Extreme Cliques in Interval Graphs*

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Abstract

A clique C is an extreme clique of an interval graph G if there exists some interval model of G in which C is the first clique. A graph G is homogeneously clique-representable if all cliques of G are extreme cliques. In this paper, we present characterizations of extreme cliques and homogeneously clique-representable graphs.

Key words: Interval graphs, characterizations, forbidden subgraphs.

1 Introduction

A graph G is an *interval graph* if there is a correspondence between $V(G)$ and a family of intervals $\mathcal{R} = \{I_v \mid v \in V(G)\}$ of the real line such that, for all distinct $u, w \in V(G)$, $I_u \cap I_w \neq \emptyset \iff (u, w) \in E(G)$. Such a family $\mathcal{R}$ is called an *interval model* of G. The class of interval graphs is a well-known class [4]. Figure 1 illustrates an interval graph and one of its interval models. We assume that all interval extreme points are distinct and denote the left and right extreme points of an interval I_v respectively by $\ell(I_v)$ and $r(I_v)$.

Note that any set of vertices corresponding to intervals crossed by some vertical line induces a complete subgraph. However, this complete set is not necessarily maximal. A *clique* is a maximal set that induces a complete graph.

*This work has been partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, and by the Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro, FAPERJ, Brasil.

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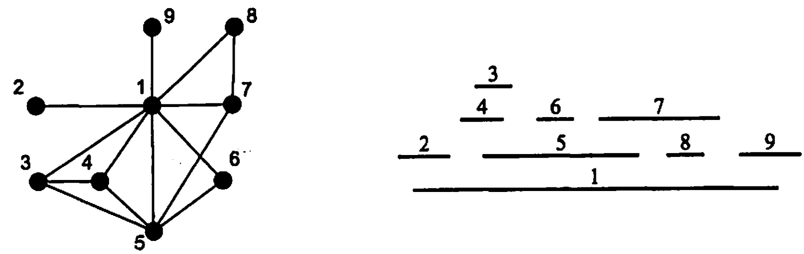

Figure 1: Interval graph and interval model.

Given an interval model, consider a set of vertical lines, each one corresponding to a clique of the graph. Then, the *clique order* of that model is the linear order on the set of cliques of the graph such that the clique C_i precedes the clique C_j in that order if and only if the vertical line corresponding to C_i is at the left of the vertical line corresponding to C_j in the model. Figure 2 displays the clique order of the model given in Figure 1.

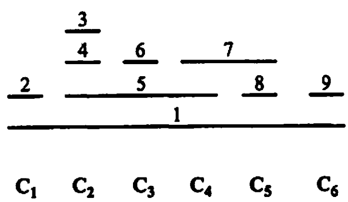

Figure 2: Interval model whose clique order is $C_1 \prec C_2 \prec \ldots \prec C_6$.

Not every linear order on the set of cliques is a clique order. For example, it is easy to see that a linear order in which there exist cliques C_1, C_2 and C_3 (in this order) such that $v \in C_1$ and $v \in C_3$ but $v \notin C_2$ cannot be a clique order, since the interval associated to v would be interrupted by the clique C_2 in any supposed model with that linear order of the cliques. Therefore, a necessary condition to any clique order is that all cliques containing some vertex v are consecutive within this order. In fact, this is a sufficient condition as well, according to a well-known characterization by Fulkerson and Gross [2]:

Theorem 1 (Fulkerson and Gross, 1965). *A graph is an interval graph if and only if there exists a linear order of its cliques such that, for each vertex v of the graph, the cliques containing v are consecutive within the order.*

Another well-known characterization of interval graphs is due to Booth and Lueker [1].

Theorem 2 (Booth and Lueker, 1976). *A graph G is an interval graph if and only if there exists a PQ-tree representing G.*

A *PQ-tree* of an interval graph G is a special ordered tree whose set of leaves is the set of cliques of G and each internal node is either a P *node* or a Q *node*. P nodes have at least two child nodes and Q nodes have at least three child nodes. The special property of a PQ-tree is that the linear order on the cliques obtained by reading its leaves from left to right is a clique order. Conversely, for any clique order $\mathcal{C}$ of G, there exists a sequence of operations on the tree such that after the application of them, the new order obtained by reading the leaves from left to right is precisely $\mathcal{C}$. Each operation is either a permutation of the children of a P node or a reversal of the children of a Q node. So, in some sense, a PQ-tree encodes all possible clique orders of the associated interval graph.

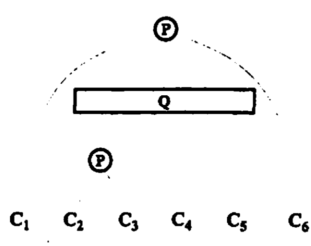

Figure 3: A PQ-tree of the graph in Figure 1.

A vertex v of an interval graph is an *extreme vertex* if there exists an interval model in which $\ell(I_v)$ is the *first* interval extreme point, i.e., the leftmost extreme point of the model. For instance, Figure 1 shows that the interval I_2 is an extreme vertex. It is easy to see that it is not the unique: I_9 is also an extreme vertex (just consider the reversal of the model in Figure 1).

A clique C is an *extreme clique* of an interval graph if there exists an interval model $\mathcal{R}$ such that C is the first clique within the clique order of $\mathcal{R}$. For example, the model in Figure 2 shows that the clique C_1 is an extreme clique, and by taking the reversal of that model, we see that so is

C_6. However, not every clique is an extreme clique. It is easily shown that there is no clique order of G in which C_4 is the first clique.

The first characterization of extreme vertices by forbidden subgraphs in interval graphs is due to Gimbel [3].

Theorem 3 (Gimbel, 1988). *A vertex v of an interval graph G is an extreme vertex if and only if G contains none of the forbidden induced subgraphs depicted in Figure 4 with v in the indicated position.*

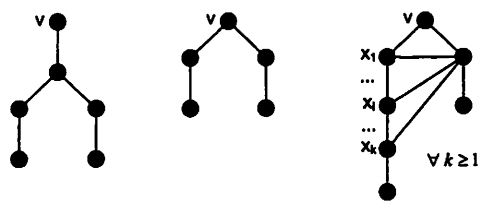

Figure 4: Forbidden induced subgraphs for an extreme vertex.

The remaining of the paper is as follows. In Section 2, we show a characterization of extreme cliques by forbidden induced subgraphs. In Section 3, we define and characterize the class of graphs in which all their cliques are extreme cliques.

2 Characterization of Extreme Cliques

Let $G = (V, E)$ be a graph. We denote by $G[W]$, $W \subseteq V$, the induced subgraph of G by W. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set $\{w \in V \mid (v, w) \in E\}$. A vertex $v \in V$ is said to be a *simplicial vertex* if $N(v) \cup \{v\}$ is a clique of G. If a clique contains a simplicial vertex, then it is called a *simplicial clique*. A vertex v is *universal* to a set of vertices $W \subseteq V$ if v is adjacent to each vertex $w \in W$. A *flip* of $\mathcal{R}$ is the new model obtained by reversing horizontally the intervals of $\mathcal{R}$ through a vertical line that crosses $\mathcal{R}$. A *submodel* $\mathcal{R}'$ of $\mathcal{R}$ is an interval model obtained by the removal of some intervals of $\mathcal{R}$. A *submodel* of $\mathcal{R}$ *induced* by the set of vertices W is the submodel obtained by the removal in $\mathcal{R}$ of the intervals that are not associated to vertices of W. For convenience, a vertex v and its associated interval in a model may be used interchangeably if no ambiguity occurs. So, for example, a vertex v universal to a submodel $\mathcal{R}$ means that v is universal to the set of vertices which have their associated intervals in $\mathcal{R}$.

The following lemma is clear.

Lemma 4. *If C is an extreme clique, then C is a simplicial clique.*

We show next another property, which will be useful in the characterization of extreme cliques.

Lemma 5. *Let C be a simplicial clique of an interval graph G such that for every simplicial vertex $s \in C$, G does not contain the induced subgraphs in Figure 5. Let $\mathcal{R}$ be an interval model of G in which C is neither the first nor the last clique of $\mathcal{R}$ and let C_1 and C_2 be the cliques which immediately precedes and succeeds C in $\mathcal{R}$, respectively. Then, $C_1 \cap C \supseteq C_2 \cap C$ or $C_1 \cap C \subseteq C_2 \cap C$.*

Proof. Suppose for a contradiction that the claim is false. Then there exist $u_1 \in C_1 \cap C$ such that $u_1 \notin C_2$, and $u_2 \in C_2 \cap C$ such that $u_2 \notin C_1$. Since C, C_1 and C_2 are pairwise distinct, there exist $a \in C_1$ and $b \in C_2$ such that $a, b \notin C$. Since C is an extreme clique, by Lemma 4, there exists a simplicial vertex $s \in C$. Therefore, there exists in G the subgraph $G[\{s, u_1, a, u_2, b\}]$ of type (b), a contradiction. □

Next result presents an extreme clique characterization by forbidden subgraphs.

Theorem 6. *Let G be an interval graph and C be a clique of G. Then C is an extreme clique if and only if C is simplicial and G contains none of the induced subgraphs in Figure 5, where s is a simplicial vertex of C.*

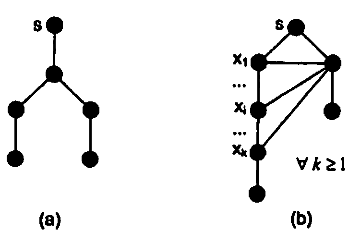

Figure 5: Forbidden induced subgraphs for an extreme clique.

Proof. If C is an extreme clique, then let $\mathcal{R}$ be an interval model of G such that C is the first clique of $\mathcal{R}$. By Lemma 4, there exists a simplicial vertex $s \in C$. Thus C is a simplicial clique. We show that G does not contain the forbidden subgraphs of Figure 5. Suppose, by way of contradiction, that G contains the induced subgraph (a) or (b) of Figure 5. Since C is the first clique of $\mathcal{R}$, the model "grows" to the right of s. Try to build $\mathcal{R}$ in the following cases:

- G contains the induced subgraph of type (a) (Figure 6): The unique possible model of the subgraph $G[\{s, a, c, e\}]$ is the one shown in the figure. Therefore, the interval b must be located in the model between the intervals s and c. Now the interval d must intersect b, but not a, which is impossible.

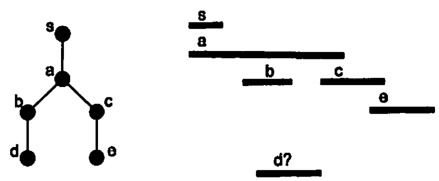

Figure 6: Subgraph of type (a).

- G contains the induced subgraph of type (b) (Figure 7): The unique possible model of the subgraph $G[\{s, a, x_1, \ldots, x_k\}]$ is the one shown in the figure. Therefore, the interval b must be located in the model at the right of x_k and intersecting a. Now, the interval c must intersect x_k, but not a, which is impossible.

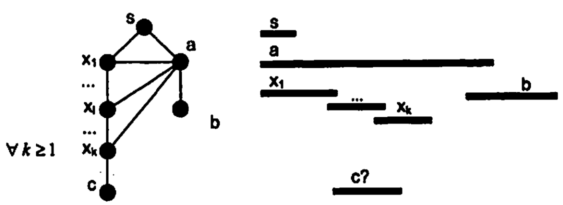

Figure 7: Subgraph of type (b).

Conversely, let C be a simplicial clique of G such that there do not exist in G induced subgraphs of type (a) and (b) with a simplicial vertex $s \in C$. Let $\mathcal{R}$ be an interval model of G. We show that it is possible to transform $\mathcal{R}$ into a new model in which C is the first clique.

If C is the first clique of $\mathcal{R}$, no transformation is needed. If C is the last clique of $\mathcal{R}$, the transformation consists simply in flipping $\mathcal{R}$. If C is neither the first nor the last clique of $\mathcal{R}$, let C_1 and C_2 be the cliques which immediately precedes and succeeds C in $\mathcal{R}$, respectively. By Lemma 5, we know that $C_1 \cap C \supseteq C_2 \cap C$ or $C_1 \cap C \subseteq C_2 \cap C$. Since C_1, C and C_2 are distinct cliques, there exist $a \in C_1$ and $b \in C_2$ such that $a \notin C$ and $b \notin C$. We obtain our transformation by induction on the number k of cliques of G.

Consider the case $k = 3$. Assume without loss of generality that $C_1 \cap C \supseteq C_2 \cap C$. Exchanging the positions of C_1 and C we obtain a new interval model of G in which C becomes the first clique. Note that this exchange is possible because $C_1 \cap C \supseteq C_2 \cap C$.

Now, suppose that the claim is true for any graph G with less than $k > 3$ cliques.

If G is disconnected, then let G^C be the connected component of G that contains the clique C and let $\mathcal{R}^C$ be the submodel of $\mathcal{R}$ induced by $V(G^C)$. Since the number of cliques of G^C is less than k, then by induction hypothesis we obtain from $\mathcal{R}^C$ another model $\mathcal{R}'^C$ in which C is the first clique. Removing from $\mathcal{R}$ the submodel $\mathcal{R}^C$ and adding $\mathcal{R}'^C$ to the left of the model, we obtain an interval model of G in which C is the first clique. Otherwise, if G is connected, let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the submodels induced by the union of all cliques that are at the left and at the right of C in $\mathcal{R}$, respectively.

If $C_1 \cap C \supset C_2 \cap C$ or $C_1 \cap C \subset C_2 \cap C$, suppose without loss of generality that $C_1 \cap C \supset C_2 \cap C$. Otherwise, $C_1 \cap C = C_2 \cap C$.

For both cases, note that, if each vertex $v \in C_1 \cap C_2$ were universal to $\mathcal{R}_1$, then we could obtain a model of G in which C is the first clique by moving $\mathcal{R}_2$ flipped to the left of $\mathcal{R}_1$, and then flipping the entire model. It is easy to check that the resulting model is an interval model of the same graph, but now with C as the first clique. Otherwise, let $w \in C_1 \cap C_2$ be the vertex that is not universal to $\mathcal{R}_1$ with the rightmost left extreme point in $\mathcal{R}$. Let x_1 be the interval of $\mathcal{R}_1$ such that it does not intersect w, and choose that with the rightmost right extreme point. Since G is connected, then let x_2 be the interval that intersects x_1 and w with the rightmost right extreme point.

Consider the following cases:

1. $C_1 \cap C \supset C_2 \cap C$ (Figure 8): Then there exists a vertex $u \in C_1 \cap C$ such that $u \notin C_2 \cap C$. We show that there does not exist a path in G between x_1 and u with only vertices associated to intervals in $\mathcal{R}_1$ which are not in $C_2 \cap C$. Suppose, by way of contradiction, that there exists such a path and pick a minimum path $P = x_1, x_2, \ldots, x_m, u$ of such a type. Since P is minimum, two vertices of P are adjacent if and only if they are consecutive in P. If $x_m \in C$, G contains the forbidden subgraph $G[\{s, w, b, x_m, \ldots, x_2, x_1\}]$ of type (b), a contradiction. Otherwise, then G contains the forbidden subgraph $G[\{s, w, b, u, x_m, \ldots, x_2, x_1\}]$ of type (b). Therefore, there does not

exist such a path.

Thus, there exist consecutive cliques C_1' preceding C_2' between C_1 and the rightmost clique which contains x_1 such that $C_1' \cap C_2' \subseteq C_1 \cap C_2$. By the choice of w ($w \in C_1 \cap C_2$ with the rightmost left extreme point in $\mathcal{R}$), we have $C_1' \cap C_2' = C_1 \cap C_2$.

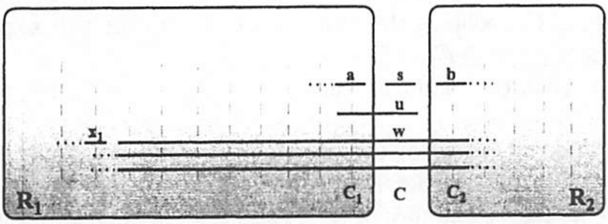

Figure 8: Case in which $C_1 \cap C \supset C_2 \cap C$.

Next, let G' be the graph obtained from G by removing the vertices belonging to any clique from C_2' until C_1, except those in $C_1 \cap C_2$. Let $\mathcal{R}_C$ be the submodel of $\mathcal{R}$ induced by the cliques from C_2' until C_1, after having removed the vertices of $C_1 \cap C_2$. Let $\mathcal{R}_{G'}$ be the submodel of $\mathcal{R}$ obtained by the removal from $\mathcal{R}$ of the intervals that are in $\mathcal{R}_C$. It is clear by construction that $\mathcal{R}_{G'}$ is an interval model of G'. Furthermore, since C_1' and C_1 are distinct cliques, G' has less than k cliques. Since the required properties of C also hold for G', the induction hypothesis implies that we can obtain a model $\mathcal{R}'$ from $\mathcal{R}_{G'}$ in which C is the first clique. The model obtained by the insertion into $\mathcal{R}'$ of $\mathcal{R}_C$ flipped between C and the remaining of $\mathcal{R}'$ is clearly an interval model of G, in which C is the first clique.

2. $C_1 \cap C = C_2 \cap C$ (Figure 9): if each vertex $v \in C_1 \cap C_2$ were universal to $\mathcal{R}_2$, then we could obtain a model in which C would be the first clique, moving $\mathcal{R}_1$ flipped to the right of $\mathcal{R}_2$. It is clear that the resulting model would be an interval model of the same graph, but with C as the first clique. Otherwise, let $w' \in C_1 \cap C_2$ be the vertex which is not universal to $\mathcal{R}_2$ with the leftmost right extreme point of $\mathcal{R}$. Let $y_1 \in \mathcal{R}_2$ be the interval which does not intersect w' with the leftmost left extreme point. Since G is connected, let y_2 be the interval that intersects y_1 and w' with the leftmost left extreme point. We show that at least one of the following statements is true: (i) the set of the vertices common to both the rightmost clique that contains x_1 and its immediate successor in $\mathcal{R}$ is a subset of $C_1 \cap C_2$; (ii) the set of the vertices common to both the leftmost clique that contains y_1 and its immediate predecessor in $\mathcal{R}$ is a subset of $C_1 \cap C_2$.

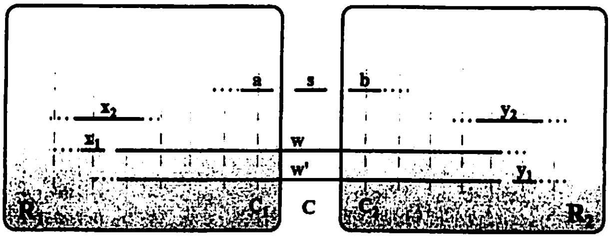

Figure 9: Case in which $C_1 \cap C = C_2 \cap C$.

Suppose, by way of contradiction, that $x_2 \notin C_1 \cap C_2$ and $y_2 \notin C_1 \cap C_2$. Since $C_1 \cap C = C_2 \cap C$, then $x_2, y_2 \notin C$. If $w = w'$, then there exists the forbidden subgraph $G[\{s, w, x_2, x_1, y_2, y_1\}]$ of type (a), a contradiction. Otherwise, if $w \neq w'$, because of the choice of w and w' we have $(x_2, w') \in E(G)$ and $(w, y_2) \in E(G)$. If $(w, y_1) \notin E(G)$, then there exists the forbidden subgraph $G[\{s, w, x_2, x_1, y_2, y_1\}]$ of type (a). Therefore $(w, y_1) \in E(G)$. If $(x_1, w') \notin E(G)$, then there would exist the forbidden subgraph $G[\{s, w', x_2, x_1, y_2, y_1\}]$ of type (a), a contradiction as well. Thus, $(x_1, w') \in E(G)$. Therefore there exists the forbidden subgraph $G[\{s, w, y_1, w', x_1\}]$ of type (b), a contradiction.

Consequently, our claim is true. Without loss of generality, suppose that the set of the common vertices to both the rightmost clique that contains x_1 and that clique's immediate successor in $\mathcal{R}$, denoted respectively by C_1' and C_2', is a subset of $C_1 \cap C_2$. Finally, we build the graph G' and complete the proof as in the previous case.

This completes the proof. □

3 Homogeneously Clique-Representable Graphs

After having characterized extreme vertices in [3], Gimbel characterizes the family of graphs whose vertices are all extreme vertices, called *homogeneously representable graphs*. Extending this concept to extreme cliques, we define a graph G to be *homogeneously clique-representable* if all cliques of G are extreme cliques. In this section, our goal is to provide a characterization of such graphs. Note that the class of homogeneously clique-representable graphs is a subclass of that of homogeneously representable graphs. And this inclusion is proper: a path of order 4 is homogeneously representable, but not homogeneously clique-representable, and this example is the smallest graph which separates these two classes.

A *bull* is the subgraph obtained in Figure 5 (*b*) for $k = 1$. In [3], the following characterization is presented:

Theorem 7 (Gimbel, 1988). *An interval graph is homogeneously representable if and only if it contains neither a path of order* 5, *nor a bull as an induced subgraph.*

The following lemma warrants that a connected interval graph is not homogeneously clique-representable when it does not contain a universal vertex.

Lemma 8. *Let G be an interval graph. If G is homogeneously clique-representable, then for each connected component G_i of G, there exists some vertex in G_i which is universal to G_i.*

Proof. Let $\mathcal{R}$ be an interval model of G, and $\mathcal{R}_i$ be the submodel of $\mathcal{R}$ induced by G_i. Let $C_1 \prec \ldots \prec C_q$ be the clique order of $\mathcal{R}_i$. Let I_v be the interval with the rightmost right extreme point such that $v \in C_1$. Let k be the largest integer such that $v \in C_k$. Suppose there is no universal vertex in G_i. Then, $k < q$. Since the cliques C_k and C_{k+1} are distinct, there exists $u \in C_{k+1}$ such that $u \notin C_k$. Since G_i is connected, there exists some vertex $w \in C_k \cap C_{k+1}$. Note that $w \notin C_1$ or the choice of I_v would be a contradiction. Since any extreme clique is a simplicial clique, there exist simplicial vertices $s' \in C_1$ and $s \in C_k$. Consequently, G contains the forbidden subgraph $G[\{s, w, u, v, s'\}]$ of type (*b*) in Figure 5, a contradiction. Therefore, each connected component G_i of G contains a universal vertex in G_i. □

The recursive procedure below produces an interval model on condition that, in each stage, it is applied to graphs having a universal vertex.

Procedure *Model*(G)

1. If $V(G) = \{v\}$, then return $\mathcal{R} = \{I_v\}$.

2. If G is disconnected with connected components $G_1, \ldots, G_\omega$, let $\mathcal{R}_i$ be the return of *Model*(G_i). Return $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_\omega$, such that $\mathcal{R}_i$ is entirely at the left of $\mathcal{R}_{i+1}$, for all $1 \leq i < \omega$.

3. If G does not contain a universal vertex, return $FAIL$. Otherwise, let u be some universal vertex of G and $\mathcal{R}'$ be *Model*($G \setminus u$). Return $\mathcal{R}$ as being $\mathcal{R}'$ added by the interval I_u universal to $\mathcal{R}'$.

Next theorem characterizes the homogeneously clique-representable graphs.

Theorem 9. *Let G be an interval graph. The following affirmatives are equivalent:*

(a) *G is a homogeneously clique-representable graph.*

(b) *G contains no induced path of order 4.*

(c) *Interval models of G can be obtained by executing Model(G).*

(d) *No PQ-tree of G contains a Q node.*

Proof. (a) $\iff$ (b): Let G be an interval graph. Suppose that G is homogeneously clique-representable. By Lemma 4, it follows that each clique of G is simplicial. Suppose there exists an induced path v_1, v_2, v_3, v_4 in G, and consider a clique C that contains v_2 and v_3. Since C contains a simplicial vertex s, there exists the forbidden subgraph $G[\{s, v_1, v_2, v_3, v_4\}]$ of type (b) in Figure 5, a contradiction. Conversely, suppose that G does not contain an induced path of order 4. By way of contradiction, suppose there exist consecutive cliques C_1, C, C_2 such that C is not simplicial. Since C_1 and C are distinct cliques, then there exist $v_1 \in C_1 \setminus C$ and $v_3 \in C \setminus C_1$. Similarly, there exist $v_4 \in C_2 \setminus C$ and $v_2 \in C \setminus C_2$. Since C is not simplicial, then $v_2 \in C_1$ and $v_3 \in C_2$. Therefore, G contains the induced path $G[\{v_1, v_2, v_3, v_4\}]$, a contradiction. Thus, every clique of G is simplicial. Since none of the simplicial vertices are contained in the forbidden subgraphs, otherwise G would contain an induced path of order 4, all cliques of G are extreme cliques.

(a) $\iff$ (c): It is clear that G is homogeneously clique-representable if and only if each connected component of G is homogeneously clique-representable. Furthermore, if G is a connected interval graph, then G is homogeneously clique-representable if and only if $G \setminus u$ is homogeneously clique-representable, where u is a universal vertex of G, whose existence is guaranteed by Lemma 8. Therefore, an interval graph G is homogeneously clique-representable if and only if the procedure *Model*(G) terminates successfully.

(a) $\iff$ (d): Let T be some PQ-tree of G. Note that G is homogeneously clique-representable if and only if there exists no clique C of G such that there is no PQ-tree T' of G such that: (i) T' is obtained from T by reversals of the children of Q nodes and permutations of the children of P nodes, and (ii) the clique order corresponding to T' has C as its first clique. On the other hand, there exists such a clique C if and only if there exists a Q node in T, since a Q node has at least three children, meaning that at least one clique can not be an extreme clique (a clique descendant of some "internal" child node of such a Q node). □

As a remark, the equivalence (a) $\iff$ (b) of the previous theorem implies that the class of homogeneously clique-representable graphs coincides with that of trivially perfect graphs, defined by Golumbic [4].

4 Conclusion

In [3] both characterizations of extreme vertices and of homogeneously representable graphs by forbidden subgraphs are presented. In this work, the goal has been to extend these characterizations to extreme cliques. Actually, we have characterized extreme cliques of an interval graph by forbidden subgraphs. In addition, the concept of homogeneously clique-representable graphs has been introduced and characterized.

It is possible to obtain the characterization of extreme cliques presented in this work from the characterization of extreme vertices given in Theorem 3. However, our proof is interesting because it is self contained, in a way that differently from the characterization of extreme vertices, it does not use the forbidden subgraphs of interval graphs. Furthermore, it leads to an algorithm that when applied to an interval model in which the considered extreme clique C is not the first one, it produces another interval model of the same graph, having C as the first.

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