The spectral characterization of graphs of index less than 2 with no Z_n as a component

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Abstract

A graph is said to be determined by its adjacency spectrum (or to be a DS graph, for short) if there is no other non-isomorphic graph with the same adjacency spectrum. Although all connected graphs of index less than 2 are known to be determined by their adjacency spectra, the classification of DS graphs of index less than 2 is not complete yet. The purpose of this paper is to characterize all DS graphs of index less than 2 with no Z_n as a component.

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1. Introduction

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Let G be an undirected finite simple graph with n vertices and the adjacency matrix $A(G)$. Since $A(G)$ is a real symmetric matrix, its eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of G. The multiset of eigenvalues of $A(G)$ is called the *adjacency spectrum* (or *A-spectrum*, for short) of G. The maximum eigenvalue of $A(G)$ is called the *index* of G. Two graphs are said to be *cospectral with respect to adjacency matrix* (or *A-cospectral*, for short) if they have the same A-spectra. A graph is said to be *determined* (DS for short) *by its A-spectrum* if there is no other non-isomorphic graph with the same A-spectrum. There have been some attempts to characterize graphs having an index at most a given number. In [7] all graphs of index at most 2 are identified. Subsequently, graphs of index not exceeding $\sqrt{2+\sqrt{5}}$ are determined in [1, 2]. Most of the connected graphs of index at most 2 are known to be DS with respect to the adjacency matrix (see [6, 9, 10]). Moreover, in [4], all connected DS graphs of index at most $\sqrt{2+\sqrt{5}}$ are identified. Although the classification of DS graphs of index less than 2 is not complete, some important results are known. All connected graphs of index less than 2 are known to be determined by their A-spectra [4]. In [9], it has been shown that the disjoint union of k disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$ is determined by its A-spectrum as well as the its L-spectrum, where $n_1, n_2, \ldots, n_k$ are integers at least 2. Moreover, in [6], Shen and others showed that $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ is determined by its A-spectrum, where $n_1, n_2, \ldots, n_k$ are integers at least 2. Recently, we characterized all DS graphs of index less than 2 with no path as a component [5]. For more information about DS graphs we refer the interested reader to [8, 9]. In this paper, we consider graphs of index less than 2, and among them we identify those DS graphs which does not have any Z_n as a component.

2. Graphs of index less than 2

In [7], all connected graphs of index less than 2 are identified. Moreover, all of them are known to be determined by their A-spectra [4].

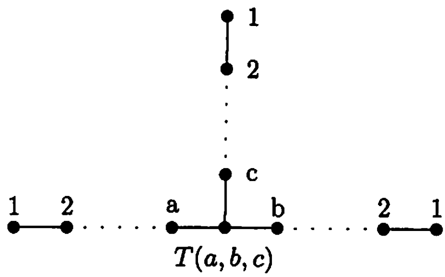

Fig. 1

Notation. The path and cycle with n vertices are denoted by P_n and C_n, respectively. For $a, b, c \geq 1$, we denote the graph shown in Fig. 1, by $T(a, b, c)$. In particular, $Z_n (n \geq 2)$ stands for $T(1, n-1, 1)$. We denote the graphs $T(1,2,2)$, $T(1,2,3)$ and $T(1,2,4)$ by T_i for $i = 1,2,3$, respectively.

Theorem 1.[7] *The list of all connected graphs of index less than 2 includes precisely the following graphs (see Fig. 1):*
i) $P_n, Z_n (n \geq 2)$,
ii) T_i *for* $i = 1,2,3$.

Let G be a graph of index less than 2. Then G can be represented in a unique way as a linear combination of the form

$$P_{i_1} + P_{i_2} + \cdots + P_{i_l} + Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1 T_1 + t_2 T_2 + t_3 T_3.$$

The A-spectrum of the union of two graphs is obviously the union of their spectra (having in view the multiplicities of the eigenvalues). The expressions $G_1 + G_2$ and $\hat{G}_1 + \hat{G}_2$ will denote the union of the graphs G_1 and G_2 and the union of their adjacency spectra, respectively. The expressions kG and $k\hat{G}$ denote the union of k copies of G and $\hat{G}$, respectively. If $\hat{G}_1 \subseteq \hat{G}_2$, the expression $\hat{G}_2 - \hat{G}_1$ denote the difference of systems of numbers of $\hat{G}_1$ from $\hat{G}_2$.

The A-spectra of graphs of index less than 2 are known [3], and we have:

$$\hat{P}_n = \{2\cos\frac{j\pi}{n+1} \mid j = 1,2,\ldots,n\},$$

$$\hat{Z}_n = \{2\cos\frac{(2j+1)\pi}{2(n+1)} \mid j = 0,1,2,\ldots,n\} + \{0\},$$

$$\hat{T}_1 = \{2\cos\frac{j\pi}{12} \mid j = 1,4,5,7,8,11\},$$

$$\hat{T}_2 = \{2\cos\frac{j\pi}{18} \mid j = 1,5,7,9,11,13,17\},$$

$$\hat{T}_3 = \{2\cos\frac{j\pi}{30} \mid j = 1,7,11,13,17,19,23,29\}.$$

Lemma 1.[9]. *For $n \times n$ symmetric matrices A and B, the following are equivalent:*

i) *A and B are cospectral,*

ii) $\mathrm{tr}(A^i) = \mathrm{tr}(B^i)$ *for $i = 1,\ldots,n$.*

If A is the adjacency matrix of a graph, then $\mathrm{tr}(A^i)$ gives the total number of closed walks of length i. So by the above lemma, two A-cospectral graphs have the same number of closed walks of a given length i. In particular, they have the same number of edges and triangles. The following lemma gives some A-cospectral graphs of index less than 2.

Lemma 2.[3] *The following can be obtained from the above quoted facts:*

i) $\hat{Z}_n + \hat{P}_n = \hat{P}_{2n+1} + \hat{P}_1$,

ii) $\hat{T}_1 + \hat{P}_3 + \hat{P}_5 = \hat{P}_1 + \hat{P}_2 + \hat{P}_{11}$,

iii) $\hat{T}_2 + \hat{P}_5 + \hat{P}_8 = \hat{P}_{17} + \hat{P}_2 + \hat{P}_1$,

iv) $\hat{T}_3 + \hat{P}_{14} + \hat{P}_9 + \hat{P}_5 = \hat{P}_{29} + \hat{P}_4 + \hat{P}_2 + \hat{P}_1$.

Theorem 2.[3] *The A-spectrum of any graph of index less than 2 can be represented in a unique way as a linear combination of the form*

$$\delta_1\hat{P}_1 + \delta_2\hat{P}_2 + \cdots + \delta_m\hat{P}_m.$$

The number m is bounded by a function of the number of vertices.

Lemma 3.[9] *Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r}$ be a graph of index less than 2 and let $1 < i_1 \leq i_2 \leq \cdots \leq i_r$. Then G can be determined by its Aspectrum.*

2. Main results

In this section, we give some A-cospectral graphs. The results will be used to characterize DS graphs of index less than 2 with no Z_n as a component.

Lemma 4. *Let a be a non-negative integer. We have:*
i) $\hat{H}_1 = \hat{P}_8 + \hat{P}_{11} + \hat{T}_2 = \hat{P}_2 + \hat{P}_{17} + \hat{Z}_5$,
ii) $\hat{H}_2 = \hat{P}_9 + \hat{P}_{11} + \hat{P}_{14} + \hat{T}_3 = \hat{P}_2 + \hat{P}_4 + \hat{P}_{29} + \hat{Z}_5$,
iii) $\hat{H}_3 = \hat{P}_9 + \hat{P}_{14} + \hat{P}_{17} + \hat{T}_3 = \hat{P}_4 + \hat{P}_8 + \hat{P}_{29} + \hat{T}_2$,
iv) $\hat{H}_4 = \hat{T}_2 + \hat{P}_5 = \hat{Z}_8 + \hat{P}_2$,
v) $\hat{H}_5 = \hat{T}_1 + \hat{P}_3 = \hat{Z}_5 + \hat{P}_2$,
vi) $\hat{H}_6 = \hat{T}_3 + \hat{P}_5 + \hat{P}_9 = \hat{Z}_{14} + \hat{P}_2 + \hat{P}_4$,
vii) $\hat{H}_7 = \hat{P}_5 + \hat{P}_7 + \hat{T}_1 = \hat{P}_2 + \hat{P}_{11} + \hat{Z}_3$,
viii) $\hat{H}_8 = \hat{P}_5 + \hat{P}_{14} + \hat{P}_{19} + \hat{T}_3 = \hat{P}_2 + \hat{P}_4 + \hat{P}_{29} + \hat{Z}_9$,
ix) *If $a \geq 2$, then $\hat{H}_9 = \hat{P}_1 + \hat{P}_{2a+1} = \hat{P}_a + \hat{Z}_a$.*

Proof. Using Lemma 2, we can represent each side of these equations as a linear combination of the form $\delta_1\hat{P}_1 + \delta_2\hat{P}_2 + \cdots + \delta_m\hat{P}_m$. It is clear that the two sides of each equation have the same representation. ☐

Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r} + t_1T_1 + t_2T_2 + t_3T_3$ be a graph of index less than 2. Suppose that G has p_i components of type P_i. Then for some non-negative integer r, G can be represented as

$$G = \sum_{i=1}^{r} p_iP_i + t_1T_1 + t_2T_2 + t_3T_3.$$

Let H be A-cospectral to G. Since G has index less than 2, H can be represented in a unique way as a linear combination of the form

$$H = P_{j_1} + P_{j_2} + \cdots + P_{j_t} + Z_{c_1} + Z_{c_2} + \cdots + Z_{c_k} + \bar{t}_1T_1 + \bar{t}_2T_2 + \bar{t}_3T_3.$$

Again, suppose that H has $\bar{p}_i$ components of type P_i and $\bar{z}_i$ components of type Z_i. Then for some non-negative integers $\bar{r}$ and $\bar{l}$, H can be represented as

$$H = \sum_{i=1}^{\bar{r}} \bar{p}_i P_i + \sum_{i=2}^{\bar{l}} \bar{z}_i Z_i + \bar{t}_1 T_1 + \bar{t}_2 T_2 + \bar{t}_3 T_3.$$

Since G and H are A-cospectral and bipartite graphs of index less than 2, by Theorem 2, their spectra can be represented in a unique way as a linear combination of the form

$$\delta_1 \hat{P}_1 + \delta_2 \hat{P}_2 + \cdots + \delta_m \hat{P}_m. \tag{1}$$

By Lemma 1, H and G have the same numbers of vertices and edges. Since their components are trees, they have the same number of components and we have:

$$\sum_{i=1}^{r} p_i + t_1 + t_2 + t_3 = \sum_{i=1}^{\bar{r}} \bar{p}_i + \sum_{i=2}^{\bar{l}} \bar{z}_i + \bar{t}_1 + \bar{t}_2 + \bar{t}_3. \tag{2}$$

Using Theorem 2, to represent $\hat{G}$ and $\hat{H}$ in terms of linear combinations of $\hat{P}_i$'s, we can calculate δ_i for any i.

$$\delta_5 = p_5 - t_1 - t_2 - t_3 = \bar{p}_5 + \bar{z}_2 - \bar{z}_5 - \bar{t}_1 - \bar{t}_2 - \bar{t}_3. \tag{3}$$

$$\delta_1 = p_1 + t_1 + t_2 + t_3 = \bar{p}_1 + \sum_{i=2}^{\bar{l}} \bar{z}_i + \bar{t}_1 + \bar{t}_2 + \bar{t}_3. \tag{4}$$

By the relations (2) and (4) we have $\sum_{i=2}^{r} p_i = \sum_{i=2}^{\bar{r}} \bar{p}_i$.

$$\delta_2 = p_2 + t_1 + t_2 + t_3 = \bar{p}_2 - \bar{z}_2 + \bar{t}_1 + \bar{t}_2 + \bar{t}_3. \tag{5}$$

$$\delta_3 = p_3 - t_1 = \bar{p}_3 - \bar{z}_3 - \bar{t}_1. \tag{6}$$

$$\delta_4 = p_4 + t_3 = \bar{p}_4 - \bar{z}_4 + \bar{t}_3. \tag{7}$$

$$\delta_{11} = p_{11} + t_1 = \bar{p}_{11} + \bar{z}_5 - \bar{z}_{11} + \bar{t}_1. \tag{8}$$

$$\delta_8 = p_8 - t_2 = \bar{p}_8 - \bar{z}_8 - \bar{t}_2. \tag{9}$$

$$\delta_{14} = p_{14} - t_3 = \bar{p}_{14} - \bar{z}_{14} - \bar{t}_3. \tag{10}$$

$$\delta_{29} = p_{29} + t_3 = \bar{p}_{29} - \bar{z}_{29} + \bar{z}_{14} + \bar{t}_3. \tag{11}$$

$$\delta_9 = p_9 - t_3 = \bar{p}_9 + \bar{z}_4 - \bar{z}_9 - \bar{t}_3. \tag{12}$$

$$\delta_{17} = p_{17} + t_2 = \bar{p}_{17} + \bar{z}_8 - \bar{z}_{17} + \bar{t}_2. \tag{13}$$

$$\delta_7 = p_7 = \bar{p}_7 + \bar{z}_3 - \bar{z}_7. \tag{14}$$

Lemma 5. *Let* $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r} + t_1T_1 + t_2T_2 + t_3T_3$ *be a graph of index less than 2. If* G *has* P_1 *as a component, then* G *can be determined by its spectrum if and only if* G *does not have any components where the spectra of their unions is* $\hat{H}_9$.

Proof. Suppose that G has p_i components of type P_i. Then for some non-negative integer r, G can be represented as

$$G = \sum_{i=1}^{r} p_iP_i + t_1T_1 + t_2T_2 + t_3T_3.$$

It is clear that if G has some components where the spectra of their unions is $\hat{H}_9$, then G has a A-cospectral mate and so it can not be determined by its A-spectrum. Now suppose that G does not have any components where the spectra of their unions is $\hat{H}_9$. We show that if G has P_1 as a component, then G is determined by its A-spectrum. Let H be A-cospectral to G. Since G has index less than 2, for some non-negative integers $\bar{r}$ and $\bar{l}$, H can be represented as

$$H = \sum_{i=1}^{\bar{r}} \bar{p}_iP_i + \sum_{i=2}^{\bar{l}} \bar{z}_iZ_i + \bar{t}_1T_1 + \bar{t}_2T_2 + \bar{t}_3T_3.$$

Without loss of generality we can suppose that at most one of the t_i and $\bar{t}_i$ (p_i and $\bar{p}_i$) is nonzero. So we can suppose that $p_1 > 0$ and $\bar{p}_1 = 0$. Since G dose not have any components where the spectrum of their unions is $\hat{H}_9$, for any $n \geq 1$ we have $p_{(2n+1)} = 0$. Let $S = \{1, 2, 4, 5, 8, 14, \}$ and let $x \in N - S$ be a natural number. Then for $n \geq 1$ we have:

$$\delta_{(2^nx+2^n-1)} = p_{(2^nx+2^n-1)} = \bar{p}_{(2^nx+2^n-1)} + \bar{z}_{(2^{n-1}x+2^{n-1}-1)} - \bar{z}_{(2^nx+2^n-1)} = 0.$$

Therefore $\bar{p}_{(2^nx+2^n-1)} = \bar{z}_{(2^nx+2^n-1)} - \bar{z}_{(2^{n-1}x+2^{n-1}-1)} \geq 0$. It means that the sequence $\{\bar{z}_{(2^nx+2^n-1)}\}$ is an increasing sequence. If $\bar{z}_x > 0$, then $\bar{z}_{(2^nx+2^n-1)} > 0$ for any $n \geq 0$. But this contradicts to the fact that G is a finite graph. So $\bar{z}_x = 0$. Since $p_3 = \bar{z}_3 = 0$, then by (6), $\bar{p}_3 = \bar{t}_1 - t_1 \geq 0$. Again, since $p_{11} = \bar{z}_{11} = 0$, by (8) we have $\bar{p}_{11} = t_1 - \bar{t}_1 - \bar{z}_5 \geq 0$. So $\bar{p}_{11} = \bar{p}_3 = t_1 = \bar{t}_1 = \bar{z}_5 = 0$. Since $p_9 = \bar{z}_9 = 0$ then by (12),

$\bar{p}_9 = \bar{t}_3 - \bar{z}_4 - t_3 \geq 0$. Again, we have $p_{29} = \bar{z}_{29} = 0$. So by (11), $\bar{p}_{29} = t_3 - \bar{t}_3 - \bar{z}_{14} \geq 0$. So $\bar{p}_{29} = \bar{p}_9 = t_3 = \bar{t}_3 = \bar{z}_4 = \bar{z}_{14} = 0$. Since $p_{17} = \bar{z}_{17} = 0$, then by (13), $\bar{p}_{17} = t_2 - \bar{t}_2 - \bar{z}_8$ and we can suppose that $\bar{t}_2 = 0$. Again, since $p_5 = 0$, by (3) we have $\bar{p}_5 = \bar{z}_5 - \bar{z}_2 - t_2 \geq 0$ and so $\bar{p}_5 = \bar{p}_2 = t_2 = \bar{z}_2 = \bar{z}_5 = \bar{z}_8 = \bar{p}_{17} = \bar{p}_8 = 0$. Since for $1 \leq i \leq 3$, $\bar{t}_i = t_i = 0$ and for $n \geq 1$, $p_{(2n+1)} = 0$, then for any natural number y we have:

$$\delta_{(2^n y+2^n-1)} = p_{(2^n y+2^n-1)} = \bar{p}_{(2^n y+2^n-1)} + \bar{z}_{(2^{n-1}y+2^{n-1}-1)} - \bar{z}_{(2^n y+2^n-1)} = 0.$$

Hence $\bar{p}_{(2^n y+2^n-1)} = \bar{z}_{(2^n y+2^n-1)} - \bar{z}_{(2^{n-1}y+2^{n-1}-1)} \geq 0$. Again, the sequence $\{\bar{z}_{(2^n y+2^n-1)}\}$ is an increasing sequence. If $\bar{z}_y > 0$, then $\bar{z}_{(2^n y+2^n-1)} > 0$ for any $n \geq 0$. Which contradicts to this fact that G is a finite graph. So $\bar{z}_y = 0$ and so by (4), $p_1 = \bar{p}_1$, which is not possible. □

Theorem 3. *Let $G = \sum_{i=1}^{r} p_i P_i + t_1 T_1 + t_2 T_2 + t_3 T_3$ be a graph of index less than 2. Then G can be determined by its spectrum if and only if G does not have any components where the spectrum of their unions is $\hat{H}_i$ $(i = 1, \ldots, 9)$.*

Proof. It is clear that if G has some components where the spectrum of their unions is $\hat{H}_i$ $(i = 1, \ldots, 9)$, then G has a A-cospectral mate and so it can not be determined by its A-spectrum. Now suppose that G does not have any components where the spectrum of their unions is $\hat{H}_i$ for $i = 1, \ldots, 9$. We show that G is determined by its A-spectrum. Let $H = \sum_{i=1}^{\bar{r}} \bar{p}_i P_i + \sum_{i=2}^{\bar{l}} \bar{z}_i Z_i + \bar{t}_1 T_1 + \bar{t}_2 T_2 + \bar{t}_3 T_3$ be A-cospectral to G. Without loss of generality we can suppose that at most one of the t_i and $\bar{t}_i$ (p_i and $\bar{p}_i$) is nonzero. Since G does not have any components where the spectrum of their unions is $\hat{H}_9$, if $p_1 > 0$, then by Lemma 5, G can be determined by its spectrum. So we can suppose that $p_1 = 0$. By the relations (4) and (5), we have $p_2 = \bar{p}_2 - \bar{p}_1 - \bar{z}_2 - \sum_{i=2}^{\bar{l}} \bar{z}_i$. So $p_2 = 0$, $\bar{p}_2 = \bar{p}_1 + \bar{z}_2 + \sum \bar{z}_i$ and $\bar{p}_2 \geq \bar{z}_5$. Hence by (3), (4), $p_5 \geq \bar{p}_5$ and we can suppose that $\bar{p}_5 = 0$ and $p_5 = \bar{p}_2 - \bar{z}_5$. We have the following cases:

i) Let $t_1 > 0$ and $\bar{t}_1 = 0$. Since G does not have any components where the spectrum of their unions is $\hat{H}_5$, $p_3 = 0$ and so by (6), $\bar{p}_3 = \bar{z}_3 - t_1$. If $t_2 > 0$ and $\bar{t}_2 = 0$, then since G does not have any components where the spectrum of their unions is $\hat{H}_4$, $p_5 = 0$ and so $\bar{p}_2 = \bar{z}_5$. Therefore, for $i \neq 5$, $\bar{p}_1 = \bar{z}_i = 0$ and $\bar{p}_3 = -t_1 < 0$, which is impossible. So $t_2 = 0$ and $\bar{t}_2 \geq 0$. If $t_3 = 0$, then by (4), we have $t_1 = \bar{p}_1 + \sum_{i=2}^{l} \bar{z}_i + \bar{t}_2 + \bar{t}_3 \geq \bar{z}_3$. On the other hand, $\bar{p}_3 = \bar{z}_3 - t_1 \geq 0$. So $t_1 = \bar{p}_1 + \sum_{i=2}^{l} \bar{z}_i + \bar{t}_2 + \bar{t}_3 = \bar{z}_3$. Hence for $i \neq 3$, $\bar{p}_3 = \bar{p}_1 = \bar{z}_i = \bar{t}_2 = \bar{t}_3 = 0$, $t_1 = \bar{z}_3$. By (8), $p_{11} = 0$ and $\bar{p}_{11} = t_1$. In a similar way by (3), (5) and (14) we have $t_1 = p_5 = p_7 = \bar{p}_2 = \bar{p}_{11} = \bar{z}_3 > 0$. So G has some components where the spectrum of their unions is $\hat{H}_7$. This is impossible. Hence $t_3 > 0$ and $\bar{t}_3 = 0$. By (7), $p_4 = 0$ and $\bar{p}_4 = t_3 + \bar{z}_4$. Since G dose not have any components where the spectrum of their unions is $\hat{H}_6$, we have $p_5 p_9 = 0$. If $p_5 = 0$, then by (3) and (4) we have $\bar{p}_1 + \bar{p}_5 + \bar{z}_2 - \bar{z}_5 + \sum_{i=2}^{l} \bar{z}_i = 0$. Therefore, for $i \neq 5$, $\bar{p}_5 = \bar{p}_1 = \bar{z}_i = 0$ and by (6), $\bar{p}_3 = -t_1 < 0$, this is impossible. So $p_9 = 0$ and by (12), $\bar{p}_9 = \bar{z}_9 - \bar{z}_4 - t_3$. Since G dose not have any components where the spectrum of their unions is $\hat{H}_5$, $p_3 = 0$ and by (6), $\bar{p}_3 = \bar{z}_3 - t_1 \geq 0$. Hence by (4), $t_3 \geq \bar{p}_1 + \sum_{i=2}^{l} \bar{z}_i - \bar{z}_3 + \bar{t}_2$. On the other hand by (12), $t_3 = \bar{z}_9 - \bar{z}_4 - \bar{p}_9$. So for $i \neq 3, 9$, we have $\bar{z}_i = \bar{p}_9 = \bar{p}_1 = \bar{t}_2 = 0$. Again, using the previous equations we have $p_7 = \bar{p}_{11} = \bar{z}_3 = t_1$, $p_{14} = \bar{p}_4 = \bar{p}_{29} = t_3$, $p_5 = \bar{p}_2 = t_1 + t_3$, and $\bar{p}_8 = \bar{p}_{17} = \bar{p}_{14} = \bar{p}_9 = \bar{p}_7 = \bar{p}_3 = \bar{p}_1 = p_2 = p_4 = p_3 = p_9 = p_{17} = p_{11} = p_8 = 0$. Since $p_{19} = \bar{p}_{19} + \bar{z}_9 - \bar{z}_{19}$ we have $p_{19} = \bar{z}_9 = t_3$ and $\bar{p}_{19} = 0$. So G has some components where the spectrum of their unions is $\hat{H}_8$. This is a contradiction.

ii) Let $\bar{t}_1 \geq 0$ and $t_1 = 0$. By (6), $p_3 = 0$ and $\bar{p}_3 = \bar{z}_3 + \bar{t}_1$. Let $t_2 > 0$ and $\bar{t}_2 = 0$. Since G does not have any components where the spectrum of their unions is $\hat{H}_4$, $p_5 = 0$ and so $\bar{p}_2 = \bar{z}_5$. Therefore, for $i \neq 5$, $\bar{p}_1 = \bar{z}_i = 0$. Again, using the above quoted facts we have $\bar{p}_8 = \bar{p}_{11} = p_3 = p_{17} = 0$, $p_8 = \bar{p}_{17} = t_2$, $p_{11} = \bar{t}_1 + \bar{z}_5$ and $\bar{p}_3 = \bar{t}_1$. Since G does not have any components where the spectrum of their unions is $\hat{H}_1$, we have $\bar{t}_1 = \bar{z}_5 = 0$. By (3), we have $t_2 = \bar{t}_3$. Again, using the previous facts $p_4 = p_8 = p_{29} = \bar{p}_9 = \bar{p}_{14} = \bar{p}_{17} = t_2 = \bar{t}_3$. So G has some components where the spectrum of their unions is $\hat{H}_3$, this is impossible. Now let $t_2 = 0$ and $\bar{t}_2 \geq 0$. Then

$\bar{p}_5 = p_8 = 0$, $\bar{p}_8 = \bar{z}_8 + \bar{t}_2$ and $p_5 = \bar{p}_2 - \bar{z}_5$. If $t_3 = 0$, then by Lemma 3, G can be determined by its A-spectrum. So we can suppose that $t_3 > 0$ and $\bar{t}_3 = 0$. Hence by (7), $p_4 = 0$ and $\bar{p}_4 = \bar{z}_4 + t_3$. Since G dose not have any components where the spectrum of their unions is $\hat{H}_6$, $p_5 p_9 = 0$. If $p_5 = 0$, then $\bar{p}_2 = \bar{z}_5$. Therefore, for $i \neq 5$, $\bar{p}_1 = \bar{z}_i = 0$ and so by the previous equations we have $\bar{p}_{11} = \bar{p}_9 = \bar{p}_{14} = \bar{p}_{17} = p_{29} = 0$, $p_{11} = \bar{t}_1 + \bar{z}_5 = t_3 - \bar{t}_2$, $p_{17} = \bar{t}_2$ and $p_{14} = p_9 = \bar{p}_{29} = t_3$. So $p_{11} + p_{17} = \bar{t}_3 > 0$ and at least one of the numbers p_{11} or p_{17} is nonzero. Therefore, G has some components where the spectrum of their unions is either $\hat{H}_2$ or $\hat{H}_3$. Again, which is a contradiction. If $p_9 = 0$, then $\bar{p}_9 = \bar{z}_9 - \bar{z}_4 - t_3$ and by (4), $\bar{p}_9 = \bar{z}_9 - \bar{z}_4 - t_3 = \bar{z}_9 - \bar{z}_4 - (\bar{p}_1 + \sum_{i=2}^{l} \bar{z}_i + \bar{t}_1 + \bar{t}_2) \geq 0$. Therefore by the previous facts for $i \neq 9$, we have $\bar{p}_1 = \bar{z}_i = \bar{t}_1 = \bar{t}_2 = 0$ and $p_5 = \bar{p}_4 = \bar{z}_9 = \bar{p}_{29} = \bar{p}_2 = p_{14} = t_3$. Since $p_{19} = \bar{p}_{19} + \bar{z}_9 - \bar{z}_{19}$ we have $p_{19} = t_3$ and $\bar{p}_{19} = 0$. So G has some components where the spectrum of their unions is $\hat{H}_8$. This is impossible. □

From Theorem 3 we have the following corollaries.

Corollary 1. *[9] Let $G = P_{i_1} + P_{i_2} + \cdots + P_{i_r}$ be a graph of index less than 2. Then G can be determined by its spectrum if and only if G does not have any components where the spectrum of their unions is $\hat{H}_9$.*

Corollary 2. *Let $G = t_1 T_1 + t_2 T_2 + t_3 T_3$. Then G can be determined by its A-spectrum.*

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