

Chromaticity of Turán Graphs with Certain Matching or Star Deleted

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ABSTRACT

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G . A graph G is chromatically unique if for any graph H , $P(H, \lambda) = P(G, \lambda)$ implies H is isomorphic to G . In this paper, we study the chromaticity of Turán graphs with deleted edges that induce a matching or a star. As a by-product, we obtain new families of chromatically unique graphs.

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1 Introduction

All graphs considered in this paper are finite and simple. For a graph G , we denote by $P(G; \lambda)$ (or $P(G)$), the chromatic polynomial of G . Two graphs G and H are said to be *chromatically equivalent* (simply χ -*equivalent*), denoted $G \sim H$ if $P(G) = P(H)$. A graph G is said to be *chromatically unique* (simply χ -*unique*), if $H \sim G$ implies that $H \cong G$. A family \mathcal{G} of graphs is said to be chromatically-closed (simply χ -*closed*) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. Many families of χ -unique graphs are known (see [4, 5]).

For a graph G , let $e(G)$, $v(G)$, $t(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles and chromatic number of G . By \overline{G} , we denote

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the complement of G . Let O_n be an edgeless graph with n vertices. Also let $Q(G)$ and $K(G)$ respectively be the number of induced subgraphs C_4 and complete subgraphs K_4 in G . Suppose S is a set of s edges of G . Denote by $G - S$ the graph obtained from G by deleting all edges in S , and by $\langle S \rangle$ the graph induced by S . For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \dots \leq p_t$, let $K(p_1, p_2, \dots, p_t)$ be a complete t -partite graph with partition sets V_i such that $|V_i| = p_i$ for $i = 1, 2, \dots, t$. The Turán graph, denoted $T = K(t_1 \times p, t_2 \times (p + 1))$ is the unique complete t -partite graph having $t_1 \geq 1$ partite sets of size p and t_2 partite sets of size $p + 1$ (where $t_1 + t_2 = t$). It is shown in [2] that Turán graphs are χ -unique. In this paper, we study the chromaticity of Turán graph with deleted edges that induced a matching or a star. As a by-product, we obtain new families of χ -unique graphs.

2 Preliminary results and notations

Let $\mathcal{K}^{-s}(p_1, p_2, \dots, p_t)$ be the family $\{K(p_1, p_2, \dots, p_t) - S \mid S \subset E(G) \text{ and } |S| = s\}$. For $p_1 \geq s + 1$, we denote by $K_{i,j}^{-K(1,s)}(p_1, p_2, \dots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \dots, p_t)$ where the s edges in S induced a $K(1, s)$ with center in V_i and all the end-vertices in V_j , and by $K_{i,j}^{-sK_2}(p_1, p_2, \dots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \dots, p_t)$ where the s edges in S induced a matching with end-vertices in V_i and V_j . For convenience, we let $T^{-s} = \mathcal{K}^{-s}(t_1 \times p, t_2 \times (p + 1))$, $T_{i,j}^{-K(1,s)} = K_{i,j}^{-K(1,s)}(t_1 \times p, t_2 \times (p + 1))$ and $T_{i,j}^{-sK_2} = K_{i,j}^{-sK_2}(t_1 \times p, t_2 \times (p + 1))$.

For a graph G and a positive integer k , a partition $\{A_1, A_2, \dots, A_k\}$ of $V(G)$ is called a k -independent partition in G if each A_i is a non-empty independent set of G . Let $\alpha(G, k)$ denote the number of k -independent partitions in G . If G is of order n , then $P(G, \lambda) = \sum_{k=1}^n \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1) \dots (\lambda - k + 1)$ (see [8]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \dots$, if $G \sim H$.

For a graph G with n vertices, the polynomial $\sigma(G, x) = \sum_{k=1}^n \alpha(G, k)x^k$ is called the σ -polynomial of G (see [1]), and the polynomial $h(G, x) = \sum_{k=1}^n \alpha(\overline{G}, k)x^k$ is called the adjoint polynomial of G (see [6]). Clearly, the conditions $P(G, \lambda) = P(H, \lambda)$, $\sigma(G, x) = \sigma(H, x)$ and $h(\overline{G}, x) = h(\overline{H}, x)$ are equivalent for any graphs G and H .

For disjoint graphs G and H , $G + H$ denotes the disjoint union of G and H ; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy \mid x \in V(G) \text{ and } y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t -partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [9].

Lemma 2.1. (Koh and Teo [4]) *Let G and H be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \dots$, and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [1]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \dots, n_t), x) = \prod_{i=1}^t \sigma(O_{n_i}, x).$$

The above lemma is equivalent to the following:

Remark. *Let G and H be two disjoint graphs. Then*

$$h(\overline{G} + \overline{H}, x) = h(\overline{G}, x)h(\overline{H}, x).$$

In particular,

$$h(\overline{K(n_1, n_2, \dots, n_t)}, x) = \prod_{i=1}^t h(K_{n_i}, x).$$

Lemma 2.3. (Liu [7]) *Let G be a graph with $e \in E(G)$. If $e = uv$ does not belong to any triangle of G , then*

$$h(G, x) = h(G - e, x) + xh(G - \{u, v\}, x),$$

where $G - e$ (respectively $G - \{u, v\}$) denote the graph obtained from G by deleting the edge e (respectively the vertices u and v).

Denote by $\beta(G)$ the minimum real root of $h(G, x)$.

Lemma 2.4. (Zhao [10]) *Let G be a connected graph such that G contains H as a proper subgraph. Then*

$$\beta(G) < \beta(H).$$

Suppose $G = K(p_1, p_2, \dots, p_t)$ and $H = G - S$ for a set S of s edges of G . Define $\alpha_k(H) = \alpha(H, k) - \alpha(G, k)$ for $k \geq t + 1$.

Lemma 2.5. (Zhao [10]) *Let $G = K(p_1, p_2, \dots, p_t)$ and $H = G - S$. If $p_1 \geq s + 1$, then*

$$s \leq \alpha_{t+1}(H) = \alpha(H, t + 1) - \alpha(G, t + 1) \leq 2^s - 1,$$

$\alpha_{t+1}(H) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H) = 2^s - 1$ if and only if $\langle S \rangle = K(1, s)$.

Lemma 2.6. (Dong et al. [3]) *Let p_1, p_2 and s be positive integers with $3 \leq p_1 \leq p_2$, then*

(i) $K_{1,2}^{-K(1,s)}(p_1, p_2)$ is χ -unique for $1 \leq s \leq p_2 - 2$,

(ii) $K_{2,1}^{-K(1,s)}(p_1, p_2)$ is χ -unique for $1 \leq s \leq p_1 - 2$, and

(iii) $K^{-sK_2}(p_1, p_2)$ is χ -unique for $1 \leq s \leq p_1 - 1$

In [10], Zhao obtained the following results on Turán graph with s edges deleted for $t_1 + t_2 \geq 5$.

Lemma 2.7. (Zhao [10]) *Let $s \geq 1$ and $t_1 \geq 1$. If $p \geq s + 2$, then T^{-s} is χ -closed.*

Lemma 2.8. (Zhao [10]) *Suppose $s \geq 1$ and $p \geq s + 2$, then*

(i) every $T_{i,j}^{-K(1,s)}$ is χ -unique for any (i, j) where $1 \leq i \neq j \leq t$ and $|V_i| = |V_j| = p$, or $|V_i| = p$, $|V_j| = p + 1$, or $|V_i| = p + 1$, $|V_j| = p$, or $|V_i| = |V_j| = p + 1$.

(ii) $T_{1,2}^{-sK_2}$ is χ -unique if $t_1 = 2$.

Note that Lemmas 2.7 and 2.8 hold for $t = 2, 3$ and 4 (see the proofs of Theorem 6.6.2 to Theorem 6.6.4 in [10]). Observe that if $G \in T^{-s}$ such that $\alpha_{t+1}(G) = s$, then Lemma 2.7 also holds for $p > 2 + \log_2 s$ (see the proof of Theorem 6.6.2 in [10]).

Lemma 2.9. *If $p > 2 + \log_2 s$ and $G \in T^{-s}$ such that $\alpha_{t+1}(G) = s$, then T^{-s} is χ -closed.*

It follows that Lemma 2.8(ii) also holds for $p > 2 + \log_2 s$.

By Lemmas 2.5 and 2.8, every graph $G \in T^{-s}$ with $\alpha_{t+1}(G) = 2^s - 1$ is χ -unique if $p \geq s + 2$. However, only one family of graphs G is known to be χ -unique when $\alpha_{t+1}(G) = s$. We now give a necessary condition for two graphs G and H in T^{-s} (with $\alpha_{t+1}(G) = s$) to be χ -equivalent.

Suppose $F = K(p_1, p_2, \dots, p_t)$. For $G = F - S$, denote by $t_i(G)$ the number of triangles in F that contain i deleted edges in S for $i = 1, 2, 3$. Suppose $G \in \mathcal{T}^{-s}$. An edge $e = uv$ in S is of Type A (respectively, Type B and Type C) if $u \in V_i, v \in V_j$ for $1 \leq i < j \leq t_1$ (respectively, for $1 \leq i \leq t_1, t_1 + 1 \leq j \leq t$, and for $t_1 + 1 \leq i < j \leq t$). Denote by $s_1(G)$ (respectively, $s_2(G)$ and $s_3(G)$) the number of Type A (respectively, Type B and Type C) edges in S .

Lemma 2.10. *Suppose G and H are two graphs in \mathcal{T}^{-s} with $\alpha_{t+1}(G) = s$. If $G \sim H$, then $s_2(G) + 2s_3(G) + t_2(G) = s_2(H) + 2s_3(H) + t_2(H)$.*

Proof. Let $G = F - S$ and $H = F - S'$ be two graphs in \mathcal{T}^{-s} with $\alpha_{t+1}(G) = s$. If $H \sim G$, then Lemma 2.1 implies that $\alpha_{t+1}(H) = s$ and $t(G) = t(H)$. Observe that both $\langle S \rangle$ and $\langle S' \rangle$ contain no K_3 subgraphs. So, $t_3(G) = t_3(H) = 0$. Note that $|S| = |S'| = s = \sum_{i=1}^3 s_i(G) = \sum_{i=1}^3 s_i(H)$. We now consider $t(G)$ and $t(H)$. Note that both G and H has t_1 and t_2 partite sets of size p and $p + 1$ respectively. Clearly,

$$\begin{aligned} t(G) &= t(F) - t_1(G) + t_2(G) \\ &= t(F) - s_1(G) ((t_1 - 2)p + t_2(p + 1)) - \\ &\quad s_2(G) ((t_1 - 1)p + (t_2 - 1)(p + 1)) - \\ &\quad s_3(G) (t_1 p + (t_2 - 2)(p + 1)) + t_2(G) \\ &= t(F) - s(t_1 p + t_2(p + 1)) + 2sp + \\ &\quad s_2(G) + 2s_3(G) + t_2(G). \end{aligned}$$

Similarly,

$$t(H) = t(F) - s[t_1 p + t_2(p + 1)] + 2sp + s_2(H) + 2s_3(H) + t_2(H).$$

It follows immediately that $s_2(G) + 2s_3(G) + t_2(G) = s_2(H) + 2s_3(H) + t_2(H)$. \square

In what follows, we let F be the Turán graph with $\chi(F) = t \geq 2$.

3 Turán graph with a matching deleted

For a graph $G \in \mathcal{K}^{-s}(p_1, p_2, \dots, p_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively, Type 2, and Type 3) if the vertices of the induced C_4 are in exactly two (respectively, three, and four) partite sets of $V(G)$. An example of induced C_4 of Type 1, 2 and 3 is shown in Figure 1.

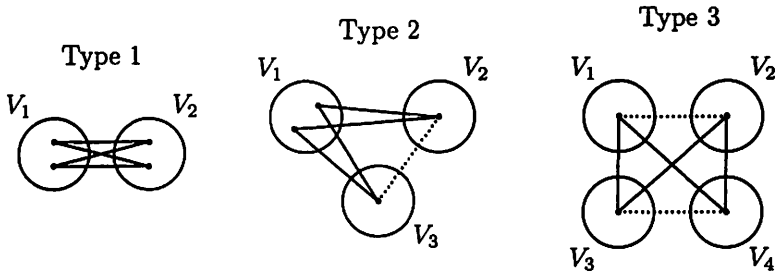


Figure 1. Three types of induced C_4 in a t -partite graph.
 Vertices joined by dotted lines are not adjacent.

Let S_{ij} ($1 \leq i \leq t, 1 \leq j \leq t$) be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geq 0$. We also say S is *ideal* if and only if $S = S_{ij}$ for some i and j , $1 \leq i \leq t$, $1 \leq j \leq t$.

Lemma 3.1. For integer $t \geq 3$, let $F = K(p_1, p_2, \dots, p_t)$ and $G = F - S$ for a set S of s edges in F . Suppose S induces a matching in F , then

$$\begin{aligned}
 Q(G) = & Q(F) - \sum_{1 \leq i < j \leq t} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leq i < j < l \leq t} s_{ij}s_{il} - \\
 & \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl} + \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{k \notin \{i, j\}} \binom{p_k}{2} \right] + \\
 & \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k, l\}}} s_{ij}s_{kl},
 \end{aligned}$$

and

$$\begin{aligned}
 K(G) = & K(F) - \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq t \\ \{i, j\} \cap \{k, l\} = \emptyset}} p_k p_l \right] + \\
 & \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k, l\}}} s_{ij}s_{kl}.
 \end{aligned}$$

Proof. Let $Q_1(G)$ (respectively, $Q_2(G)$ and $Q_3(G)$) be the number of Type 1 (respectively, Type 2 and Type 3) induced C_4 in G . Observe that $S = \bigcup_{1 \leq i < j \leq t} S_{ij}$. Hence,

$$\begin{aligned} Q_1(G) &= \sum_{1 \leq i < j \leq t} \binom{p_i}{2} \binom{p_j}{2} - \sum_{1 \leq i < j \leq t} (p_i - 1)(p_j - 1)s_{ij} + \sum_{1 \leq i < j \leq t} \binom{s_{ij}}{2} \\ &= Q(F) - \sum_{1 \leq i < j \leq t} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leq i < j < l \leq t} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl}. \end{aligned}$$

Note that $\sum_{1 \leq i < j < l \leq t} s_{ij}s_{il} \geq 0$ and $\sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl} \geq 0$, and the equality

holds if S is ideal.

We now find $Q_2(G)$. For three distinct indices i, j, k ; $1 \leq i, j, k \leq t$, let $v_i v_j$ be an edge in S such that $v_i \in V_i$ and $v_j \in V_j$, and let v_k, v'_k be two distinct vertices in V_k . It is clear that $v_i v_k v_j v'_k v_i$ is a Type 2 induced C_4 in G . Since the number of 2-element subsets of V_k is $\binom{p_k}{2}$, we have

$$Q_2(G) = \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{k \notin \{i, j\}} \binom{p_k}{2} \right].$$

We now find $Q_3(G)$. For four distinct indices i, j, k, l ; $1 \leq i, j, k, l \leq t$, let $v_i v_j$ and $v_k v_l$ be two edges in S such that $v_a \in V_a$ for $a \in \{i, j, k, l\}$. It is clear that $v_i v_k v_j v_l v_i$ is a Type 3 induced C_4 in G . Hence,

$$Q_3(G) = \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k, l\}}} s_{ij}s_{kl}.$$

Note that $Q_3(G) = 0$ if for any two distinct edges in S , say $v_i v_j$ and $v_k v_l$, $\{i, j\} \cap \{k, l\} \neq \emptyset$. We now find $K(G)$. Observe that each K_4 subgraph in F has at most two edges in S . Let $K_m(G)$ be the number of K_4 subgraphs in F that contains m edges in S for $m = 1, 2$. Hence, $K(G) = K(F) - K_1(G) + K_2(G)$. Clearly,

$$K(F) = \sum_{1 \leq i < j < k < l \leq t} p_i p_j p_k p_l.$$

Let $v_i v_j$ be an edge in S such that $v_i \in V_i$ and $v_j \in V_j$. Then, the number of K_4 subgraphs in F that contains $v_i v_j$ is $\sum_{1 \leq k < l \leq t} p_k p_l$ where $\{i, j\} \cap \{k, l\} = \emptyset$. Hence,

$$K_1(G) = \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq t \\ \{i, j\} \cap \{k, l\} = \emptyset}} p_k p_l \right].$$

Observe that there is a one-to-one correspondence between the set of Type 3 induced C_4 in G and the set of K_4 subgraph in F that contains two edges in S . Hence, $K_2(G) = Q_3(G)$.

This completes the proof. \square

We now present two main results that extend Lemma 2.8(ii) above.

Theorem 3.1. *Suppose $s \geq 1$ and $p \geq s + 2$, then $T_{1,2}^{-sK_2}$ is χ -unique for $t_1 = 1$.*

Proof. Suppose $H \sim G = T_{1,2}^{-sK_2}$ with $t_1 = 1$. By Lemma 2.6, the theorem holds for $t = 2$. So, we assume $t \geq 3$. We shall show that $H \cong G$.

By Lemma 2.9, $H \in T^{-s}$ with $t_1 = 1$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G) = s$. Hence, $s_1(G) = s_3(G) = t_2(G) = s_1(H) = 0$, and so $s = s_2(G)$. By Lemma 2.10, we have that $s = s_2(H) + 2s_3(H) + t_2(H)$. We assert that $s_3(H) = t_2(H) = 0$. Otherwise, $s_2(H) + 2s_3(H) + t_2(H) > s$, a contradiction.

It follows that for H each deleted edge in S must have one end-vertex in V_1 , and another end-vertex in V_j for $2 \leq j \leq t$. Hence, each edge in S must be in one of S_{1j} ($2 \leq j \leq t$). Moreover, all the edges in S must induce a matching in F . Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$. Clearly,

$H \cong G$ if S is ideal. Otherwise, we may assume that $S_{1j} \neq \emptyset$ for $2 \leq j \leq k$ and $k \geq 3$. Observe that each induced C_4 in G and H is of Type 1 or 2. It is easy to see that

$$Q(G) = Q(F) - s(p-1)p + \binom{s}{2} + s(t-2) \binom{p+1}{2}.$$

By Lemma 3.1, we have

$$\begin{aligned} Q(H) &= Q(F) - s(p-1)p + \binom{s}{2} - \sum_{2 \leq j < l \leq k} s_{1j} s_{1l} + s(t-2) \binom{p+1}{2} \\ &= Q(G) - \sum_{2 \leq j < l \leq k} s_{1j} s_{1l}, \end{aligned}$$

and

$$\begin{aligned}
 K(H) &= K(F) - \binom{t-2}{2} (p+1)^2 \sum_{2 \leq j \leq k} s_{1j} \\
 &= K(F) - \binom{t-2}{2} s(p+1)^2 \\
 &= K(G).
 \end{aligned}$$

Hence, $2K(G) - Q(G) \neq 2K(H) - Q(H)$. This contradicts Lemma 2.1. Therefore, G is χ -unique and the proof is now complete. \square

Theorem 3.2. *Suppose $s \geq 1$ and $p \geq s + 2$, then $T_{1,2}^{-sK_2}$ is χ -unique for $t_1 \geq 2$.*

Proof. Suppose $H \sim G = K_{1,2}^{-sK_2}$ with $t_1 \geq 2$. We shall show that $H \cong G$.

By Lemma 2.9, $H \in T^{-s}$ with $t_1 \geq 2$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G) = s$. Hence, $s_1(G) = s$ and $s_2(G) = s_3(G) = t_2(G) = 0$. By Lemma 2.10, we have that $s_2(H) + 2s_3(H) + t_2(H) = 0$. It follows immediately that $s_2(H) = s_3(H) = t_2(H) = 0$ and so $s_1(H) = s$.

Therefore, relatively to H each deleted edge in S must have one end-vertex in V_i and another end-vertex in V_j for $1 \leq i < j \leq t_1$. Hence, each edge in S must be in one of S_{ij} ($1 \leq i < j \leq t_1$). Moreover, all the edges in S must induce a matching in F . Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$.

Clearly, $H \cong G$ if S is ideal. Otherwise, there exist i, j, k and l such that S_{ij} ($1 \leq i < j \leq t_1$) and S_{kl} ($1 \leq k < l \leq t_1$) are two disjoint non-empty subsets of S . Observe that each induced C_4 in G (respectively H) is of Type 1 or 2 (respectively Type 1, 2 or 3). It is easy to see that

$$Q(G) = Q(F) - s(p-1)^2 + \binom{s}{2} + s \left[(t_1 - 2) \binom{p}{2} + t_2 \binom{p+1}{2} \right],$$

By Lemma 3.1, we have

$$\begin{aligned}
 Q(H) &= Q(F) - s(p-1)^2 + \binom{s}{2} - \sum_{1 \leq i < j < l \leq t_1} s_{ij} s_{il} - \\
 &\quad \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq k < l \leq t_1 \\ i < k}} s_{ij} s_{kl} + s \left[(t_1 - 2) \binom{p}{2} + t_2 \binom{p+1}{2} \right] + \\
 &\quad \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij} s_{kl}
 \end{aligned}$$

$$\begin{aligned}
&= Q(G) - \sum_{1 \leq i < j < l \leq t_1} s_{ij}s_{il} - \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq k < l \leq t_1 \\ i < k}} s_{ij}s_{kl} + \\
&\quad \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl},
\end{aligned}$$

where $\sum_{1 \leq i < j < l \leq t_1} s_{ij}s_{il} + \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq k < l \leq t_1 \\ i < k}} s_{ij}s_{kl} > 0$, and $\sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl} \geq 0$.

0. Moreover,

$$\begin{aligned}
K(H) &= K(F) - \left[\binom{t_1 - 2}{2} p^2 + (t_1 - 2)t_2 \times p(p + 1) + \binom{t_2}{2} (p + 1)^2 \right] \times \\
&\quad \sum_{1 \leq i < j \leq t_1} s_{ij} + \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl} \\
&= K(F) - s \left[\binom{t_1 - 2}{2} p^2 + (t_1 - 2)t_2 \times p(p + 1) + \right. \\
&\quad \left. \binom{t_2}{2} (p + 1)^2 \right] + \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl} \\
&= K(G) + \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
2K(H) - Q(H) &= 2K(G) - Q(G) + \sum_{1 \leq i < j < l \leq t_1} s_{ij}s_{il} + \\
&\quad \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq k < l \leq t_1 \\ i < k}} s_{ij}s_{kl} + \sum_{\substack{1 \leq i < j \leq t_1 \\ 1 \leq i < k < l \leq t_1 \\ j \notin \{k, l\}}} s_{ij}s_{kl},
\end{aligned}$$

contradicting Lemma 2.1.

The proof is now complete. \square

Note that Lemma 2.8(ii) now follows directly from Theorem 3.2.

A matching subgraph sK_2 of $K(p_1, p_2, \dots, p_t)$ is called a *special sK_2* if no two vertices of sK_2 are in the same partite set of $K(p_1, p_2, \dots, p_t)$. Let $G^\# \in \mathcal{T}^{-s}$ such that $\langle S \rangle$ is a special sK_2 with each vertex belongs to a partite set of size p . For $m \leq n$, denote by $K_m \cdot K_n$ the graph obtained from $K_m + K_n$ by joining by an edge a vertex of K_m and a vertex of K_n . We now prove the chromatic uniqueness of the graph $G^\#$ in the following theorem.

Theorem 3.3. *Suppose $t_1 \geq 2s \geq 2$ and $p > 2 + \log_2 s$, then $G^\#$ is χ -unique.*

Proof. Suppose $H \sim G^\#$. We shall show that $H \cong G$.

By Lemma 2.9, $H \in \mathcal{T}^{-s}$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G^\#) = s$. Hence, $s_1(G^\#) = s$ and $s_2(G^\#) = s_3(G^\#) = t_2(G^\#) = 0$. By an argument similar to that in Theorem 3.2, we conclude that $s_1(H) = s$ and $s_2(H) = s_3(H) = t_2(H) = 0$. Hence, each deleted edge in S must have one end-vertex in V_i and another end-vertex in V_j for $1 \leq i < j \leq t_1$. Moreover, all the edges in S must induce a matching. Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$.

Claim. $\langle S \rangle$ is a special sK_2 .

Proof of the claim. Suppose the claim is not true. We shall show that $\beta(\overline{G^\#}) > \beta(\overline{H})$, contradicting $\beta(\overline{G^\#}) = \beta(\overline{H})$.

By Lemma 2.2, we have $h(\overline{G^\#}, x) = [h(K_p \cdot K_p, x)]^s [h(K_p, x)]^{t_1 - 2s} \times [h(K_{p+1}, x)]^{t_2}$ and $h(\overline{H}, x) = h(\overline{H'}, x) [h(K_{p+1}, x)]^{t_2}$. So, $h(\overline{G^\#}, x) = h(\overline{H}, x)$ implies that $h(\overline{G'}, x) = h(\overline{H'}, x)$ for $h(\overline{G'}, x) = [h(K_p \cdot K_p, x)]^s \times [h(K_p, x)]^{t_1 - 2s}$. Clearly, $\overline{H'}$ is a graph that contains a $K_p \cdot K_p$ as a proper subgraph. Hence, by Lemma 2.4, $\beta(\overline{G'}) = \beta(K_p \cdot K_p) > \beta(\overline{H'})$, a contradiction.

By the above claim, we have that $H \cong G^\#$ and the proof is now complete.

□

4 Turán graph with a star deleted

A star subgraph $K(1, s)$ of $K(p_1, p_2, \dots, p_t)$ is called a *special $K(1, s)$* if no two end-vertices of $K(1, s)$ are in the same partite set of G . Let $G_1^* \in \mathcal{T}^{-s}$ such that $\langle S \rangle$ is a special $K(1, s)$ with each vertex belongs to a partite set of size $p+1$, and for $t_2 = 1$, let $G_2^* \in \mathcal{T}^{-s}$ such that $\langle S \rangle$ is a special $K(1, s)$ with central vertex belongs to the only partite set of size $p+1$. We now prove the chromatic uniqueness of the graphs G_1^* and G_2^* in the following two theorems.

Theorem 4.1. *Suppose $1 \leq s \leq t_2 - 1$ and $p > 2 + \log_2 s$, then G_1^* is χ -unique.*

Proof. Suppose $H \sim G_1^*$. We shall show that $H \cong G_1^*$.

By Lemma 2.9, $H \in T^{-s}$ and $\alpha_{t+1}(H) = \alpha_{t+1}(G_1^*) = s$. If $s = 1$, then Lemma 2.10 implies that $s_3(H) = 1$. So, $H \cong G_1^*$. Hence, we may assume $s \geq 2$. Now, $s_1(G_1^*) = s_2(G_1^*) = 0$, $t_2(G_1^*) = \binom{s}{2}$ and $s = s_3(G_1^*)$. By Lemma 2.10, we have $2s + \binom{s}{2} = s_2(H) + 2s_3(H) + t_2(H)$. Note that $t_2(H) \leq \binom{s}{2}$. We assert that $s_1(H) = s_2(H) = 0$.

If $s_1(H) \neq 0$, then $s_2(H) + s_3(H) < s$. Hence $s_2(H) + 2s_3(H) + t_2(H) < 2s + \binom{s}{2}$, a contradiction. Therefore, $s_1(H) = 0$. Consequently, if $s_2(H) \neq 0$, we also have a similar contradiction. Hence, $s_3(H) = s$ and $t_2(H) = \binom{s}{2}$.

Claim. $\langle S \rangle$ is a special $K(1, s)$.

Proof of the claim. Suppose the claim does not hold. We now show that $t_2(H) < \binom{s}{2}$, which is a contradiction. We proceed by induction on s . If $s = 2$ and $\langle S \rangle$ is not a special $K(1, 2)$, then $t_2(H) = 0 < \binom{2}{2}$. Hence, the claim holds for $s = 2$. Assume that the claim holds for $s = n \geq 3$. Let $s = n + 1$ and $\langle S \rangle$ is not a special $K(1, n + 1)$. Suppose e is an edge in S , then $S - \{e\}$ has n edges. Let t'_2 (respectively t''_2) be the number of triangles in $K(p_1, p_2, \dots, p_t)$ that contain two edges in S and do not contain (respectively contain) the edge e . We consider two cases.

Case 1. $\langle S - \{e\} \rangle$ is a special $K(1, n)$. In this case, edge e is not adjacent to the central vertex of $K(1, n)$. Therefore, $t'_2 = \binom{n}{2}$ and $t''_2 \leq 1$. Hence, $t_2(H) \leq \binom{n}{2} + 1 < \binom{n+1}{2}$.

Case 2. $\langle S - \{e\} \rangle$ is not a special $K(1, n)$. In this case, $\max\{t_2(H)\}$ is attained if the edge e is adjacent to each edge in $\langle S - \{e\} \rangle$. By induction hypotheses, $t'_2 < \binom{n}{2}$, whereas $t''_2 \leq n$. Hence, $t_2(H) < \binom{n}{2} + n = \binom{n+1}{2}$.

Therefore, the claim holds. Consequently, we have $H \cong G_1^*$.

The proof is now complete. \square

Corollary 4.1. *Suppose $1 \leq s \leq t - 1$ and $p > 2 + \log_2 s$, then $K^{-s}(\underbrace{p, \dots, p}_t)$ is χ -unique if $\langle S \rangle$ is a special $K(1, s)$.*

Theorem 4.2. *Suppose $1 \leq s \leq t - 1$ and $p > 2 + \log_2 s$, then G_2^* is χ -unique.*

Proof. Suppose $H \sim G_2^*$. We shall show that $H \cong G_2^*$.

By Lemma 2.9, $H \in \mathcal{K}^{-s}(\underbrace{p, \dots, p}_{t-1}, p+1)$ and $\alpha_{t+1}(H) = \alpha_{t+1}(G_2^*) = s$. If

$s = 1$, then Lemma 2.10 implies that $s_2(H) = 1$. So, $H \cong G_2^*$. Hence, we may assume $s \geq 2$. Now, $s_1(G_2^*) = s_3(G_2^*) = s_3(H) = 0$, $s_2(G_2^*) = s$ and $t_2(G_2^*) = \binom{s}{2}$. By Lemma 2.10, we have $s + \binom{s}{2} = s_2(H) + t_2(H)$.

Note that $t_2(H) \leq \binom{s}{2}$. We assert that $s_1(H) = 0$. Suppose $s_1(H) \neq 0$, then $s_2(H) < s$. This means $s_2(H) + t_2(H) < s + \binom{s}{2}$, a contradiction. Hence, $s_2(H) = s$ and $t_2(H) = \binom{s}{2}$. From the claim in the proof of Theorem 4.1, we know that $\langle S \rangle$ is a special star with central vertex in V_t . Hence, $H \cong G_2^*$.

The proof is now complete. \square

Remark. Theorem 4.2 is best possible in the sense that if G_2^* contains more than one partite set of size $p+1$, then it is not χ -unique. We can see this as follows. For $t_1, t_2 \geq s \geq 2$, let G' (respectively G'') be a graph in \mathcal{T}^{-s} such that $\langle S \rangle$ is a special $K(1, s)$, $s \geq 2$, where the central vertex belongs to partite set of size p (respectively size $p+1$) and the end-vertices belong to partite sets of size $p+1$ (respectively size p). It is clear that $G' \not\cong G''$. By Lemma 2.3 and mathematical induction on s , it is easy to show that $h(\overline{G'}, x) = h(\overline{G''}, x)$ for all $p \geq 1$. Hence, G' and G'' are chromatically equivalent. It follows immediately that for integers $k \geq 1$, $\underbrace{G' \vee G' \vee \dots \vee G'}_k$ is chromatically equivalent to $\underbrace{G'' \vee G'' \vee \dots \vee G''}_k$ but they are not isomorphic.

Note that for G' with $t_1 < s$ and G'' with $t_2 < s$, we have $G' \not\sim G''$. We can show that G' (respectively, G'') is χ -unique if $t_1 < s$ (respectively, $t_2 < s$) for $s = 2, 3$. We end this paper with the following conjecture and problem.

Conjecture 4.1. *The graphs G' with $t_1 < s$ and G'' with $t_2 < s$ are χ -unique.*

Problem 4.1. *Study the chromaticity of Turán graphs with a matching or a star deleted.*

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