A Sequence Representation of the Dyck Path Poset

Yu-Shuang Li
School of Science, Yanshan University,
Qinhuangdao 066004, P. R. China
Jun Wang*†
Department of Mathematics, Shanghai Normal University,
Shanghai 200234, P.R. China

Abstract: In this paper, a sequence representation of Dyck paths is presented, which yields a sequence representation of the Dyck path poset D ordered by pattern containment. This representation makes it clear that the Dyck path poset D takes the composition poset investigated by Sagan and Vatter as a subposet, and that the pattern containment order on Dyck paths exactly agrees with a generalized subword order also presented by Sagan and Vatter. As applications of the representation, we describe the Möbius function of D and establish the Möbius inverse of the rank function of D in terms of Dyck sequences. In the end, a Sperner and unimodal subposet of D is given.

Keywords: Dyck path poset, Dyck sequence, Möbius function, Sperner, unimodal.

1 Introduction

A Dyck path is a lattice path in the first quadrant from $(0,0)$ to $(2n,0)$ consisting of up steps $(1,1)$ and down steps $(1,-1)$. Call n the *semi-length* of the path. Let $\mathcal{D}$ be the set of all Dyck paths and $\mathcal{D}_n$ the set of Dyck paths of semi-length n. It is well known (see e.g. [1]) that $|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, and thus $\sum_{n\geq 0} |\mathcal{D}_n|x^n = \frac{1-\sqrt{1-4x}}{2x}$.

Among the many sets of combinatorial objects whose cardinalities are also given by the Catalan numbers, the most classical example is that of

*Correspondence author, E-mail address: junwang@dlut.edu.cn
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complete parenthesis systems [1]. Up steps and down steps in a Dyck path correspond to left and right parentheses, respectively. Equivalently, we may encode each up step by a letter u and a down step by a letter d, thus obtaining the frequently used encoding of Dyck paths by Dyck words.

A *pyramid* in a Dyck word is a factor of the form $u^h d^h$. We refer to h as the height of the pyramid. A pyramid in a Dyck word w is maximal if, as a factor in w, it is not immediately proceeded by an u and immediately followed by a d. An *exterior pair* in w is a pair consisting of an u and its matching d (when viewed as parentheses) which do not belong in any pyramid. Thus for a Dyck path of semi-length n, the sum of the heights of its maximal pyramids plus the number of its exterior pairs equal n. The reader can refer to reference [2] for more details about these two combinatorial statistics on Dyck paths. Figure 1 illustrates the maximal pyramids and exterior pairs in a Dyck path of semi-length 10.

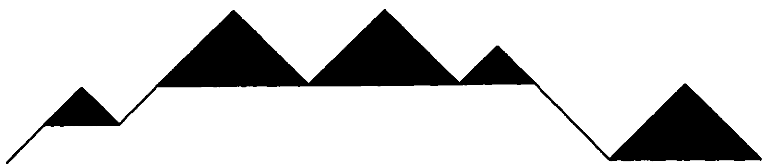

Figure 1: Maximal pyramids and exterior pairs in a Dyck path

The poset of Dyck paths of the same length ordered by inclusion has been studied in [3]. We now define the pattern containment order "$\preceq$" on $\mathcal{D}$ as follows.

Given two Dyck paths p_1 and p_2 in $\mathcal{D}$, $p_1 \preceq p_2$ if and only if p_1 is obtained from p_2 by deleting some pyramids and exterior pairs of p_2 and reconnecting the remaining steps in the obvious way. For example, in Figure 2, $p_1 \prec p_2$ since p_1 is obtained from p_2 by deleting two pyramids of height 1 and one exterior pair of p_2. Denote this poset by D. Clearly, D is ranked with the nth rank $\mathcal{D}_n$.

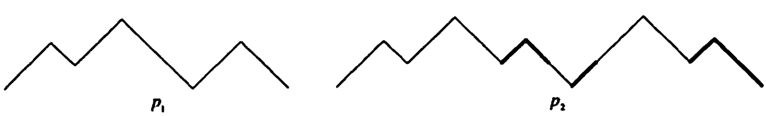

Figure 2: $p_1 \prec p_2$

In Section 2 we present a sequence representation of Dyck paths, named Dyck sequences, and describe the above pattern containment order on the set $\mathcal{D}$ in terms of Dyck sequences. As applications of the representation, in Section 3 we describe the Möbius function of D and establish the Möbius inverse of the rank function of D by using Dyck sequences. In Section 4 we find a subposet Q_τ of rank $2n-1$ with rank unimodality and Sperner property, that is, there exists an index r $(0 < r < 2n-1)$ such that the rank sequence $\{a_i\}$ of Q_τ satisfies $a_0 \le \cdots \le a_r \ge \cdots \ge a_{2n-1}$, and a_r equals the size of the maximal antichains of Q_τ. The explicit expression for the index r is also derived. At last, a conjecture in Section 5 ends the paper.

2 The sequence representation of Dyck paths

For our purpose, we factorize a Dyck path $p \in \mathcal{D}_n$ into pyramids and exterior pairs from the origin to $(2n, 0)$. Denote the maximal pyramid of height h by a number "h", the up step of an exterior pair by a symbol "$\bar{1}$", and the down step by a symbol "$\bar{0}$". The motivation of introducing $\bar{1}$ and $\bar{0}$ is to distinguish between the exterior pair and the maximal pyramid of height 1. Thus, for each Dyck path p we get a sequence, denoted by $\sigma(p)$, consisting of positive integers, $\bar{1}$'s and $\bar{0}$'s, satisfying two conditions:

(1) In $\sigma(p)$, the $\bar{1}$'s and $\bar{0}$'s appear pairwise and satisfy the parenthesis rule, that is, if $\sigma(p) = (\ldots, \bar{1}, \ldots, \bar{1}, \ldots, \bar{0}, \ldots, \bar{0}, \ldots)$, and there are no other $\bar{1}$'s and $\bar{0}$'s from the first $\bar{1}$ to the last $\bar{0}$, then the inner $\bar{1}$ matches the inner $\bar{0}$, and then the outer.

(2) Between each pair of $\bar{1}$ and $\bar{0}$ there are at least two positive integers.

Conversely, each sequence satisfying (1) and (2) corresponds to a Dyck path. We call such sequence a Dyck sequence. By $\mathcal{D}'$ we denote the set of all Dyck sequences. Given a $\sigma = (\sigma(1), \ldots, \sigma(m)) \in \mathcal{D}'$, call m the *length* $\ell(\sigma)$ of σ. Set $\rho(\sigma) = \sigma(1)' + \cdots + \sigma(m)' - 1$, where $\sigma(i)' = 1$ (0) if $\sigma(i) = \bar{1}$ ($\bar{0}$), otherwise $\sigma(i)' = \sigma(i)$. In this case, the Dyck path corresponding to σ is of the rank $\rho(\sigma)$. For example, the Dyck path illustrated by Figure 1 has a corresponding Dyck sequence $\sigma = (\bar{1}, 1, \bar{1}, 2, 2, 1, \bar{0}, \bar{0}, 2)$ and $\rho(\sigma) = 9$.

A sequence is called a generalized Dyck sequence if it satisfies the condition (1) and not necessarily the condition (2). It is easy to see that for a generalized Dyck sequence we can also obtain a Dyck path. For example, the generalized sequence $(\underbrace{\bar{1}, \ldots, \bar{1}}_{k}, h, \underbrace{\bar{0}, \ldots, \bar{0}}_{k})$ represents the Dyck path consisting of a pyramid of height $k+h$, or the Dyck sequence $(k+h)$. In fact, we can obtain a Dyck sequence σ from a generalized Dyck sequence $\tilde{\sigma}$ by replacing every such part with a positive integer and keeping other parts unchanged. By ψ we denote the mapping from the set of generalized Dyck sequences to the set $\mathcal{D}'$. We say two generalized Dyck sequences $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are equivalent if $\psi(\tilde{\sigma}_1) = \psi(\tilde{\sigma}_2)$, and call $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ generalized representations of the Dyck sequence $\psi(\tilde{\sigma}_1)$.

Set $S = \mathbb{N} \cup \{\bar{0}, \bar{1}\}$, where $\mathbb{N}$ is the set of all natural numbers. Define a partial order "$\leq$" on S such that $\mathbb{N}$ keeps the natural order, $\bar{0}$ and $\bar{1}$ are incomparable, and for $r \in \mathbb{N}$, $r \leq \bar{0}$ or $r \leq \bar{1}$ if and only if $r = 0$.

We now describe the pattern containment order on $\mathcal{D}$ in terms of Dyck sequences. For $\tau = (\tau(1), \ldots, \tau(n))$ and $\sigma = (\sigma(1), \ldots, \sigma(m)) \in \mathcal{D}'$, say $\tau \succeq \sigma$ if τ contains a generalized Dyck subsequence $\tilde{\tau}'$ such that $\psi(\tilde{\tau}') = (\tau'(1), \ldots, \tau'(m))$ satisfying $\tau'(i) \geq \sigma(i)$ in S for $i = 1, 2, \ldots, m$. For example, $(\bar{1}, \bar{1}, 4, 2, \bar{0}, \bar{0}, \bar{1}, 2, 1, \bar{0}) \succeq (5, \bar{1}, 1, 1, \bar{0})$ because $\psi(\bar{1}, 4, \bar{0}, \bar{1}, 2, 1, \bar{0}) = (5, \bar{1}, 2, 1, \bar{0}) \geq (5, \bar{1}, 1, 1, \bar{0})$. By D' we denote the poset on the set $\mathcal{D}'$ of all Dyck sequences.

Obviously, we can obtain the composition poset studied in [4] from D' by deleting such Dyck sequences containing $\bar{1}$'s and $\bar{0}$'s, that is, the Dyck path poset takes the composition poset as a subposet. Moreover, this order on Dyck paths corresponds to a generalized subword order as defined in [4]. In the next section we use this correspondence to describe the Möbius function of D.

3 The Möbius function and its application

We assume the reader is familiar with Möbius function, but all the necessary definitions and theorems we use here can be found in Stanley's text [6].

We begin by defining some relevant definitions.

For a sequence $\alpha = (\alpha(1), \ldots, \alpha(n))$, its *support* is the set $\mathrm{Supp}(\alpha) = \{i \mid \alpha(i) \neq 0\}$. Given $\sigma = (\sigma(1), \ldots, \sigma(m)) \in \mathcal{D}'$, an *expansion of* σ is a sequence $\epsilon_\sigma = (\epsilon_\sigma(1), \ldots, \epsilon_\sigma(n))$ such that if we restrict ϵ_σ to its support, we recover σ.

A *$\tilde{\sigma}$-embedding* of σ into $\tau = (\tau(1), \ldots, \tau(n))$ is an expansion $\epsilon_{\tilde{\sigma}\tau} = (\epsilon_{\tilde{\sigma}\tau}(1), \ldots, \epsilon_{\tilde{\sigma}\tau}(n))$ of a generalized representation $\tilde{\sigma}$ of σ such that $\tau(i) \geq \epsilon_{\tilde{\sigma}\tau}(i)$ in S for $1 \leq i \leq n$, and that the pairs of $\bar{1}$'s and $\bar{0}$'s appearing in $\epsilon_{\tilde{\sigma}\tau}$ agree with ones in τ. For example, if $\tau = (\bar{1}, \bar{1}, 4, 1, \bar{0}, \bar{0}, 2, 2, \bar{1}, 2, 1, \bar{0})$, $\sigma = (5, 2, \bar{1}, 1, 1, \bar{0})$ and $\tilde{\sigma} = (\bar{1}, 4, \bar{0}, 2, \bar{1}, 1, 1, \bar{0})$, then $(0, \bar{1}, 4, 0, \bar{0}, 0, 2, 0, \bar{1}, 1, 1, \bar{0})$ and $(\bar{1}, 0, 4, 0, 0, \bar{0}, 0, 2, \bar{1}, 1, 1, \bar{0})$ are both $\tilde{\sigma}$-embeddings of σ into τ, but $(0, \bar{1}, 4, 0, 0, \bar{0}, 2, 0, \bar{1}, 1, 1, \bar{0})$ is not since the first pair of $\bar{1}$ and $\bar{0}$ violates the definition. Thus $\sigma \preceq \tau$ in D' also exactly when there is a $\tilde{\sigma}$-embedding of σ into τ.

Given $\sigma = (\sigma(1), \ldots, \sigma(m)) \in \mathcal{D}'$, a *run* of k's is the maximal interval of indices $[r, t]$ such that $\sigma(r) = \sigma(r+1) = \cdots = \sigma(t) = k$. A $\tilde{\sigma}$-embedding $\eta_{\tilde{\sigma}\tau}$ of σ into τ to be *normal* if it satisfies the following conditions:
1. For $1 \leq i \leq l(\tau)$ we have $\eta_{\tilde{\sigma}\tau}(i) = \tau(i)$, $\tau(i)' - 1$ (here $\tau(i) \neq \bar{1}, \bar{0}$), or 0 (Note $\bar{1}$ matches $\bar{0}$. If $\bar{1}$ is changed to 0, so does its matching $\bar{0}$. For simplicity, we do not emphasize this rule in the sequel).
2. For every run $[r, t]$ of k's in τ we have:

(2.1) $(r,t] \subseteq \mathrm{Supp}(\eta_{\tilde{\sigma}\tau})$ if $k = 1$ or $\bar{1}$, or
(2.2) $[r,t) \subseteq \mathrm{Supp}(\eta_{\tilde{\sigma}\tau})$ if $k = \bar{0}$, or
(2.3) $r \in \mathrm{Supp}(\eta_{\tilde{\sigma}\tau})$ if $k \geq 2$.

Of the two $\tilde{\sigma}$-embeddings of σ into τ in the example of the previous paragraph, the first one is normal but the second one is not, because its second component violates (2.1), the fifth violates (2.2) and the seventh violates (2.3).

The *defect* of a normal $\tilde{\sigma}$-embedding $\eta_{\tilde{\sigma}\tau}$ of σ into τ is defined by

$$d(\eta_{\tilde{\sigma}\tau}) = \#\{i \mid \eta_{\tilde{\sigma}\tau}(i) = \tau(i)' - 1\}.$$

Our sequence representation and the results of [4] apply to give the Möbius function of D.

Proposition 3.1 *The Möbius function of D' is given by*

$$\mu(\sigma,\tau) = \sum_{\eta_{\tilde{\sigma}\tau}} (-1)^{d(\eta_{\tilde{\sigma}\tau})},$$

where the sum is over all normal $\tilde{\sigma}$-embeddings $\eta_{\tilde{\sigma}\tau}$'s of σ into τ.

Corollary 3.2 *For any Dyck sequence $\tau \in D'$, the Möbius inverse of the rank function is given by*

$$f(\tau) = \sum_{\sigma \preceq \tau} \mu(\sigma,\tau)\rho(\sigma) = \begin{cases} 1, & \text{if } \tau = (k),\ k > 1, \text{ or } \tau = (k,\ldots,k),\ k \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. $f(\tau) = \sum_{\sigma \preceq \tau} \mu(\sigma,\tau)\rho(\sigma) = \sum_{\sigma \in P_\tau} \mu(\sigma,\tau)\rho(\sigma)$, where $P_\tau = \{\sigma \in D' \mid \sigma \preceq \tau \text{ and } \mu(\sigma,\tau) \neq 0\}$, so we solve the problem only in the subposet P_τ of D'.

Case 1. If $\tau = (k)$, $k > 1$, or $\tau = (1,\ldots,1)$ and $\ell(\tau) = k$, then $f(\tau) = (k-1) - (k-2) = 1$, as desired.

Case 2. If $\tau = \underbrace{(k,\ldots,k)}_{n}$, $k > 1$. From the definition of the normal embedding and Proposition 3.1 we know that the first component k in τ only can be reduced 1 or kept fixed. Choose the first k as the special component in τ. Set

$$P_1 = \{\sigma \in P_\tau \mid \sigma = (k,\ldots) \text{ and } \ell(\sigma) > 1\},$$

$$P_2 = \{\sigma \in P_\tau \mid \sigma = (k-1,\ldots) \text{ and } \ell(\sigma) > 1\},$$

and $P_3 = \{(k),\ (k-1)\}$. A careful analysis of the situation shows that $P_\tau = P_1 \cup P_2 \cup P_3 \cong P_{\tau'} \cup P_{\tau'} \cup P_3$, where $\tau' = \underbrace{(k,\ldots,k)}_{n-1}$. Moreover,

P_i and P_j are disjoint for $i \neq j$. The bijection between P_1 and P_2 is clear: $\sigma_1 = (k, a_2, \ldots, a_m) \rightarrow (k-1, a_2, \ldots, a_m) = \sigma_2$, which implies that $\mu(\sigma_1, \tau) = -\mu(\sigma_2, \tau)$ and $\rho(\sigma_1) = \rho(\sigma_2) + 1$. See Figure 3 for an example. From the above analysis we have

$$\begin{aligned} f(\tau) &= \sum_{\sigma \in P_\tau} \mu(\sigma, \tau)\rho(\sigma) \\ &= \sum_{\sigma \in P_1} \mu(\sigma, \tau)\rho(\sigma) + \sum_{\sigma \in P_2} \mu(\sigma, \tau)\rho(\sigma) + \sum_{\sigma \in P_3} \mu(\sigma, \tau)\rho(\sigma) \\ &= \sum_{\sigma \in P_{\tau'}} \mu(\sigma, \tau') + \sum_{\sigma \in P_{\tau''}} \mu(\sigma, \tau'') + 1 \\ &= 0 + 0 + 1 \\ &= 1, \end{aligned}$$

where $\tau'' = \underbrace{(k, \ldots, k)}_{n-2}$.

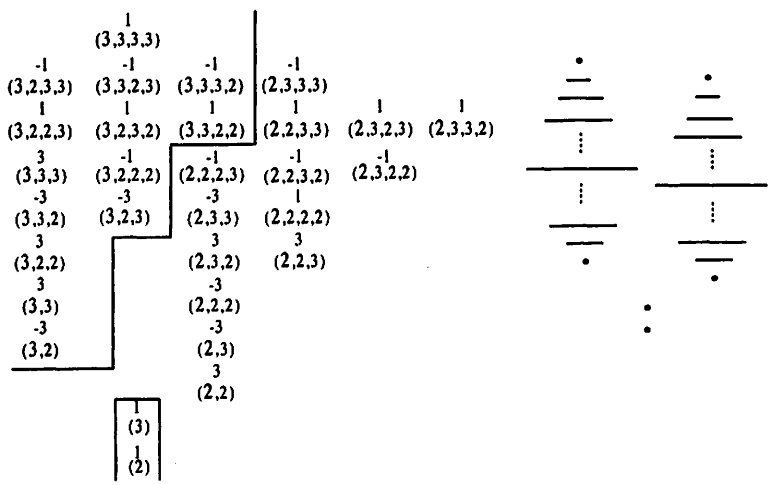

Figure 3: The decomposition of P_τ and its sketch

Case 3. If $\tau = (\tau(1), \ldots, \tau(n)) \in D'$ but $\tau \neq (k)$, $k > 1$, and $\tau \neq (k, \ldots, k)$, $k \geq 1$, we decompose P_τ in the same spirit of Case 2. Based on the form of τ, P_τ can be divided into two cases. If τ consists entirely of runs $[r, t]$ $(r < t)$ of k's, where $k > 1$, then P_τ falls into (3.2). Otherwise, into (3.1).
(3.1) Choose a proper special component $\tau(i)$ in τ (it only can be reduced 1 or kept fixed).

- If $\tau(i) \neq 1$, $\bar{1}$, or $\bar{0}$, and $\tau(i-1) \neq \tau(i) \neq \tau(i+1)$ ($\tau(i-1)$ or

$\tau(i+1)$ may be empty), set $P_1 = \{\sigma \in P_\tau \mid \sigma = (\dots, \tau(i), \dots)\}$ and $P_2 = \{\sigma \in P_\tau \mid \sigma = (\dots, \underline{\tau(i)}-1, \dots)\}$. Then $P_\tau = P_1 \cup P_2 \cong P_{\tau'} \cup P_{\tau'}$, where $\tau' = (\tau(1), \dots, \overline{\tau(i)}, \dots, \tau(n))$, the Dyck sequence obtained from τ by deleting the component $\tau(i)$ of τ.

- If $\tau(i) = \cdots = \tau(j) = 1$ where $j \geq i$, set P_1 as the above and $P_2 = \{\sigma \in P_\tau \mid \sigma \preceq (\tau(1), \dots, \overline{\tau(i)}, \dots, \tau(n))\}$. Then $P_\tau = P_1 \cup P_2 \cong P_{\tau'} \cup P_{\tau'}$, where $\tau' = (\tau(1), \dots, \overline{\tau(i)}, \dots, \overline{\tau(j)}, \dots, \tau(n))$.
- If $\tau(i) = \cdots = \tau(i+m-1) = \bar{1}$ and $\tau(j) = \cdots = \tau(j+m-1) = \bar{0}$, where $m \geq 1$, set $P_1 = \{\sigma \in P_\tau \mid \sigma = (\dots, \tau(i), \dots, \tau(j+m-1), \dots)\}$ and $P_2 = \{\sigma \in P_\tau \mid \sigma \preceq (\tau(1), \dots, \overline{\tau(i)}, \dots, \tau(j+m-1), \dots, \tau(n))\}$. Then $P_\tau = P_1 \cup P_2 \cong P_{\tau'} \cup P_{\tau'}$, where $\tau' = (\tau(1), \dots, \overline{\tau(i), \dots, \tau(i+m-1)}, \dots, \overline{\tau(j), \dots, \tau(j+m-1)}, \dots, \tau(n))$.

In a word, we can claim that $P_\tau = P_1 \cup P_2 \cong P_{\tau'} \cup P_{\tau'}$. $P_1 \cap P_2$ may be empty or not. If not, repeat those elements σ's in P_τ satisfying $\sigma \in P_1 \cap P_2$ such that $\mu(\sigma_{P_1}, \tau) + \mu(\sigma_{P_2}, \tau) = \mu(\sigma, \tau)$, where $\mu(\sigma_{P_i}, \tau)$ denotes the Möbius function between $\sigma \in P_i$ and τ. The relationships between P_1 and P_2 are same as that in Case 2. See Figure 4 for an example.

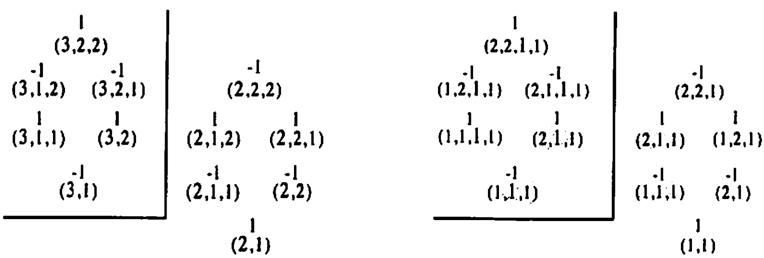

Figure 4: The decompositions of two P_τ's

(3.2) Here, we only consider such τ consisting entirely of runs $[r,t]$ $(r < t)$ of k's $(k > 1)$. $\tau(1)$ can act as the special component. Suppose $\tau(1) = \cdots = \tau(m) = p$, $\tau(m+1) = \cdots = \tau(j) = q$. Set

$$P_1 = \{\sigma \in P_\tau \mid \sigma(1) = \tau(1) \text{ and } \sigma(2) = \tau(2) \text{ or } \tau(2)-1\},$$

$$P_2 = \{\sigma \in P_\tau \mid \sigma(1) = \tau(1)-1 \text{ and } \sigma(2) = \tau(2) \text{ or } \tau(2)-1\},$$

and $P_3 = P_{\tau'}$, where $\tau' = (\overline{\tau(1)}, \dots, \tau(n))$. Then $P_\tau = P_1 \cup P_2 \cup P_3 \cong P_{\tau'} \cup P_{\tau'} \cup P_{\tau'}$. It is not hard to see that $P_1 \cap P_2$ is empty, $P_1 \cap P_3$ and $P_2 \cap P_3$ are both empty if $|p-q| > 1$, or both not if $|p-q| = 1$ (the sequence $(p, p, q, \tau(j+1), \dots, \tau(n)) \in P_1 \cap P_3$ and the sequence $(p-1, p, q, \tau(j+1), \dots, \tau(n)) \in P_2 \cap P_3$).

Evidently, $|P_1| = |P_2| = |P_3|$. When $|p-q| = 1$, we take precisely the same repetition measures as (3.1). The relationships between P_1 and P_2 are also same as that in Case 2. See Figure 5 for an example.

From the above analysis, for (3.1) we have

$$\begin{aligned} f(\tau) &= \sum_{\sigma\in P_\tau} \mu(\sigma,\tau)\rho(\sigma) \\ &= \sum_{\sigma\in P_1} \mu(\sigma,\tau)\rho(\sigma) + \sum_{\sigma\in P_2} \mu(\sigma,\tau)\rho(\sigma) \\ &= \sum_{\sigma\in P_{\tau'}} \mu(\sigma,\tau') \\ &= 0, \end{aligned}$$

and for (3.2) we have

$$\begin{aligned} f(\tau) &= \sum_{\sigma\in P_\tau} \mu(\sigma,\tau)\rho(\sigma) \\ &= \sum_{\sigma\in P_1} \mu(\sigma,\tau)\rho(\sigma) + \sum_{\sigma\in P_2} \mu(\sigma,\tau)\rho(\sigma) + \sum_{\sigma\in P_3} \mu(\sigma,\tau)\rho(\sigma) \\ &= 0 + \sum_{\sigma\in P_3} \mu(\sigma,\tau)\rho(\sigma) \\ &= \sum_{\sigma\in P_{\tau'}} \mu(\sigma,\tau')\rho(\sigma). \end{aligned}$$

From the condition $\tau(1) = \cdots = \tau(m) = p$, via $m-1$ recursions we have

$$f(\tau) = \sum_{\sigma\in P_{\tau'}} \mu(\sigma,\tau')\rho(\sigma) = \sum_{\sigma\in P_{\tau''}} \mu(\sigma,\tau'')\rho(\sigma) = f(\tau''),$$

where $\tau'' = (\overline{\tau(1),\ldots,\tau(m-1)}, \tau(m),\ldots,\tau(n))$. From (3.1) we know

$$f(\tau'') = 0.$$

□

By the Möbius inversion formula, we immediately get

$$\rho(\tau) = \sum_{\sigma\preceq\tau} f(\sigma) = \#\{\sigma \mid \sigma \preceq \tau,\ \sigma = (k),\ k > 1,\ \text{or } \sigma = (k,\ldots,k),\ k \geq 1\}.$$

4 A subposet with rank unimodality and Sperner property

From the proof of Corollary 3.2, it is easy to show that for $\tau = (k)$, $k > 1$, or $\tau = (k,\ldots,k)$, $k \geq 1$, the subposet $P_\tau = \{\sigma \in D' \mid \sigma \preceq \tau \text{ and } \mu(\sigma,\tau) \neq 0\}$

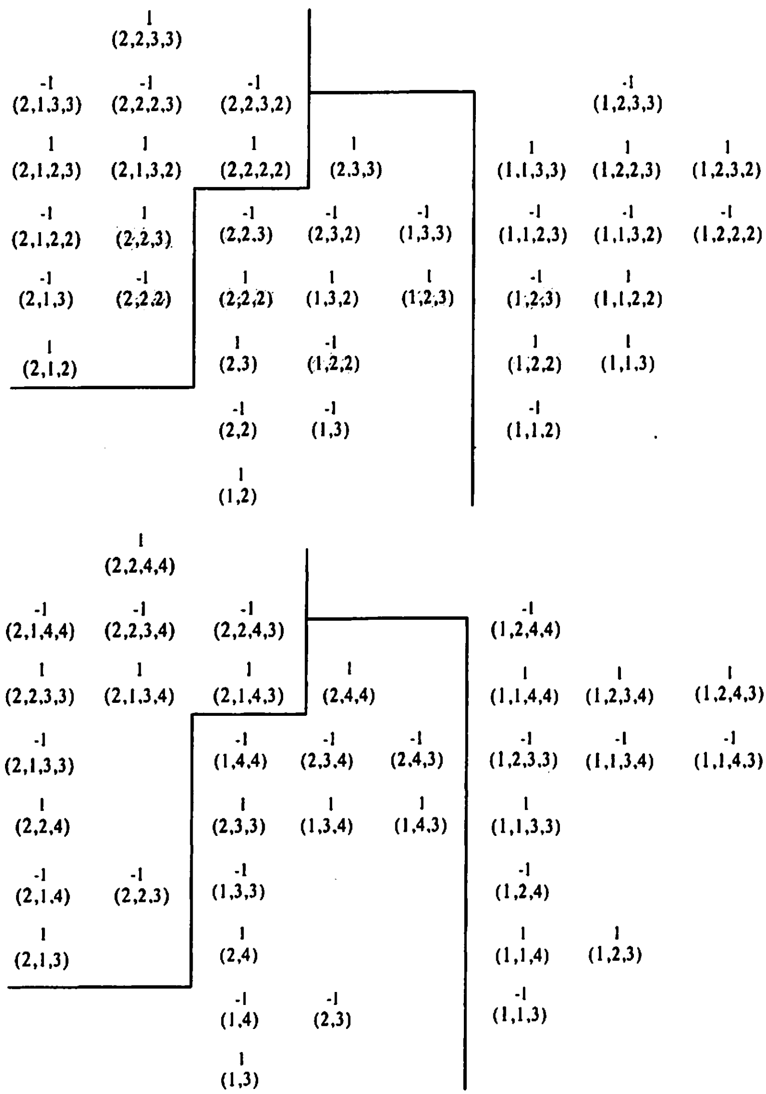

Figure 5: The decompositions of two P_{τ}'s

is rank unimodal (see Figure 3). It is natural to ask about $Q_\tau = \{\sigma \in D' \mid \sigma \preceq \tau\}$. Clearly, Q_τ is rank unimodal and Sperner for $\tau = (k)$, $k > 1$, or $\tau = (1, \ldots, 1)$ and $\ell(\tau) = k$, since it is a chain of length $k - 1$. A bit more generally, we have the following result.

Proposition 4.1 *Q_τ is rank unimodal and Sperner for $\tau = (2, \ldots, 2)$.*

Proof. Let $\ell(\tau) = n \geq 2$. Then $\rho(Q_\tau) = 2n - 1$. Set $\mathcal{Q}_i = \{\sigma \in Q_\tau : \rho(\sigma) = i\}$ and $|\mathcal{Q}_i| = a_i$ for $i = 0, 1, \ldots, 2n - 1$.

We say the map $\phi : \mathcal{Q}_i \to \mathcal{Q}_{i+1}$, or $\phi : \mathcal{Q}_{i+1} \to \mathcal{Q}_i$, is an order matching if ϕ is injective and ϕ respects the order, i.e., $\phi(\sigma) \succ \sigma$ or $\phi(\sigma) \prec \sigma$, for all $\sigma \in Q_\tau$. To show the proposition, it suffices to show that there exist order matchings $\mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots \mathcal{Q}_r \leftarrow \mathcal{Q}_{r+1} \leftarrow \cdots \mathcal{Q}_{2n-1}$ (see [7]).

For our purpose, we here regard 1's as left parentheses and 2's as right parentheses. The 1 and 2 in $\sigma \in Q_\tau$ are paired if two parentheses representing them are paired in the parenthesis sequence corresponding to σ. For example, let $\tau = (2, 2, 2, 2, 2, 2)$ and $\sigma = (2, 1, 2, 1, 1, 2)$. Then σ corresponds to the sequence of parentheses)()((), which implies that $\sigma(2)$ matches $\sigma(3)$ and $\sigma(5)$ matches $\sigma(6)$.

Define the two map ϕ and φ on Q_τ as follows:

$$\phi(\sigma) = \begin{cases} \sigma(i) = 1 \to 2, & \text{if } \ell(\sigma) = n - k \ (k \geq 0) \text{ and there are at least } 2k \text{ unpaired 2's;} \\ (1, \sigma), & \text{if } \ell(\sigma) = n - k \ (k > 0) \text{ and there are at most } 2k - 1 \text{ unpaired 2's;} \\ \sigma, & \text{otherwise } (\sigma(i) \text{ does not exist}), \end{cases}$$

where $\sigma(i) = 1 \to 2$ denotes the Dyck sequence $\phi(\sigma)$ obtained from σ by changing the leftmost unpaired $\sigma(i) = 1$ of σ to 2.

$$\varphi(\sigma) = \begin{cases} \sigma(j) = 2 \to 1, & \text{if } \ell(\sigma) = n - k \text{ and there are at least } 2k + 1 \text{ unpaired 2's;} \\ (\overline{\sigma(1)}, \ldots, \sigma(n-k)), & \text{if } \sigma(1) = 1 \text{ and there are at most } 2k \text{ unpaired 2's;} \\ \sigma, & \text{otherwise,} \end{cases}$$

where $k \geq 0$ and $\sigma(j) = 2 \to 1$ denotes the Dyck sequence $\varphi(\sigma)$ obtained from σ by changing the rightmost unpaired $\sigma(j) = 2$ of σ to 1.

Set $r = \max\{\rho(\sigma) \mid \varphi(\sigma) = \sigma\}$. It is easy to see that $r \leq \min\{\rho(\sigma) \mid \phi(\sigma) = \sigma\}$.

Now, we prove the restriction $\phi : \mathcal{Q}_i \to \mathcal{Q}_{i+1}$ is an order matching if $0 \leq i < r$. Evidently, $\phi(\sigma) \succ \sigma$ for $\sigma \in \mathcal{Q}_i$. Given $\sigma_1, \sigma_2 \in \mathcal{Q}_i$, $\sigma_1 \neq \sigma_2$.

(1) If $\phi(\sigma_1) = (1, \sigma_1)$ and $\phi(\sigma_2) = (1, \sigma_2)$, then clearly $\phi(\sigma_1) \neq \phi(\sigma_2)$.

(2) If $\phi(\sigma_1) : \sigma_1(i) = 1 \to 2$ and $\phi(\sigma_2) : \sigma_2(j) = 1 \to 2$, then clearly $\phi(\sigma_1) \neq \phi(\sigma_2)$ if $i = j$. Without loss of generality, we assume $i < j$. Suppose $\phi(\sigma_1) = \phi(\sigma_2)$. Then $\sigma_1(j) = 2$ and $\sigma_2(i) = 2$. From $\sigma_2(j) = 1 \to 2$ we know that all 1's appearing in $(\sigma_2(1), \ldots, \sigma_2(j-1))$ are paired, which derives that $\sigma_1(i) = 1$ is paired, contradicting $\sigma_1(i) = 1 \to 2$, so $\phi(\sigma_1) \neq \phi(\sigma_2)$.

(3) If $\phi(\sigma_1) = (1, \sigma_1)$ and $\phi(\sigma_2) : \sigma_2(j) = 1 \to 2$. We will do the case also by contradiction. Suppose $\phi(\sigma_1) = \phi(\sigma_2)$. Then $\sigma_2(1) = 1$. Suppose $\sigma_2(1)$ matches $\sigma_2(i) = 2$, where $i < j$ (if $i > j$, it is $\sigma_2(j)$ not $\sigma_2(1)$ that matches $\sigma_2(i)$). Then $\sigma_1(1)$ matches $\sigma_1(i) = 2$. If $\ell(\sigma_2) = n - k$ $(k \geq 0)$, then $\ell(\sigma_1) = n - (k+1)$. According to the definition of the map ϕ we know that there exist at least $2k$ unpaired 2's in σ_2, so there exist at least $2k + 2$ unpaired 2's in σ_1 ($\sigma_1(i)$ and $\sigma_1(j)$ both belong in them), contradicting $\phi(\sigma_1) = (1, \sigma_1)$, so $\phi(\sigma_1) \neq \phi(\sigma_2)$. This final contradiction finishes the proof that the restriction $\phi : \mathcal{Q}_i \to \mathcal{Q}_{i+1}$ is an order matching if $0 \leq i < r$.

Taking the similar method we can prove that the restriction $\varphi : \mathcal{Q}_{j+1} \to \mathcal{Q}_j$ is an order matching if $j \geq r$. □

While Proposition 4.1 establishes that $\mathcal{Q}_r$ is the maximal rank set of Q_τ, it leaves unresolved the numerical value of r related to the rank of τ. To do this we need to give an explicit formula for a_k. Set the subposet $Q_k = \{\sigma \mid \sigma \preceq \tau \text{ and } \ell(\sigma) = k\}$. Observe that the *Boolean algebra* B_k is a subposet of Q_k. Moreover, they have the same rank numbers. The bijection between Q_k and B_k is clear: for any $\sigma = (\sigma(1), \ldots, \sigma(k)) \in Q_k$, take $\varphi(\sigma) = \{i \mid \sigma(i) = 2\}$. Therefore

$$a_k = \begin{cases} \sum_{i \geq 0} \binom{k+1-i}{i}, & 0 \leq k \leq n-1; \\ \sum_{i \geq 0} \binom{n-i}{k-n+1+i}, & n \leq k \leq 2n-1. \end{cases}$$

Proposition 4.2 *For a fixed n the sequence $\{a_k\}$ satisfies $a_0 \leq \cdots \leq a_r \geq \cdots \geq a_{2n-1}$, where $r = \frac{3n}{2} - 2$ if n is even; $r = \frac{3n-3}{2}$ if n is odd and $1 \leq n \leq 27$, and $r = \frac{3n-5}{2}$ if n is odd and $n \geq 29$.*

Proof. Note $a_k = \sum_{i \geq 0} \binom{k+1-i}{i}$ for $0 \leq k \leq n-1$ are the Fibonacci numbers, so the subsequence $a_0, \ldots, a_{n-1}$ is strictly increasing. We only consider the remaining subsequence $a_{n-1}, \ldots, a_{2n-1}$. For convenience we rewrite a_k $(n-1 \leq k \leq 2n-1)$ as $b_{2n-1-k} = \sum_{i \geq 0} \binom{n-i}{2n-1-k-2i}$, i.e., $b_k = \sum_{i \geq 0} \binom{n-i}{k-2i}$ for $0 \leq k \leq n$.

Now, to prove Proposition 4.2 it suffices to prove that $b_0 \leq \cdots \leq b_t \geq \cdots \geq b_n$, where $t = \frac{n}{2} + 1$ if n is even; $t = \frac{n+1}{2}$ if n is odd and $1 \leq n \leq 27$, and $t = \frac{n+1}{2} + 1$ if n is odd and $n \geq 29$, consequently to get $r = 2n - 1 - t$.

Observe that the sequence $\{c_k = \binom{n}{k} + \binom{n-1}{k-2}\}$ for $0 \le k \le n+1$ is symmetric unimodal because $c_k = c_{n-k+1}$. And $c_0 \le \cdots \le c_{\frac{n}{2}} = c_{\frac{n}{2}+1} \ge \cdots \ge c_{n+1}$ if n is even; $c_0 \le \cdots \le c_{\frac{n+1}{2}} \ge \cdots \ge c_{n+1}$ if n is odd.

When n is even, the conclusion clearly holds for $n = 2, 4$. We apply induction on n. Assume $n = 2m > 4$ we have $t = m+1$. When $n = 2m+2$, from the hypothesis and the symmetric unimodality of the sequence $\{c_k\}$ for $0 \le k \le 2m+3$ we have $b_0 \le b_1 \le \cdots \le b_{m+2}$ and $b_{m+5} \ge b_{m+4} \ge \cdots \ge b_{2m+2}$. We now discuss the relationships among $b_{m+2}, b_{m+3}, b_{m+4}$ and b_{m+5}.

$$\begin{aligned}
& b_{m+2} - b_{m+3} \\
= \; & \left[\binom{2m+2}{m+2} - \binom{2m+2}{m+3}\right] \\
& -\left\{\left[\binom{2m}{m-1} - \binom{2m}{m-2}\right] + \left[\binom{2m-1}{m-3} - \binom{2m-1}{m-4}\right] + \cdots\right\} \\
= \; & \left\{\left[\binom{2m-1}{m} - \binom{2m-1}{m-3}\right] + \left[\binom{2m-2}{m-1} - \binom{2m-2}{m-4}\right] + \cdots\right\} \\
& -\left\{\left[\binom{2m-1}{m-3} - \binom{2m-1}{m-4}\right] + \left[\binom{2m-2}{m-5} - \binom{2m-2}{m-6}\right] + \cdots\right\} \\
> \; & 0.
\end{aligned}$$

$$\begin{aligned}
& b_{m+3} - b_{m+4} \\
= \; & \left[\binom{2m+2}{m+3} - \binom{2m+2}{m+4}\right] + \left[\binom{2m+1}{m+1} - \binom{2m+1}{m+2}\right] \\
& -\left\{\left[\binom{2m}{m} - \binom{2m}{m-1}\right] + \left[\binom{2m-1}{m-2} - \binom{2m-1}{m-3}\right] + \cdots\right\} \\
= \; & \left\{\left[\binom{2m}{m-1} - \binom{2m}{m-3}\right] + \left[\binom{2m-1}{m-1} - \binom{2m-1}{m-4}\right]\right. \\
& \left. + \left[\binom{2m-2}{m-2} - \binom{2m-2}{m-5}\right] + \cdots\right\} \\
& -\left\{\left[\binom{2m-1}{m-2} - \binom{2m-1}{m-3}\right] + \left[\binom{2m-2}{m-4} - \binom{2m-2}{m-5}\right] + \cdots\right\} \\
> \; & 0.
\end{aligned}$$

$$
\begin{aligned}
& b_{m+4} - b_{m+5} \\
= & \left[\binom{2m+2}{m+4} - \binom{2m+2}{m+5}\right] + \left[\binom{2m+1}{m+2} - \binom{2m+1}{m+3}\right] \\
& - \left\{\left[\binom{2m-2}{m-3} - \binom{2m-2}{m-4}\right] + \left[\binom{2m-3}{m-5} - \binom{2m-3}{m-6}\right] + \cdots\right\} \\
= & \left\{\left[\binom{2m}{m-1} - \binom{2m}{m-4}\right] + \left[\binom{2m-1}{m-2} - \binom{2m-1}{m-5}\right]\right. \\
& \left. + \left[\binom{2m-2}{m-3} - \binom{2m-2}{m-6}\right] + \cdots\right\} \\
& - \left\{\left[\binom{2m-2}{m-3} - \binom{2m-2}{m-4}\right] + \left[\binom{2m-3}{m-5} - \binom{2m-3}{m-6}\right] + \cdots\right\} \\
> & \ 0.
\end{aligned}
$$

Therefore we have $b_0 \leq \cdots \leq b_{m+1} \leq b_{m+2} \geq b_{m+3} \geq \cdots \geq b_{2m+2}$, i.e. $t = m+2$.

When $n = 2m-1$ is odd, we take the similar method. Using Maple we verify that $b_m > b_{m+1}$ holds only for $n = 2m-1 \leq 27$. □

If we change interphase 2 in τ to 11 to get τ', then $Q_{\tau'}$ is also rank unimodal because $Q_{\tau'}$ and Q_τ have the same rank numbers. We know that the unimodality is a famous problem for partitions (see e.g. [5, 8, 9]). In fact, Proposition 4.1 and 4.2 imply that the subposet of the composition poset consisting of compositions contained in $(2, 2, \ldots, 2)$ is rank unimodal.

5 A conjecture

For any $\tau = (k, \ldots, k)$, $k > 2$, the rank numbers of Q_τ are connected to binomial coefficients. From another point of view it may be more clear. We define a right factor order on all compositions: Say σ is a right factor of τ if

(1) $\tau = v_1 v_2$,

(2) $\ell(v_2) = \ell(\sigma)$,

(3) $\sigma(i) \leq v_2(i)$ for all $1 \leq i \leq \ell(\sigma)$.

Denote this poset by L. It is not difficult to verify that L is a distributive lattice. Given $\tau \in L$, set $L_\tau = \{\sigma \in L \mid \sigma \preceq \tau\}$. For $\tau = (k, \ldots, k)$, $k > 0$, we claim that L_τ is the union of some Boolean algebras. In general, the number of Boolean algebras in L_{τ_1} equals $|L_{\tau_2}|$, where $\tau_1 = \underbrace{(2k, \ldots, 2k)}_{n}$, $\tau_2 = \underbrace{(k, \ldots, k)}_{n}$, and the number of Boolean algebras in

$L_{\tau'_1}$ equals $|L_n|$ plus the number of Boolean algebras in $L_{\tau'_2}$, where $\tau'_1 = \underbrace{(2k+1,\ldots,2k+1)}_{n}$, $\tau'_2 = \underbrace{(2k+1,\ldots,2k+1)}_{n-2}$, $L_n = \{\sigma \preceq \tau'_3 \mid \ell(\sigma) = n\}$ where $\tau'_3 = \underbrace{(k+1,\ldots,k+1)}_{n}$. The connection between L_τ and Q_τ is clear: L_τ is a subposet of Q_τ. Moreover, $|L_\tau| = |Q_\tau|$ and they have the same rank numbers.

For $\tau = (k,\ldots,k)$ and $\ell(\tau) = n$, we have verified the rank unimodality of some Q_τ's for small values k and n. Generally we conjecture:

Conjecture 5.1 *Q_τ is rank unimodal if $\tau = (k,\ldots,k)$, where $k > 2$.*

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