xt-yt^2}. \tag{2}$$

Since by (2),

$$\begin{aligned}\frac{1}{1-xt-yt^2}&=\sum_{m=0}^{\infty}(x+yt)^m t^m=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n}y^n x^{m-n}t^{m+n}\\&=\sum_{m=0}^{\infty}\sum_{n=m}^{2m}\binom{m}{n-m}x^{2m-n}y^{n-m}t^n\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\binom{n-m}{m}x^{n-2m}y^m t^n.\end{aligned} \tag{3}$$

By (2) and (3), we can get

$$F_{n+1}(x,y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n-m}{m}x^{n-2m}y^m. \tag{4}$$

As we know, if $y=1$, the sequence $\{F_n(x,1)\}$ is called the Fibonacci polynomial sequence; when $x=1$, $\{F_n(1,y)\}$ can be considered as another form of the Fibonacci polynomial [1, 3]; but if $x=1$, $y=1$, $F_n(1,1)$ is known as the Fibonacci number; when $x=1$ and $y=1$ with $F_0(x,y)=2, F_1(x,y)=1$, $F_n(1,1)$ becomes the Lucas number. For convenience, we denote them by $F_n(x), f_n(y), F_n, L_n$, respectively.

These sequences play a very important role in the studies of the theory and application of mathematics. Hence, the various properties of them were investigated by many authors. For example, Y. Yuan and W. Zhang [4] obtained a calculating formula involving the Fibonacci polynomials

$$\sum_{a_1+\cdots+a_k=n}F_{a_1+1}(x)\cdots F_{a_k+1}(x)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m}. \tag{5}$$

where the summation is over all weak compositions $a_1+a_2+\cdots+a_k=n$, $k>0$, $n\geq 0$, and $[\frac{n}{2}]$ denotes the greatest integer not exceeding $\frac{n}{2}$. Later, W. Zhang [6] proved a general summations

$$\sum_{a_1+\cdots+a_{k+1}=n}F_{m(a_1+1)}\cdots F_{m(a_{k+1}+1)}=(-i)^{mn}\frac{F_m^{k+1}}{2^k k!}U_{n+k}^{(k)}\left(\frac{i^m}{2}L_m\right), \tag{6}$$

where k, $m > 0$, $n \geq 0$, i is the square root of -1 and $U_n^k(x)$ denotes the kth derivative of the Chebyshev polynomial of the second kind $U_n(x)$ with respect to x.

In this paper, by using the generating functions of Fibonacci polynomial sequences and their partial derivatives, we work out some identities, which can be considered as the generalized forms of (5). As their primary applications, we obtain several identities involving the Fibonacci numbers and Lucas numbers.

2. Summation formulas for generalized Fibonacci polynomials

In this section, we extend the identity (5) to another more general form.

Theorem 1. Let $\{F_n(x,y)\}$ be defined by (1). Then, for any positive integers k and n, we can obtain the following calculating formula

$$\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}F_{a_l+1}(x,y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m}y^m, \quad (7)$$

where $a_r \geq 0$, for $r = 1, 2, \cdots, k, and$ $[\frac{n}{2}]$ denotes the greatest integer not exceeding $\frac{n}{2}$.

Proof : Let $\frac{\partial^{(l_1+l_2)}G(t,x,y)}{\partial x^{l_1}\partial y^{l_2}}$ denotes the l_1th partial derivative of $G(t,x,y)$ for x, and the l_2th for y; $F_n^{(l_1,l_2)}(x,y)$ denotes the l_1th partial derivative of $F_n(x,y)$ for x, and the l_2th for y. Then by induction from (2), we can easily deduce that

$$\begin{aligned}\frac{\partial G^{2l+1}(t,x,y)}{\partial x^{l+1}\partial y^l} &= \frac{(2l+1)!t^{1+3l}}{(1-xt-yt^2)^{2l+2}}=\sum_{n=0}^{\infty}F_{n+1}^{(l+1,l)}(x,y)t^n \\ &= \sum_{n=l}^{\infty}F_{n+2l+2}^{(l+1,l)}(x,y)t^{n+2l+1}=\sum_{j=0}^{\infty}F_{j+3l+2}^{(l+1,l)}(x,y)t^{j+3l+1}; \quad (8)\\ \frac{\partial G^{2l}(t,x,y)}{\partial x^l\partial y^l} &= \frac{(2l)!t^{3l}}{(1-xt-yt^2)^{2l+1}}=\sum_{n=0}^{\infty}F_{n+1}^{(l,l)}(x,y)t^n \\ &= \sum_{n=l}^{\infty}F_{n+2l+1}^{(l,l)}(x,y)t^{n+2l}=\sum_{j=0}^{\infty}F_{j+3l+1}^{(l,l)}(x,y)t^{j+3l}. \quad (9)\end{aligned}$$

Then we can find that

$$\sum_{n=0}^{\infty} t^n \sum_{a_1+\cdots+a_k=n} \prod_{l=1}^{k} F_{a_l+1}(x,y) = (\sum_{n=0}^{\infty} F_{n+1}(x,y)t^n)^k$$
$$= \frac{1}{(1-xt-yt^2)^k} = I. \quad (10)$$

When $k = 2l + 2 (l \geq 0)$, we get from (8)

$$I = \frac{1}{(2l+1)!t^{1+3l}} \frac{\partial^{2l+1} G(t,x,y)}{\partial x^{l+1} \partial y^l} = \frac{1}{(2l+1)!} \sum_{n=0}^{\infty} F_{n+3l+2}^{(l+1,l)}(x,y)t^n. \quad (11)$$

Extracting the coefficients of t^n on both sides of (10) and (11), we obtain the identity

$$\sum_{a_1+\cdots+a_k=n} \prod_{l=1}^{k} F_{a_l+1}(x,y) = \frac{1}{(2l+1)!} F_{n+3l+2}^{(l+1,l)}(x,y). \quad (12)$$

From (4), we can deduce that $F_{j+3l+2}^{(l+1,l)}(x,y)$ is

$$F_{n+3l+2}^{(l+1,l)}(x,y) = \frac{\partial^{2l+1}}{\partial x^{l+1} \partial y^l} \left(\sum_{m=0}^{[\frac{n+3l+1}{2}]} \binom{n+3l+1-m}{m} x^{n+3l+1-2m} y^m \right)$$
$$= \sum_{m=l}^{[\frac{n+2l}{2}]} \frac{(n+3l+1-m)!}{(m-l)!(n+2l-2m)!} x^{n+2l-2m} y^{m-l}$$
$$= \sum_{m=0}^{[\frac{n}{2}]} \frac{(n+2l+1-m)!}{m!(n-2m)!} x^{n-2m} y^m$$
$$= \sum_{m=0}^{[\frac{n}{2}]} \frac{(n+k-1-m)!}{m!(n-2m)!} x^{n-2m} y^m. \quad (13)$$

Combining (12) and (13), we can have the identity (7) .

The proof when $k = 2l+1$ $(l \geq 0)$ is similar to that when $k = 2l+2$ $(l \geq 0)$, and therefore is omitted here.

Letting y=1 in Theorem 1, we may immediately deduce that

Corollary 1.[4] For any positive integers k and n, we have

$$\sum_{a_1+\cdots+a_k=n} \prod_{l=1}^{k} F_{a_l+1}(x) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} x^{n-2m}.$$

Setting $x = 1$ in Theorem 1, we can get

Corollary 2. For any positive integers k and n, we obtain

$$\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k} f_{a_l+1}(y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}y^m.$$

Let $y \to 0$ in Corollary 2, then $\alpha \to 1$, $\beta \to 0$, $f_{a_r}(y) \to 1$. By these, we can immediately obtain the well-known result

Corollary 3. $\sum_{a_1+a_2+\cdots+a_k=n} 1 = \binom{n+k-1}{k-1}$.

Corollary 4. For any positive integers k, n and j, we have

$$\sum_{a_1+\cdots+a_k=n+k}\prod_{l=1}^{k} F_{(2j+1)a_l}=F_{2j+1}^{k}\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}L_{2j+1}^{n-2m}, \quad (14)$$

where $a_r > 0$, for $r = 0, 1, \cdots, k$.

Proof : Taking $x = L_{2j+1}$ in $F_{a_r}(x)$, and noting that

$$L_n^2 - 5F_n^2 = 4(-1)^n \ \ and \ \ (\frac{1 \pm \sqrt{5}}{2})^n = \frac{L_n \pm \sqrt{5}F_n}{2},$$

we have

$$\begin{aligned} F_{a_r}(L_{2j+1}) &= \frac{1}{\sqrt{L_{2j+1}^2+4}}\left(\left(\frac{L_{2j+1}+\sqrt{L_{2j+1}^2+4}}{2}\right)^{a_r} - \left(\frac{L_{2j+1}-\sqrt{L_{2j+1}^2+4}}{2}\right)^{a_r}\right) \\ &= \frac{1}{\sqrt{5}F_{2j+1}}\left(\left(\frac{L_{2j+1}+\sqrt{5}F_{2j+1}}{2}\right)^{a_r} - \left(\frac{L_{2j+1}-\sqrt{5}F_{2j+1}}{2}\right)^{a_r}\right) \\ &= \frac{1}{\sqrt{5}F_{2j+1}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{(2j+1)a_r} - \left(\frac{1-\sqrt{5}}{2}\right)^{(2j+1)a_r}\right) = \frac{F_{(2j+1)a_r}}{F_{2j+1}}. \end{aligned}$$

Then (14) follows by replacing x by L_{2j+1} in Corollary 1.

Second Proof : Substituting $y = \frac{1}{L_{2j+1}^2}$ in $f_{a_r}(y)$, we can have

$$f_{a_r}(\frac{1}{L_{2j+1}^2}) = \frac{1}{\sqrt{1+\frac{4}{L_{2j+1}^2}}}\left(\left(\frac{1+\sqrt{1+\frac{4}{L_{2j+1}^2}}}{2}\right)^{a_r} - \left(\frac{1-\sqrt{1+\frac{4}{L_{2j+1}^2}}}{2}\right)^{a_r}\right)$$

$$= \frac{1}{\sqrt{5}L_{2j+1}^{a_r-1}F_{2j+1}}\left(\left(\frac{L_{2j+1}+\sqrt{5}F_{2j+1}}{2}\right)^{a_r} - \left(\frac{L_{2j+1}-\sqrt{5}F_{2j+1}}{2}\right)^{a_r}\right)$$

$$= \frac{F_{(2j+1)a_r}}{L_{2j+1}^{a_r-1}F_{2j+1}}.$$

Then taking $y = \frac{1}{L_{2j+1}^2}$ in Corollary 2 yields (14).

Corollary 5. For any positive integer j, when $n+k \equiv 0(mod\ 2)$, then

$$\sum_{a_1+\cdots+a_k=n+k}\prod_{l=1}^{k} F_{(2j)a_l} = \frac{L_{2j}^k}{(\sqrt{5})^k}\sum_{m=0}^{[\frac{n}{2}]}(\sqrt{5}F_{2j})^{n-2m}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1} \quad (15)$$

where $n, k > 0$, $a_r > 0$ and $a_r \equiv 0(mod\ 2)$, for $r = 1, 2, \ldots, k$.

Proof : When $a_r \equiv 0(mod\ 2)$, for $r = 1, 2, \ldots, k$, setting $x = \sqrt{5}F_{2j}$ in Corollary 1 or $y = \frac{1}{(\sqrt{5}F_{2j})^2}$ in Corollary 2, similar to the proof of Corollary 4, we can derive the result.

3. Weighted summation formulas involving Fibonacci polynomials

In this section, we can derive several weighted summation formulas involving the Fibonacci polynomials.

Theorem 2: For any positive integers k and n, we can have the formula

$$\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+1)F_{a_l+1}(x)$$

$$= \sum_{m=0}^{min\{[\frac{n}{2}],k-1\}}\sum_{j=0}^{[\frac{n-2m}{2}]}\frac{n-2m+2k-1}{n-2m-j}\binom{k-1}{m}\binom{n-2m-j}{j}\binom{n-2m+2k-2-j}{2k-1}x^{n-2m-2j},$$

where $a_r \geq 0$, for $r = 1, 2, \cdots, k$.

Proof : When $y = 1$, setting $F(t,x) = \sum_{n=0}^{\infty}(n+1)F_{n+1}(x)t^n$, we can deduce from (2) that

$$F(t,x) = \frac{1+t^2}{(1-xt-t^2)^2}. \tag{16}$$

Let $\frac{\partial^l F(t,x)}{\partial x^l}$ denotes the lth partial derivative for x and $F_n^{(l)}(x)$ denotes the lth derivative for x. Then by induction from (16), we can obtain that

$$\frac{\partial^l F(x,t)}{\partial x^l} = \sum_{n=0}^{\infty}(n+l+1)F_{n+l+1}^{(l)}(x)t^{n+l} = \frac{(1+t^2)(l+1)!t^l}{(1-xt-t^2)^{l+2}}. \tag{17}$$

Then noting that $(1+t^2)^{k-1} = \sum_{m=0}^{k-1}\binom{k-1}{m}t^{2m}$, we can obtain from (16) and (17) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} t^n \sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+1)F_{a_l+1}(x) \\
&= \frac{(1+t^2)^k}{(1-xt-t^2)^{2k}} = \frac{(1+t^2)^{k-1}}{(2k-1)!}\sum_{n=0}^{\infty}(n+2k-1)F_{n+2k-1}^{(2k-2)}(x)t^n \\
&= \frac{1}{(2k-1)!}\sum_{n=0}^{\infty}\sum_{m=0}^{\min\{[\frac{n}{2}],k-1\}}\binom{k-1}{m}(n-2m+2k-1)F_{n-2m+2k-1}^{(2k-2)}(x)t^n.
\end{aligned} \tag{18}$$

Comparing the coefficients of t^n on both sides of (18) can yield

$$\begin{aligned}
&\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+1)F_{a_l+1}(x) \\
&= \frac{1}{(2k-1)!}\sum_{m=0}^{\min\{[\frac{n}{2}],k-1\}}\binom{k-1}{m}(n-2m+2k-1)F_{n-2m+2k-1}^{(2k-2)}(x).
\end{aligned} \tag{19}$$

When $y = 1$, we can obtain from (3) that

$$\begin{aligned}
&\frac{n-2m+2k-1}{(2k-1)!}F_{n-2m+2k-1}^{(2k-2)}(x) \\
&= \sum_{j=0}^{[\frac{n-2m}{2}]}\frac{n-2m+2k-1}{n-2m-j}\binom{n-2m-j}{j}\binom{n-2m+2k-2-j}{2k-1}x^{n-2m-2j}.
\end{aligned} \tag{20}$$

Combining (19) and (20), we can arrive at the result.

Similarly, considering the generating function $H(t,x) = \sum_{n=0}^{\infty}(n+2)F_{n+1}(x)t^n$, we can have the following

Theorem 3: For any positive integers k and n, we get

$$\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+2)F_{a_l+1}(x)=\sum_{j=0}^{\min\{k,n\}}\sum_{i=0}^{\lfloor\frac{n-j}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n-j-2i}{2}\rfloor}(-1)^{i+j}\frac{n-j-2i+2k-1}{n-j-2i-m}$$
$$2^{k-j}\binom{k}{j}\binom{n-j-2i-m}{m}\binom{n-j-2i+2k-2-m}{2k-1}x^{n-2i-2m},$$

where $a_r \geq 0$, for $r = 1, 2, \cdots, k$.

Proof : From (2), we may have

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}F_{n+1}(x)t^{n+2}\right)=\frac{\partial}{\partial t}\left(\frac{t^2}{1-xt-t^2}\right).$$

Hence

$$H(t,x)=\frac{2-xt}{(1-xt-t^2)^2}. \tag{21}$$

Recalling that

$$(2-xt)^k=\sum_{j=0}^{k}\binom{k}{j}(-xt)^j2^{k-j} \ \ and \ \ \frac{1}{1+t^2}=\sum_{i=0}^{\infty}(-t^2)^i,$$

we can derive from (17) and (21) that

$$\sum_{n=0}^{\infty}t^n\sum_{a_1+\cdots+a_k=n}\prod_{l=1}^{k}(a_l+2)F_{a_l+1}(x)$$
$$=\frac{(2-xt)^k}{(1-xt-t^2)^{2k}}=\frac{(2-xt)^k}{(1+t^2)(2k-1)!}\sum_{n=0}^{\infty}(n+2k-1)F_{n+2k-1}^{(2k-2)}(x)t^n$$
$$=\frac{1}{(2k-1)!}\sum_{n=0}^{\infty}\sum_{j=0}^{\min\{k,n\}}\sum_{i=0}^{\lfloor\frac{n-j}{2}\rfloor}(-1)^{i+j}\binom{k}{j}x^j2^{k-j}(n-j-2i+2k-1)F_{n-j-2i+2k-1}^{(2k-2)}(x)t^n.$$

By taking the similar steps from (19) to (20), we can deduce the result.

Letting $x = L_{2j+1}$ in Theorem 2 and in Theorem 3, respectively, and recalling $F_{a_r}(L_{2j+1}) = \frac{F_{(2j+1)a_r}}{F_{2j+1}}$, we can get the following

Corollary 6. For any positive integers k, n and j, we have

$$\sum_{a_1+\cdots+a_k=n+kl=1}\prod^{k} a_l F_{(2j+1)a_l} = F_{2j+1}^{k} \sum_{m=0}^{\min\{[\frac{n}{2}],k-1\}} \sum_{h=0}^{[\frac{n-2m}{2}]} \frac{n-2m+2k-1}{n-2m-j} \binom{k-1}{m}\binom{n-2m-h}{h}\binom{n-2m+2k-2-h}{2k-1} L_{2j+1}^{n-2(m+h)},$$

$$\sum_{a_1+\cdots+a_k=n+kl=1}\prod^{k} (a_l+1)F_{(2j+1)a_l} = F_{2j+1}^{k} \sum_{h=0}^{\min\{k,n\}} \sum_{i=0}^{[\frac{n-h}{2}]} \sum_{m=0}^{[\frac{n-h-2i}{2}]} (-1)^{i+h} \frac{n-h-2i+2k-1}{n+h-2i-m} 2^{k-h}\binom{k}{h}\binom{n-h-2i-m}{m}\binom{n-h-2i+2k-2-m}{2k-1} L_{2j+1}^{n-2(i+m)},$$

where $a_r > 0$, for $r = 0, 1, \cdots, k$.

In the same way, taking $x = \sqrt{5}F_{2j}$ in Theorem 2 and in Theorem 3, respectively, and recalling $F_{a_r}(\sqrt{5}F_{2j}) = \frac{\sqrt{5}F_{(2j)a_r}}{L_{2j}}$, when $a_r \equiv 0(mod\ 2)$, we can get the following

Corollary 7. For any positive integer j, when $n+k \equiv 0(mod\ 2)$, we have

$$\sum_{a_1+\cdots+a_k=n+kl=1}\prod^{k} a_l F_{(2j)a_l} = \frac{L_{2j}^{k}}{(\sqrt{5})^k} \sum_{m=0}^{\min\{[\frac{n}{2}],k-1\}} \sum_{h=0}^{[\frac{n-2m}{2}]} \frac{n-2m+2k-1}{n-2m-h} \binom{k-1}{m}\binom{n-2m-h}{h}\binom{n-2m+2k-2-h}{2k-1} (\sqrt{5}F_{2j})^{n-2(m+h)},$$

$$\sum_{a_1+\cdots+a_k=n+kl=1}\prod^{k} (a_l+1)F_{(2j)a_l} = \frac{L_{2j}^{k}}{(\sqrt{5})^k} \sum_{h=0}^{\min\{k,n\}} \sum_{i=0}^{[\frac{n-h}{2}]} \sum_{m=0}^{[\frac{n-h-2i}{2}]} (-1)^{i+h} \frac{n-h-2i+2k-1}{n-h-2i-m} 2^{k-h}\binom{k}{h}\binom{n-h-2i-m}{m}\binom{n-h-2i+2k-2-m}{2k-1} (\sqrt{5}F_{2j})^{n-2(i+m)},$$

where $n, k > 0$, $a_r > 0$ and $a_r \equiv 0(mod\ 2)$, for $r = 1, 2, \ldots, k$.

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