ON THE GENERALIZED FIBONACCI AND PELL SEQUENCES BY HESSENBERG MATRICES

E. KILIC[1] AND D. TASCI[2]

ABSTRACT. In this paper, we consider the generalized Fibonacci and Pell Sequences and then show the relationships between the generalized Fibonacci and Pell sequences, and the Hessenberg permanents and determinants.

1. INTRODUCTION

The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

where $F_0 = 0,\ F_1 = 1$. The Pell Sequence, $\{P_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.2}$$

where $P_0 = 0$, $P_1 = 1$.

The well-known Fibonacci and Pell numbers can be generalized as follow: Let A be nonzero, relatively prime integers such that $D = A^2+4 \neq 0$. Define sequence $\{u_n\}$ by, for all $n \geq 2$ (see [17]),

$$u_n = Au_{n-1} + u_{n-2} \tag{1.3}$$

where $u_0 = 0,\ u_1 = 1$. If $A = 1$, then $u_n = F_n$ (the nth Fibonacci number). If $A = 2$, then $u_n = P_n$ (the nth Pell number).

If the roots of the equation $t^2 - At - 1 = 0$ are σ and γ, then the Binet formula of $\{u_n\}$ is given by for $n \geq 0$

$$u_n = \frac{\sigma^n - \gamma^n}{\sigma - \gamma}.$$

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The sequence $\{u_n\}$ have studied by several authors (see [6], [1]). The following identities can be found in [6], [1]:

$$u_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} A^{n-2k}. \tag{1.4}$$

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [15] define a $n \times n$ super diagonal $(0,1)$ −matrix $F(n,k)$ for $n > k \geq 2$, and show that the permanent of $F(n,k)$ equals to the generalized order-k Fibonacci numbers. Also he give some relations involving the permanents of some $(0,1)$ −Circulant matrices and the usual Fibonacci numbers. In [10], the authors present a nice result involving the permanent of an $(-1,0,1)$-matrix and the Fibonacci Number F_{n+1}. The authors then explore similar directions involving the positive subscripted Fibonacci and Lucas Numbers as well as their uncommon negatively subscripted counterparts. Finally the authors explore the generalized order-k Lucas numbers, (see [20] and [9] for more detail the generalized Fibonacci and Lucas numbers), and their permanents. In [12] and [13], the authors gave the relations involving the generalized Fibonacci and Lucas numbers and the permanent of the $(0,1)$ −matrices. The results of Minc, [15], and the result of Lee, [12], on the generalized Fibonacci numbers are the same because they use the same matrix. However, Lee proved the same result by a different method, contraction method for the permanent (for more detail of the contraction method see [2]). In [14], Lehmer proves a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary. Also in [18] and [19], the authors define a family of tridiagonal matrices $M(n)$ and show that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [5] and[4], the family of tridiagonal matrices $H(n)$ and the authors show that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3. The determinants, [3], now generate the Lucas sequence L_n. In [7], the authors find the families of $(0,1)$ −matrices such that permanents of the matrices, equal to the sums of Fibonacci and Lucas numbers. In [8], the authors define two tridiagonal matrices and then give the relationships the permanents and determinants of these matrices and the second order linear recurrences. In [11], the authors define two generalized doubly stochastic matrices and then show the relationships between the generalized doubly stochastic permanents and second order linear recurrences.

A lower *Hessenberg* matrix, $A_n = (a_{ij})$, is an $n \times n$ matrix where $a_{j,k} = 0$ whenever $k > j+1$ and $a_{j,j+1} \neq 0$ for some j. Clearly,

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & 0 \\ a_{31} & a_{32} & a_{33} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{n,n} \end{bmatrix}.$$

Also, in [5], the authors consider the above general lower Hessenberg matrix and then give following determinant formula: for $n \geq 2$,

$$\det A_n = a_{n,n} . \det A_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} a_{m,r} \prod_{j=r}^{n-1} a_{j,j+1} \det A_{r-1} \right).$$

Furthermore, the authors consider the Fibonacci sequence, $\{F_n\}$, and then give an example: Let

$$D_n = \begin{bmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 2 \end{bmatrix}_{n \times n}$$

and then state that the determinants of the first few matrices are $\det D_1 = 2$, $\det D_2 = 3$ and $\det D_3 = 5$, and, it runs out that this sequence is precisely $\{F_n\}$ starting at $n = 3$.

In this paper, we consider the generalized Fibonacci sequence $\{u_n\}$ and then we show the relationships between the Hessenberg determinants and permanents, and the generalized Fibonacci sequence $\{u_n\}$. Consequently, our results are more general in fact that the generalized Fibonacci sequence.

2. On The Generalized Fibonacci Sequence By Hessenberg Matrices

In this section we define a $n \times n$ lower Hessenberg matrix and then show that its determinant and permanents produce the terms of generalized Fibonacci sequence $\{u_n\}$.

We define the $n \times n$ lower Hessenberg matrix $H_n = (h_{ij})$ with $h_{ii} = A^2+1$ for all i and 1 otherwise. Clearly

$$H_n = \begin{bmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & A^2+1 & 1 & \cdots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \cdots & 1 & A^2+1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & A^2+1 \end{bmatrix}. \quad (2.1)$$

Also we define another the $n \times n$ lower Hessenberg matrix $T_n = (t_{ij})$ with $t_{ii} = A^2+1$ for $1 \leq i \leq n-1$, $t_{nn} = 1$ and 1 otherwise. Clearly

$$T_n = \begin{bmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & A^2+1 & 1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 1 & \dots & & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \\ 1 & 1 & \dots & \dots & 1 & 1 \end{bmatrix}. \quad (2.2)$$

Then we start with the following Lemma.

Lemma 1. *Let the $n \times n$ Hessenberg matrices H_n and T_n have the forms (2.1) and (2.2). Then, for $n \geq 3$*

$$\det T_n = A^2 \det H_{n-2}.$$

Proof. We use elementary operations of determinant. Subtracting the $(n-1)$s row from the nth row and then expanding with respect to last row gives

$$\det T_n = \begin{vmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & A^2+1 & 1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 1 & \dots & & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \\ 0 & 0 & \dots & 0 & -A^2 & 0 \end{vmatrix}$$

$$= A^2 \begin{vmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & A^2+1 & 1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 1 & \dots & & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \end{vmatrix}.$$

Considering the definition of the matrix H_n and expanding with respect to last column, we obtain

$$
\begin{aligned}
\det T_n &= A^2 \begin{vmatrix} A^2+1 & 1 & 0 & \dots & 0 & 0 \\ 1 & A^2+1 & 1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 1 & \dots & A^2+1 & 1 & 0 \\ 1 & 1 & & 1 & A^2+1 & 1 \\ 1 & 1 & \dots & 1 & 1 & A^2+1 \end{vmatrix} \\
&= A^2 \det H_{n-2}.
\end{aligned}
$$

So the proof is complete. □

Now we give our main result with the following Theorem.

Theorem 1. *Let the hessenberg matrix H_n has the form (2.1). Then, for $n > 0$*

$$
\begin{aligned}
\det H_n &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} A^{2n-2k} \\
&= A^{n-1} u_{n+2}
\end{aligned}
$$

where u_n is the nth term of the sequence $\{u_n\}$ and A be as before.

Proof. We will use the induction method to prove that $\det H_n = A^{n-1} u_{n+2}$. If $n = 1$, then we have

$$
\begin{aligned}
\det H_1 &= \det \left[A^2+1\right] = \sum_{k=0}^{1} \binom{2-k}{k} A^{2-2k} \\
&= \binom{2}{0} A^2 + \binom{1}{1} A^0 = A^2 + 1 = u_3.
\end{aligned}
$$

If $n = 2$, then we have

$$
\begin{aligned}
\det H_2 &= \det \begin{bmatrix} A^2+1 & 1 \\ 1 & A^2+1 \end{bmatrix} \\
&= \sum_{k=0}^{\lfloor \frac{3}{2} \rfloor} \binom{3-k}{k} A^{4-2k} = \left[\binom{3}{0} A^4 + \binom{2}{1} A^2\right] \\
&= A^4 + 2A^2 = A u_4.
\end{aligned}
$$

We suppose that the equation holds for n. That is,

$$\det H_n = A^{n-1} u_{n+2}.$$

Then we show that the equation holds for $n+1$. If we compute the $\det H_{n+1}$ by laplace expansion of determinant with respect to last column, then we have

$$\det H_{n+1} = \begin{vmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & A^2+1 & 1 & \cdots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \cdots & 1 & A^2+1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & A^2+1 \end{vmatrix}$$

$$= \left(A^2+1\right)\begin{vmatrix} A^2+1 & 1 & \cdots & 0 & 0 \\ 1 & A^2+1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 1 & 1 & 1 & A^2+1 & 1 \\ 1 & 1 & \cdots & 1 & A^2+1 \end{vmatrix}$$

$$- \begin{vmatrix} A^2+1 & 1 & \cdots & 0 & 0 \\ 1 & A^2+1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 1 & 1 & 1 & A^2+1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}.$$

From the definitions of the matrices H_n and T_n, we may write

$$\det H_{n+1} = \left(A^2+1\right)\det H_n - \det T_n.$$

Using the result of Lemma 1, we can write the last equation as

$$\det H_{n+1} = \left(A^2+1\right)\det H_n - A^2 \det H_{n-2}$$

and by our assumption we obtain

$$\begin{aligned} \det H_{n+1} &= \left(A^2+1\right)A^{n-1}u_{n+2} - A^2A^{n-3}u_n \\ &= \left(A^{n+1}+A^{n-1}\right)u_{n+2} - A^{n-1}u_n. \end{aligned}$$

From the recurrence relation of the sequence $\{u_n\}$, we write the last equation as follow

$$\begin{aligned} \det H_{n+1} &= \left(A^{n+1}+A^{n-1}\right)\left(Au_{n+1}+u_n\right) - A^{n-1}u_n \\ &= A^{n+2}u_{n+1} + A^n u_{n+1} + A^{n+1}u_n + A^{n-1}u_n - A^{n-1}u_n \\ &= A^{n+2}u_{n+1} + A^n u_{n+1} + A^{n+1}u_n \\ &= A^{n+1}\left(Au_{n+1}+u_n\right) + A^n u_{n+1} \\ &= A^{n+1}u_{n+2} + A^n u_{n+1} = A^n\left(Au_{n+2}+u_{n+1}\right) \\ &= A^n u_{n+3} \end{aligned}$$

or

$$\det H_{n+1} = \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} A^{2n+2-2k}.$$

So the proof is complete. □

For example, when $A = 1$, the sequence $\{u_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$, and by Theorem 1, we have that

$$\det H_n = \begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 2 \end{vmatrix} = F_n$$

which is given in [5].

A matrix A is called *convertible* if there is an $n \times n$ $(1, -1)$ $-$matrix H such that $perA = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be a $(1, -1)$ $-$matrix of order n, defined by

$$S = \begin{bmatrix} 1 & -1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -1 & 1 \\ 1 & 1 & 1 & \dots & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

We denote the matrices $H_n \circ S$ by B_n, respectively. Thus

$$B_n = \begin{bmatrix} A^2+1 & -1 & 0 & \dots & 0 & 0 \\ 1 & A^2+1 & -1 & \dots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & -1 \\ 1 & 1 & 1 & \dots & 1 & A^2+1 \end{bmatrix}. \tag{2.3}$$

Then we have the following Theorem without proof.

Theorem 2. *Let the $n \times n$ Hessenberh matrix B_n has the form (2.3). Then, for $n > 0$*

$$\begin{aligned} perB_n &= A^{n-1} u_{n+2} \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} A^{2n-2k} \end{aligned}$$

where u_n is the nth term of the sequence $\{u_n\}$.

For example, when $A = 2$, the sequence $\{u_n\}$ is reduced to the Pell sequence $\{P_n\}$, and by Theorem 2, we have

$$
\begin{aligned}
perB_n &= per\begin{bmatrix} 5 & -1 & 0 & \dots & 0 \\ 1 & 5 & -1 & \ddots & 0 \\ 1 & 1 & 5 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & \dots & 1 & 5 \end{bmatrix}_{n\times n} \\
&= \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n+1-k}{k} 2^{2n-2k} = 2^{n-1}P_{n+2}.
\end{aligned}
$$

3. On The Terms u_{2n+1} and u_{2n}

In this section, we define two lower Hessenberg matrices and then we show that their determinants equal to the terms u_{2n+1} and u_{2n}.

Firstly, we define a $n \times n$ lower Hessenberg matrix $W_n = (w_{ij})$ with $w_{ii} = A^2 + 1$ for all i, $w_{i,i+1} = -1$, $w_{ij} = A^2$ for $i > j$ and 0 otherwise. That is,

$$
W_n = \begin{bmatrix} A^2+1 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix}. \tag{3.1}
$$

Then we have the following Theorem.

Theorem 3. *Let the $n\times n$ lower Hessenberg matrix W_n has the form (3.1). Then, for $n > 1$*

$$\det W_n = u_{2n+1}$$

where u_n is the nth term of the sequence $\{u_n\}$.

Proof. We will use the induction method to prove that $\det W_n = u_{2n+1}$. If $n = 1$, then we have

$$\det W_1 = \det\left[A^2+1\right] = A^2 + 1 = u_3.$$

If $n = 2$, then we have

$$\begin{aligned} \det W_2 &= \det \begin{bmatrix} A^2+1 & -1 \\ A^2 & A^2+1 \end{bmatrix} \\ &= A^4 + 3A^2 + 1 = u_5. \end{aligned}$$

Now we suppose that the equation holds for n. That is,

$$\det W_n = u_{2n+1}.$$

Then we show that the equation holds for $n+1$. Thus using elementary row operations of determinant with subtracting the $(n+1)$st row from the nth row gives

$$\det W_{n+1} = \begin{vmatrix} A^2+1 & -1 & 0 & \cdots & 0 & 0 \\ A^2 & A^2+1 & -1 & \cdots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & A^2+2 \end{vmatrix}.$$

Also if we compute the above determinant by Laplace expansion of determinant with respect to the last column, then we have

$$\begin{aligned} \det W_{n+1} &= (A^2+2) \begin{vmatrix} A^2+1 & -1 & 0 & \cdots & 0 \\ A^2 & A^2+1 & -1 & \cdots & \vdots \\ A^2 & A^2 & A^2+1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 \end{vmatrix} \\ &+ \begin{vmatrix} A^2+1 & -1 & 0 & \cdots & 0 & 0 \\ A^2 & A^2+1 & -1 & \ddots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}. \end{aligned}$$

Using again the same Laplace expansion of determinant and by the definition of the matrix W_n, we can write that

$$\det W_{n+1} = (A^2+2)\det W_n - \det W_{n-2}.$$

Now by our assumption and the recurrence relation of the sequence $\{u_n\}$, we may write that

$$\begin{aligned}\det W_{n+1} &= \left(A^2+2\right)u_{2n+1} - u_{2n-1} \\ &= \left(A^2+1\right)u_{2n+1} + u_{2n+1} - u_{2n-1} \\ &= \left(A^2+1\right)u_{2n+1} + Au_{2n} + u_{2n-1} - u_{2n-1} \\ &= \left(A^2+1\right)u_{2n+1} + Au_{2n} \\ &= A^2u_{2n+1} + u_{2n+1} + Au_{2n} \\ &= A\left(Au_{2n+1} + u_{2n}\right) + u_{2n+1} \\ &= Au_{2n+2} + u_{2n+1} \\ &= u_{2n+3}.\end{aligned}$$

So the proof is complete. □

Second, we define a $n \times n$ lower Hessenberg matrix $V_n = (v_{ij})$ with $v_{ii} = A^2+1$ for $2 \le i \le n$, $v_{11} = A^2$, $v_{ij} = A^2$ for $i > j$, $v_{i,i+1} = -1$ and 0 otherwise. Clearly

$$V_n = \begin{bmatrix} A^2 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix}. \tag{3.2}$$

Now we have the following Theorem.

Theorem 4. *Let the $n \times n$ lower Hessenberg matrix V_n has the form (3.2). Then, for $n > 0$*

$$\det V_n = Au_{2n}$$

where u_n is the nth term of the sequence $\{u_n\}$.

Proof. We will use the induction method to prove that $\det V_n = Au_{2n}$. If $n = 1$, then

$$\det V_1 = \det\left[A^2\right] = A^2 = A.A = Au_2.$$

If $n = 2$, then we have

$$\begin{aligned}\det V_2 &= \det\begin{bmatrix} A^2 & -1 \\ A^2 & A^2+1 \end{bmatrix} \\ &= A^4 + 2A^2 = A\left(A^3+2A\right) \\ &= Au_4.\end{aligned}$$

We suppose that the equation holds for n. That is,

$$\det V_n = Au_{2n}.$$

Then we show that the equation holds for $n+1$. Thus, if we compute the $\det V_{n+1}$ by Laplace expansion of determinant with respect to the first row, then we have

$$\begin{aligned}
\det V_{n+1} &= \begin{vmatrix} A^2 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{vmatrix} \\
&= A^2 \begin{vmatrix} A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & \dots & A^2 & A^2+1 \end{vmatrix} \\
&\quad + \begin{vmatrix} A^2 & -1 & 0 & \dots & 0 \\ A^2 & A^2+1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & \dots & A^2 & A^2+1 \end{vmatrix}.
\end{aligned}$$

Considering the definitions of the matrices V_n and W_n, we may write that

$$\det V_{n+1} = A^2 \det W_n + \det V_n.$$

Also by our assumption and the recurrence relation of the sequence $\{u_n\}$, we write

$$\begin{aligned}
\det V_{n+1} &= A^2 u_{2n+1} + Au_{2n} \\
&= A\left(Au_{2n+1} + u_{2n}\right) \\
&= Au_{2n+2}.
\end{aligned}$$

So the proof is complete. □

Let S be the $(1,-1)$ −matrix of order n as before. We denote the matrices $W_n \circ S$ and $V_n \circ S$ by G_n and K_n, respectively. Clearly

$$G_n = \begin{bmatrix} A^2+1 & 1 & 0 & \cdots & 0 & 0 \\ A^2 & A^2+1 & 1 & \cdots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 & 1 \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+1 \end{bmatrix}_{n\times n} \tag{3.3}$$

and

$$K_n = \begin{bmatrix} A^2 & 1 & 0 & \cdots & 0 & 0 \\ A^2 & A^2+1 & 1 & \cdots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 & 1 \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2+1 \end{bmatrix}_{n\times n} . \tag{3.4}$$

Then we have the following Theorems without proof.

Theorem 5. *Let the $n \times n$ lower Hessenberg matrix G_n has the form (3.3). Then, for $n > 0$*

$$perG_n = u_{2n+1}$$

where u_n is the nth term of the sequence $\{u_n\}$.

Theorem 6. *Let the $n \times n$ lower Hessenberg matrix K_n has the form (3.4). Then, for $n > 0$*

$$perK_n = Au_{2n}$$

where u_n is the nth term of the sequence $\{u_n\}$.

For example, when $A = 1$, the sequence $\{u_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and by the above results

$$\det \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & -1 & \cdots & \vdots & 0 \\ 1 & 1 & 1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ 1 & 1 & \cdots & 1 & 2 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{bmatrix}_{n\times n} = F_{2n+1}$$

and when $A = 2$, the sequence $\{u_n\}$ is reduced to the Pell sequence $\{P_n\}$ and

$$per \begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ 4 & 5 & 1 & \dots & \vdots & 0 \\ 4 & 4 & 5 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 4 & 4 & \dots & 4 & 5 & 1 \\ 4 & 4 & 4 & \dots & 4 & 5 \end{bmatrix} = 2P_{2n}.$$

Using the identity (1.4) and the above Theorems, we give following representations:

$$\det W_n = perG_n = \sum_{k=0}^{n} \binom{2n-k}{k} A^{2n-2k}$$

and

$$\det V_n = perK_n = \sum_{k=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-1-k}{k} A^{2n-2k}.$$

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[1]TOBB ECONOMICS AND TECHNOLOGY UNIVERSITY 06560 ANKARA TURKEY
E-mail address: ekilic@etu.edu.tr

[2]GAZI UNIVERSITY, MATHEMATICS DEPARTMENT, 06500 ANKARA TURKEY