Note on the choosability of bipartite graphs *

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Abstract. Let B be a bipartite graph. We obtain two new results as follows. (1) Suppose that $u \in V(B)$ is a vertex such that $N_B(u)$ contains at least $|N_B(u)| - 1$ odd vertices. Let $f : V(B) \to \mathbb{N}$ be the function such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B) \backslash u$. Then B is f-choosable. (2) Suppose that $u \in V(B)$ is a vertex such that every vertex in $N_B(u)$ is odd, and $v \in V(B)$ is an odd vertex that is not adjacent to u. Let $f : V(B) \to \mathbb{N}$ be the function such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil$ and $f(w) = \lceil d_B(w)/2 \rceil + 1$ for $w \in V(B) \backslash \{u, v\}$. Then B is f-choosable.

Key words: L-coloring, f-critical, f-choosable

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1. Introduction

Let $G = (V, E)$ be a simple graph. A *list assignment* L of G is a mapping that assigns to each $v \in V$ a set $L(v)$ of colors. An *L-coloring* of G is a proper coloring c of the vertices such that $c(v) \in L(v)$ for each $v \in V$. Let $\mathbb{N}$ denote the set of positive integers, and let $f : V \to \mathbb{N}$ be a function. G is *f-choosable* if, for any list assignment L of G such that $|L(v)| \geq f(v)$

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for each $v \in V$, G has an L-coloring. For $k \in \mathbb{N}$, G is *k-choosable* if G is f-choosable when $f(v) = k$ for each $v \in V$.

Let B be a bipartite graph. Alon and Tarsi [1] showed that B is $(\lceil \Delta(B)/2 \rceil + 1)$-choosable, where $\Delta(B)$ is the maximum degree of B. Wang and Huang [2] proved the following variant of this result, in which an *odd vertex* is a vertex of odd degree, and $d_B(v)$ is the degree of v in B.

Theorem 1. *Suppose that $u \in B$ is odd and let $f : V(B) \to \mathbb{N}$ be such that $f(u) = \lceil d_B(u)/2 \rceil$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B) \backslash u$. Then B is f-choosable.*

We prove two results as follows. (1) Suppose that $u \in V(B)$ is a vertex such that $N_B(u)$ contains at least $|N_B(u)| - 1$ odd vertices, where $N_B(u)$ is the set of the vertices in B which are adjacent to u. Let $f : V(B) \to \mathbb{N}$ be such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B) \backslash u$. Then B is f-choosable. (2) Suppose that $u \in V(B)$ is a vertex such that every vertex in $N_B(u)$ is odd, and $v \in V(B)$ is an odd vertex that is not adjacent to u. Let $f : V(B) \to \mathbb{N}$ be such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil$ and $f(w) = \lceil d_B(w)/2 \rceil + 1$ for $w \in V(B) \backslash \{u, v\}$. Then B is f-choosable.

2. The main results

Let $G_1, \ldots, G_m$ be disjoint graphs. For each $i \in \{1, \ldots, m\}$, let u_i and v_i be two different vertices of G_i. We denote by $T_{uv}[G_1 \ldots G_m]$ the new graph obtained by identifying all the vertices u_i and all the vertices v_i into new vertices u^* and v^*, respectively, as in Fig. 1.

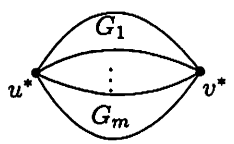

Fig. 1

If H is a subgraph of a graph G, and L is a list assignment of G, let $L|H$ denote L restricted to the vertices of H.

Lemma 2. *Let u_i and v_i be two distinct vertices of graph G_i and suppose that $f_i : V(G_i) \to \mathbb{N}$ is such that $f_i(u_i) = 1$ $(1 \leq i \leq m)$. Let $T_m^* = T_{uv}[G_1 \ldots G_m]$ and suppose that $f_T : V(T_m^*) \to \mathbb{N}$ is such that $f_T(u^*) = 1$ and $f_T(v^*) = \sum_{1 \leq i \leq m}(f_i(v_i)-1)+1$ and $f_T(v) = f_i(v)$ for $v \in V(G_i)\backslash\{u^*, v^*\}$. Then T_m^* is f_T-choosable if G_i is f_i-choosable $(1 \leq i \leq m)$.*

Proof Suppose that L is a list assignment of T_m^* satisfying $|L(v)| = f_T(v)$ $(v \in V(T_m^*))$. Let T_i be a set of colors $c_i \in L(v^*)$ such that G_i has an $L|G_i$-coloring, in which v^* is colored with c_i $(1 \leq i \leq m-1)$. Let $S_i = L(v^*)\backslash T_i$. Then $|S_i| \leq f_i(v_i) - 1$ since G_i is f_i-choosable. Define a list assignment L_m of G_m by setting $L_m(v_m) = L(v_m)\backslash \cup_{1 \leq i \leq m-1} S_i$ and $L_m(v) = L(v)$ for $v \in V(G_m)\backslash v_m$. Since $|L_m(v_m)| \geq f_m(v_m)$ and G_m is f_m-choosable, G_m has an L_m-coloring c_m. Since $v_m = v^*$ is given a color $c_m(v_m)$ that is not in any S_i, and hence is in every T_i $(1 \leq i \leq m-1)$, it follows from the definition of T_i that this coloring can be extended to an L-coloring of T_m^*. □

Theorem 3. *Suppose u is a vertex of a bipartite graph B such that $N_B(u)$ contains at least $|N_B(u)|-1$ odd vertices, and let $f : V(B) \to \mathbb{N}$ be such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil+1$ for $v \in V(B)\backslash u$. Then B is f-choosable.*

Proof Let $m = |N_B(u)| - 1$ and let $v_1, \ldots, v_m$ be distinct odd vertices in $N_B(u)$. Let $B_0 = B\backslash\{uv_i | 1 \leq i \leq m\}$. For $1 \leq i \leq m$ let G_i be a copy of K_2 with vertices $\overline{u}_i$ and $\overline{v}_i$, and let B_i be the graph obtained from G_i and B_{i-1} by identifying $\overline{u}_i$ with u and $\overline{v}_i$ with v_i, respectively. It is clear that $B = B_m$. Next we use induction to prove that B is f-choosable.

Let $g_i : V(G_i) \to \mathbb{N}$ be such that $g_i(\overline{u}_i) = 1$ and $g(\overline{v}_i) = 2$; then G_i is g_i-choosable $(1 \leq i \leq m)$. Let $f_i : V(B_i) \to \mathbb{N}$ be such that $f_i(u) = 1$ and $f_i(v) = \lceil d_{B_i}(v)/2 \rceil + 1$ for $v \in V(B_i)\backslash u$ $(0 \leq i \leq m)$. By Theorem 1, B_0 is f_0-choosable. Since $d_{B_{i-1}}(v_i)$ is even, $\lceil d_{B_i}(v_i)/2 \rceil = \lceil d_{B_{i-1}}(v_i)/2 \rceil + 1$ $(1 \leq i \leq m)$. It follows from Lemma 2 that if B_{i-1} is f_{i-1}-choosable then B_i is f_i-choosable. Since $f_m = f$, It follows that B is f-choosable. □

Lemma 4. *Let u be a vertex of graph G and suppose that $f : V(G) \to \mathbb{N}$ is a function such that $f(u) = 1$. Suppose that $g : V(G)\backslash u \to \mathbb{N}$ is the function such that $g(v) = f(v)-1$ if $v \in N_G(u)$ and $g(v) = f(v)$ otherwise. Then G is f-choosable if and only if $G\backslash u$ is g-choosable.*

Proof "If" : Let L be a list assignment of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. Color u with the color $c(u) \in L(u)$ and define a list assignment L' of $G\backslash u$ by setting $L'(v) = L(v)\backslash c(u)$ if $v \in N_G(u)$ and

$L'(v) = L(v)$ otherwise. Then $|L'(v)| \geq g(v)$ for each $v \in V(G)\backslash u$, and so $G\backslash u$ has an L'-coloring since $G\backslash u$ is g-choosable. Thus G has an L-coloring.

"Only If" : Suppose that $G\backslash u$ is not g-choosable. Then there is a list assignment L' of $G\backslash u$ with $|L'(v)| = g(v)$ for each $v \in V(G)\backslash u$ such that $G\backslash u$ has no L'-coloring. Choose $c \notin \cup_{v \in N_G(u)} L'(v)$ and define a list assignment L of G by setting $L(u) = \{c\}$ and $L(v) = L'(v) \cup \{c\}$ for $v \in N_G(u)$ and $L(v) = L'(v)$ for $v \in V(G)\backslash(N_G(u) \cup \{u\})$. Then $|L(v)| = f(v)$ for each $v \in V(G)$. Clearly G has no L-coloring. □

Let G be a graph, and let f, $g_u : V(G) \to \mathbb{N}$ be such that $g_u(u) = f(u) - 1$ and $g_u(v) = f(v)$ for $v \in V(G)\backslash u$. Suppose that G is f-choosable. Then G is *f-critical at $u \in V(G)$* if G is not g_u-choosable.

Lemma 5. *Let u_i and v_i be two nonadjacent vertices of graph G_i, and f_i, T_m^* and f_T be as in Lemma 2 ($1 \leq i \leq m$). Suppose that G_i is f_i-critical at v_i ($1 \leq i \leq m-1$). Then, for any vertex $\widetilde{v} \in V(G_m)$ that is not adjacent to u_m, G_m is f_m-critical at $\widetilde{v}$ if and only if T_m^* is f_T-critical at $\widetilde{v}$.*

Proof Since $f_m(u_m) = f_T(u^*) = 1$, it is clear that G_m is f_m-critical at u_m and T_m^* is f_T-critical at $u^* = u_m$. Thus we may assume that $\widetilde{v} \neq u_m$.

"If" : We now prove that T_m^* is not f_T-critical at $\widetilde{v}$ if G_m is not f_m-critical at $\widetilde{v}$. Let L be a list assignment of T_m^* such that $|L(\widetilde{v})| = f_T(\widetilde{v}) - 1$ and $|L(v)| = f_T(v)$ for each $v \in V(T_m^*)\backslash\widetilde{v}$. Then we can obtain that T_m^* has an L-coloring as in the proof of Lemma 2.

"Only if" : Since G_i is f_i-choosable ($1 \leq i \leq m$), T_m^* is f_T-choosable by Lemma 2. By Lemma 4, for each i for which G_i is f_i-critical at v_i, there exists a list assignment L_i of $G_i\backslash u_i$ such that

$$\left.\begin{array}{ll} |L_i(v)| = f_i(v) - 1 & \text{if } v \in N_{G_i}(u_i) \cup \{v_i\}, \\ |L_i(v)| = f_i(v) & \text{if } v \in V(G_i\backslash u_i)\backslash(N_{G_i}(u_i) \cup \{v_i\}), \\ G_i\backslash u_i \text{ has no } L_i\text{-coloring.} & \end{array}\right\} \quad (1)$$

Suppose first that $\widetilde{v} = v_m$. Then G_i is f_i-critical at v_i for $1 \leq i \leq m$, and list assignment L_i satisfying (1) exist for all i and can be chosen so that $L_i(v_i) \cap L_j(v_j) = \emptyset$ if $i \neq j$ ($i, j \in \{1, 2, \ldots, m\}$). Let c be a color not used by any L_i and define a list assignment L of T_m^* by setting

$$\left.\begin{array}{ll} L(u^*) = \{c\},\ L(v^*) = \cup_{1 \leq i \leq m} L_i(v_i), & \\ L(v) = L_i(v) \cup \{c\} & \text{if } v \in N_{G_i}(u_i), \\ L(v) = L_i(v) & \text{if } v \in V(G_i)\backslash(N_{G_i}(u_i) \cup \{u^*, v^*\}). \end{array}\right\} \quad (2)$$

for each $i \in \{1, 2, \ldots, m\}$, so that $|L(v)| = f_T(v)$ for each vertex $v \in T_m^*\backslash\widetilde{v}$ and $L(\widetilde{v}) = f_T(\widetilde{v}) - 1$. Clearly T_m^* has no L-coloring. This shows that T_m^* is f_T-critical at $\widetilde{v}(= v_m = v^*)$.

Suppose now that $\widetilde{v} \in V(G_m)\backslash\{u_m, v_m\}$. Since G_i is f_i-critical at v_i for $1 \leq i \leq m-1$, there exists a list assignment L_i of $G_i\backslash u_i$ satisfying (1) for all such i. And since G_m is f_m-critical at $\widetilde{v}$, there exists a list assignment L_m of $G_m\backslash u_m$ satisfying (1) when i is replaced by m and both occurrences of v_i in (1) are replaced by $\widetilde{v}$. The list assignment L_i can be chosen so that $L_i(v_i) \cap L_j(v_j) = \emptyset$ if $i \neq j$ $(i, j \in \{1, 2, \ldots, m\})$. As before, let c be a color not used by any L_i and define a list assignment L of T_m^* by (2). Clearly T_m^* has no L-coloring, and so T_m^* is f_T-critical at $\widetilde{v}$. □

Lemma 6. *Let u be a vertex of a bipartite graph B such that every neighbor of u is odd and suppose that $f : V(B) \to \mathbb{N}$ is the function such that $f(u) = 1$ and $f(w) = \lceil d_B(w)/2 \rceil + 1$ for $w \in V(B)\backslash u$. If $v \in V(B)$ is not adjacent to u and B is f-critical at v, then $d_B(v)$ is even.*

Proof Let $T_2^* = T_{uv}[BB]$ and suppose that $f_T : V(T_2^*) \to \mathbb{N}$ is the function such that $f_T(u^*) = 1$ and $f_T(v^*) = 2f(v) - 1$ and $f_T(w) = f(w)$ for $w \in V(T_2^*)\backslash\{u^*, v^*\}$. By Lemma 5, T_2^* is f_T-critical at v^*. This implies that T_2^* is not g-choosable, where $g : V(T_2^*) \to \mathbb{N}$ is the function such that $g(v^*) = 2(f(v) - 1)$ and $g(v) = f(v)$ for $v \neq v^*$. If $d_B(v)$ is odd then $g(v^*) = 2(f(v) - 1) = 2\lceil d_B(v)/2 \rceil = d_B(v) + 1 = d_{T_2^*}(v^*)/2 + 1$. Noting that T_2^* is still a bipartite graph, we can obtain that T_2^* is g-choosable by Theorem 3. This contradiction shows that $d_B(v)$ is even. □

As one consequence of Lemma 6, we have

Theorem 7. *Suppose that u and v be two nonadjacent vertices of a bipartite graph B such that every neighbor of u is odd and so is v. Let $f : V(B) \to \mathbb{N}$ be the function such that $f(u) = 1$ and $f(v) = \lceil d_B(v)/2 \rceil$ and $f(w) = \lceil d_B(w)/2 \rceil + 1$ for $w \in V(B)\backslash\{u, v\}$. Then B is f-choosable.*

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References

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