THE LABELINGS OF A VARIATION OF BANANA TREES

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ABSTRACT. In this paper, it is shown that a variation of banana trees is odd graceful, and it is also proved that the variation of banana is graceful and $\hat{\rho}$-labeling in some cases.

1. INTRODUCTION

The problem of finding a graceful labeling and $\hat{\rho}$-labeling of the vertices of a graph was introduced by [1] in a seminal paper in the mid-1960s. the conjecture that all trees are graceful was stated [1, 6]. since then although there have been some promising new approaches to the problem [4, 5] and a number of variations on graceful labeling have been proposed (see [6] for comprehensive survey), this " Graceful Tree Conjecture" has been remained as one of the most well-known open problems in graph theory. In this paper, we will consider such a variation of banana trees (banana trees are intioduced by Chen et al [3]). Following the definition of "odd graceful" by Gnanajothi [2], we will show that a variation of banana trees is odd graceful and the variation of banana trees is graceful and $\hat{\rho}$-labeling in some cases.

2. BASIC NOTATIONS

Definition 1. A banana tree is a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars).

Definition 2. Let $G(V, E)$ be a graph with q edges and let f: $V(G) \to \{0, 1, \cdots, q\}$ be an injection. The vertex labeling is called a graceful labeling if to each edge uv the absolute value $|f(u) - f(v)|$ is assigned as its label and the resulting edge labels are mutually distinct in $\{1, 2, \cdots, q\}$. A graph G with a graceful labeling is called a graceful graph.

Definition 3. Let $G(V, E)$ be a graph with q edges and let f: $V(G) \to \{0, 1, \cdots, q\}$ be an injection. The vertex labeling is called $\hat{\rho}$-labeling if to each edge uv the absolute value $|f(u) - f(v)|$ is assigned as its label and the resulting edge labels are mutually distinct in $\{1, 2, \cdots, q\}$ or $\{1, 2, \cdots, q-1, q+1\}$. A graph G is called $\hat{\rho}$-labeling if it has a $\hat{\rho}$-labeling.

Definition 4. Let $G(V, E)$ be a graph with q edges and let f: $V(G) \to \{0, 1, \cdots, 2q-1\}$ be an injection. The vertex labeling is called a graceful labeling if to each edge uv the absolute value $|f(u) - f(v)|$ is assigned as its label and the resulting edge labels are mutually distinct in $\{1, 3, \cdots, 2q-1\}$. A graph G with a odd graceful labeling is called a odd graceful graph.

Definition 5. A variation of banana trees is obtained by putting a vertex on one leaf edge (the edge incident to this leaf) of each of any number of stars and then connecting v to all new vertices inserted (v is not in any of the stars). A variation of banana tree is by T_n and illustrated in Fig.1.

The set of vertices labelings in T_n is denoted by X, the set of edges labelings in T_n is donoted by Y.

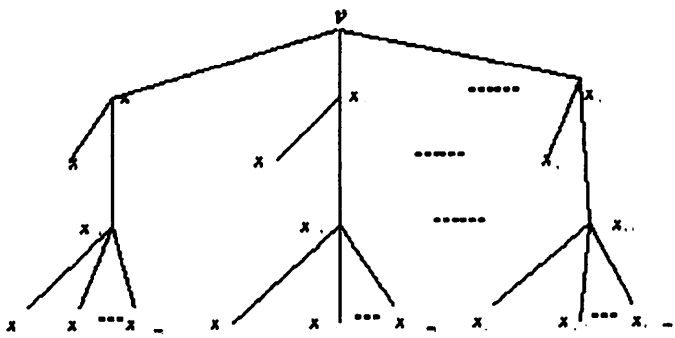

Fig. 1. the variation of banana tree

3. ODD GRACEFULNESS

Theorem 1 . *T_n is odd graceful.*

Notice: In the all following sections, $|E| = \sum_{i=1}^{n} l_i + n$, supposing $l_i \geq l_j (i < j)$.

Proof. Let

$V_0 = (x_{1,0}, x_{2,0}, \cdots, x_{n,0})$, $V_1 = (x_{1,1}, x_{2,1}, \cdots, x_{n,1})$;

$V_{l_1} = (x_{1,l_1}, x_{2,l_2}, \cdots, x_{n,l_n})$, $V_k = (x_{1,k}, x_{2,k}, \cdots, x_{n,k})$;

f: $V_i \to X_i$, $|X_i| \geq |X_J|$ $(0 \leq i < j \leq l_1)$.

Where $x_{m_k,k}$ is the last vertex of V_k, and $x_{m_k+1,k}$ does not exist.

Let $X^1 = X_0 \cup X_1 \cup X_{l_1}$, $X^2 = \bigcup_{k=2}^{l_1-1} X_k$,

where $X_i = f(V_i)$, $i = 0, 1, \cdots, l_1$.

We odd gracefully label T_n as follows:

$f(v) = 0$;

$f(V_0) = \{2, 6, 10, \cdots, 4n-2\}$, in the order;

$f(V_1) = \{2|E| - 2n + 1, 2|E| - 2n + 3, \cdots, 2|E| - 1\}$, in the order;

$f(V_{l_1}) = \{2|E| - 4n + 2, 2|E| - 4n + 6, \cdots, 2|E| - 2\}$, in the order; then

$X^1 = \{2, 6, 10, \cdots, 4n-2\} \cup \{2|E| - 2n + 1, 2|E| - 2n + 3, \cdots, 2|E| - 1\} \cup \{2|E| - 4n + 2, 2|E| - 4n + 6, \cdots, 2|E| - 2\}$.

There are $3n$ different numbers in X^1. For the remaining vertices, let

$f(V_2) = \{2|E| - 4n + 1, 2|E| - 4n + 3, \cdots, 2|E| - 2n - 1\}$, in the order,

$f(V_3) = \{(2|E| - 2n - 1) - (4n - 2), (2|E| - 2n - 1) - (4n - 2) + 2, \cdots, (2|E| - 2n - 1) - (4n - 2) + 2m_3 - 2\}$, in the order,

$\cdots \quad \cdots \quad \cdots$

$f(V_{k+1}) = \{f(x_{m_k,k}) - f(x_{m_k,0}), f(x_{m_k,k}) - f(x_{m_k,0}) + 2, \cdots, f(x_{m_k,k}) - f(x_{m_k,0}) + 2m_{k+1} - 2\}$, in the order,

$k = 2, 3, \cdots, l_1 - 2; m_{k+1} = |X_{k+1}|$,

where $x_{m_k,k}$ is the last vertex of V_k, and $x_{m_k+1,k}$ does not exist.

the vertex labelings in X_k are different, the minimum label in X_k is

$f(x_{1,k}) = f(x_{m_{k-1},k-1}) - f(x_{m_{k-1},0})$,

the maximum label in X_k is $f(x_{m_k,k})$, that is

$f(x_{1,k}) = f(x_{m_{k-1},k-1}) - f(x_{m_{k-1},0}) + 2(m_k - 1) \leq f(x_{1,k-1}) - 2$,

so $X_p \cap X_q = \phi \ (p \neq q)$.

Ther are $\sum_{i=1}^{n} l_i - 2n$ different numbers in X^2, the minimum labeling is $f(x_{1,l_1-1})$, the maximum labeling is not larger than $f(x_{n,2})$.

In $X = \{0\} \cup X^1 \cup X^2$, there are $\sum_{i=1}^{n} l_i + n + 1$ numbers, and those numbers are different, the maximum labeling is $2|E| - 1$, the minimum labeling is zero.

The set of labels on edges $x_{k,1}v$, $x_{k,1}x_{k,l_k}$ and $x_{k,1}x_{k,0}$ $(k = 1, 2, \cdots, n)$ is

$Y^1 = \{2|E| - 1, 2|E| - 3, \cdots, 2|E| - (2n - 1)\} \cup \{1, 3, \cdots, 2n - 1\} \cup \{2|E| - (2n + 1), 2|E| - (2n + 3), \cdots, 2|E| - (4n + 1)\}$.

The set of labels on edges $x_{m,k}x_{m,0}$ is $Y^2 = \cup_{k=2}^{l_1-1} Y_k$, where

$Y_k = \{f(x_{1,k}) - 2(l - 1) - 2;\ l = 1, 2, \cdots, m_i;\ k = 2, 3, \cdots, l_1 - 1\}$.

From Y_k $(k = 2, 3, \cdots, l_1 - 1\}$, we can get $Y_p \cap Y_q = \phi \ (p \neq q)$.

There are $\sum_{i=1}^{n} l_i - 2n$ different odd numbers in Y_2, and the maximum one is

$$f(x_{1,1}) - f(x_{1,0}) - 2 = 2|E| - 4n - 1,$$

the minimum one is

$$2|E| - 4n - 1 - 2(\sum_{i=1}^{n} l_i - 2n - 1) = 2n + 1.$$

Hence, the labels of edges in $\cup_{k=2}^{l_1-1} Y_k$ are from $2|E| - 4n - 1$ to $2n + 1$; the followings come out from the above results

$$Y^1 \cap Y^2 = \phi, \ Y = Y^1 \cup Y^2 = \{1, 3, \cdots, 2|E| - 1\};$$

further T_n is odd graceful.

For example:

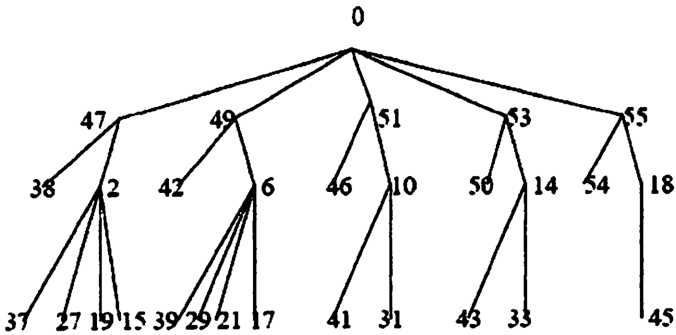

Fig. 2. $n = 28$

4. GRACEFULNESS

Theorem 2 . *If* $l_1 = l_2 = \cdots = l_n = l$, T_n *is graceful.*

Proof. We gracefully label T_n as follows:

$f(v) = 0,\ f(x_{m,0}) = 2 + (l+1)(m-1),\ f(x_{m,l_m}) = f(x_{m,0}) - 1,$
$f(x_{m,k}) = n(l+1) - (l+1)(m-1) - (k-1)$
$(m =, 1, 2, \cdots, n;\ k = 1, 2, \cdots, l_m - 1);$
$X_{m,1} = \{f(x_{m,l_m}), f(x_{m,0})\} = \{1 + (l+1)(m-1), 2 + (l+1)(m-1)\},$
$X_{m,2} = \{f(x_{m,k})\} = \{n(l+1) - (l+1)(m-1),\ n(l+1) - (l+1)(m-1) - 1, \cdots, n(l+1) - (l+1)(m-1) - (l-2)\}.$

Obviously, $X_{p,1} \cap X_{q,1} = \phi$, $X_{p,2} \cap X_{q,2} = \phi$,

$$n(l+1) - (p-1)(l+1) - (k-1) - 2 - (q-1)(l+1) = (l+1)(n-p-q+2) - (l+1),$$

since $1 \le k \le l-1$, the above value is not zero; similarly, the number

$$n(l+1) - (p-1)(l+1) - (k-1) - 2 - (q-1)(l+1)$$

is not zero too, hence, $X_{P,1} \cap X_{q,2} = \phi$.

Then for the set of the vertex labels, we have

$$X = \{0\} \cup (\cup_{m=1}^{n} (X_{m,1} \cup X_{m,2})) = \{0, 1, 2, \cdots, |E|\},$$

and for the set of the labels of edges, we have

$Y = Y_0 \cup (\cup_{m=1}^{n} Y_m),$
$Y_0 = \{|E|, |E| - (l+1), \cdots, l+1\},$
$Y_m = \{(l+1)(n-m+1) - 1, (l+1)(n-m+1) - 2, \cdots, (l+1)(n-m+1) - l\}.$

Similarto the above, we can obtain

$$Y_p \cap Y_q = \phi\ (p \ne q),\ Y = \{1, 2, \cdots, |E|\}.$$

Hence $T_n(l_1 = l_2 = \cdots = l_n = l)$ is graceful.

Theorem 3 . *If* $l_m = l_{m+1} + 2\ \ (m = 1, 2, \cdots, n-1)$, T_n *is graceful.*

Proof. By $|E| = n(n + l_n)$, we gracefully label T_n as follows:

$f(v) = |E| - l_1$, $f(x_{1,0}) = |E| - 1$, $f(x_{1,l_1}) = |E|$,

$f(x_{1,m}) = m - 1$ $(m = 1, 2, \cdots, l_1 - 1)$, $f(x_{k,l_k}) = f(x_{k-1,l_{k-1}}) - 2$,

$f(x_{k,0}) = f(x_{k,l_k}) - 1$, $f(x_{k,1}) = f(x_{k-1,l_{k-1}-1}) + 1$,

$f(x_{k,2}) = f(x_{k-1,l_k-1}) + 2$,

$f(x_{k,m}) = f(x_{k,2}) + (m-2)$ $(k = 2, 3, \cdots, n-1;\ m = 3, , 4, \cdots, l_k - 1)$,

$f(x_{n,1}) = f(x_{n-1,l_{n-1}-1}) + 1$, $f(x_{n,0}) = f(x_{n-1,0}) - 1$,

$f(x_{n,2}) = f(x_{n-1,l_{n-1}-1}) + 3$, $f(x_{n,l_n}) = f(x_{n-1,0}) - 2$,

$f(x_{n,m}) = f(x_{n,2}) + (m - 2)$ $(m = 3, 4, \cdots, l_n - 1)$.

There $f(x_{k,0})$, $f(x_{k,l_k})(k = 1, 2, \cdots, n)$ are different, the maximum is $|E|$ and the minimum is $|E| - 2n + 1$, and the total number of them is $2n$; further more

$f(x_{k,m})$ $(k = 1, 2, \cdots, n;\ m = 1, 2, \cdots, l_k - 1)$

are in $\cup_{k=1}^{n} X_k$, their labels sets are

$X_1 = \{0, 1, \cdots, l_1 - 2\}$,

$$X_k = \{\sum_{p=1}^{k-1} l_p - (k-1), \sum_{p=1}^{k-1} l_p - (k-1) + 1, \cdots, \sum_{p=1}^{k} l_p - (k+1)\}$$
$$(k = 2, 3, \cdots, n-1),$$

$$X_n = \{\sum_{p=1}^{n-1} l_p - n + 1, \sum_{p=1}^{n-1} l_p - n + 3, \sum_{p=1}^{n-1} l_p - n + 4, \cdots, \sum_{p=1}^{n} l_p - n\}.$$

The numbers in X_k are different and $X_p \cap X_q = \phi$ $(p \neq q)$; the minimum is zero, the maximum is $|E| - 2n$. Since $f(v) = \sum_{p=1}^{n-1} l_p - n + 2$, the vertex labels are different if and only if the vertices are different, and the total number is $|E| + 1$.

The sets of edge labels in the i-star and the sets of labels of edges connected to the i-star are

$\{1, 2, \cdots, l_n + 1\}$, $\{l_n + 2, l_n + 3, \cdots, l_n + l_{n-1} + 2\}, \cdots, \{|E|, |E| - 1, \cdots, |E| - l_1\}$.

Hence, T_n is graceful.

Theorem 4 . *If $n = 2, 3$, then T_n is graceful.*

In the following proof, X_i denotes the set of the vertex labels in the i-star, Y_i denotes the of the edge labels in the i-star and the labels of edges connected to the i-star, here $i = 1, 2, \cdots, n$.

Lemma 1. *T_2 is graceful.*

Proof. We gracefully label T_2 as follows:

$f(v) = 2$,

$X_1 = \{f(x_{1,0}), f(x_{1,1}), \cdots, f(x_{1,l_1})\} = \{0, |E| - l_1 + 2, |E|, |E| - 1, \cdots, |E| - l_1 + 3, 1\} = \{0, l_2 + 4, |E|, |E| - 1, \cdots, l_2 + 5, 1\}$,
$X_2 = \{f(x_{2,0}), f(x_{2,1}), \cdots, f(x_{2,l_2})\} = \{4, |E| - l_1 + 1, |E| - l_1, \cdots, |E| - l_1 - l_2 + 2, 3\} = \{4, l_2 + 3, l_2 + 2, \cdots, 5, 3\}$,

$$X_1 \cap X_2 = \phi,$$

$$\{2\} \cup X_1 \cup X_2 = \{0, 1, \cdots, |E|\};$$

$Y_1 = \{|E|, |E| - 1, \cdots, |E| - l_1\}$, $Y_2 = \{|E| - l_1 - 1, |E| - l_1 - 2, \cdots, |E| - l_1 - l_2 - 1\}$,

$$Y_1 \cup Y_2 = \{1, 2, \cdots, |E|\}, \quad Y_1 \cap Y_2 = \phi.$$

Lemma 2. *T_3 is graceful.*

Proof. We gracefully label T_3 as follows:
$f(v) = 2$,
$X_1 = \{f(x_{1,0}), f(x_{1,1}), \cdots, f(x_{1,l_1})\} = \{0, |E| - l_1 + 2, |E|, |E| - 1, \cdots, |E| - l_1 + 3, 1\} = \{0, l_2 + l_3 + 5, |E|, |E| - 1, \cdots, l_2 + l_3 + 6, 1\}$,
$X_2 = \{f(x_{2,0}), f(x_{2,1}), \cdots, f(x_{2,l_2})\} = \{4, |E| - l_1 + 1, |E| - l_1, \cdots, |E| - l_1 - l_2 + 3, 3\} = \{4, l_2 + l_3 + 4, l_2 + l_3 + 3, \cdots, l_3 + 6, 3\}$,
$X_3 = \{f(x_{3,0}), f(x_{3,1}), \cdots, f(x_{3,l_3})\} = \{l_3+5, l_3+3, l_3+2, \cdots, 5, l_3+4\}$,

$$X_p \cap X_q = \phi, \ (p \neq q),$$

$$\{2\} \cup X_1 \cup X_2 \cup X_3 = \{0, 1, \cdots, |E|\}.$$

$Y_1 = \{|E|, |E| - 1, \cdots, |E| - l_1\}$, $Y_2 = \{|E| - l_1 - 1, |E| - l_1 - 2, \cdots, |E| - l_1 - l_2 - 1\}$, $Y_3 = \{1, 2, \cdots, |E| - l_1 - l_2 - 2\}$,

$$Y_1 \cup Y_2 \cup Y_3 = \{1, 2, \cdots, |E|\}, \quad Y_p \cap Y_q = \phi \quad (p \neq q).$$

Hence, T_3 is graceful.

5. $\hat{\rho}$-LABELING

Theorem 5 . *With $l_1 = l_2 = \cdots = l_n = l > 2$, for T_n, there exists a $\hat{\rho}$-labeling.*

Proof. We gracefully label T_n as follows:
$f(v) = 2, \ f(x_{1,l_1}) = 1, \ f(x_{1,1}) = |E| - l + 1, \ f(x_{1,0}) = 0,$
$f(x_{1,2}) = |E| + 1, \ f(x_{1,k}) = |E| - 1 - (k - 3) \ (k = 3, 4, \cdots, l),$
$f(x_{m,0}) = 4 + (l + 1)(m - 2), \ f(x_{m,l_m}) = f(x_{m,0}) - 1,$
$f(x_{m,k}) = |E| - l - (l + 1)(m - 2) - (k - 1)$

$$(m = 2, 3, \cdots, n; \ k = 1, 2, \cdots, l - 1).$$

It follows from theorem 2. that the conclusion holds.

Theorem 6 . *With $l_m = l_{m+1} + 2$ $(m = 1, 2, \cdots, n - 1)$, for T_n, there exists a $\hat{\rho}$-labeling.*

In the proof of theorem 3., if only let $f(x_{1,t_1}) = |E| + 1$, then we can prove this theorem.

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