Spanning eulerian subgraphs in N^2-locally connected claw-free graphs

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Abstract

A graph G is N^m*-locally connected* if for every vertex v in G, the vertices not equal to v and with distance at most m to v induce a connected subgraph in G. In this note, we first present a counterexample to the conjecture that every 3-connected, N^2-locally connected claw-free graph is hamiltonian and then show that both connected N^2-locally connected claw-free graph and connected N^3-locally connected claw-free graph with minimum degree at least three have connected even [2, 4]-factors.

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1 Introduction

We use [1] for terminology and notations not defined here, and consider finite simple graphs only. Let G be a graph. Denote by $d_G(v)$ the degree of a vertex $v \in V(G)$. For a vertex v of G, the neighborhood of v, *i.e.*, the set of all vertices that are adjacent to v, will be called the neighborhood of the *first type* of v in G and denoted by $N_1(v, G)$, or briefly, $N_1(v)$ or $N_G(v)$. We define the neighborhood of the *second type* of v in G (denoted by $N_2(v, G)$, or briefly, $N_2(v)$) as the subgraph of G induced by the edge subset $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$. We say that a vertex v is *locally connected* if $G[N(v)]$ is connected; and G is *locally connected* if every vertex of G is locally connected. Analogously, a vertex v is *N_2-locally connected* if $N_2(v)$ is connected; and G is called *N_2-locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. The following theorems give the hamiltonicity of the locally and N_2-locally connected graph.

Theorem 1.1 *(Oberly and Sumner, [9]) Every connected locally connected claw-free graph on at least three vertices is hamiltonian.*

The following was conjectured by Ryjáček [11] and recently proved affirmatively in [5].

Theorem 1.2 *(Lai, Shao and Zhan, [5]) Every 3-connected N_2-locally connected claw-free graph is hamiltonian.*

For a graph G and a vertex $v \in V(G)$, for any positive integer m we denote $N^m(v, G)$ (or simply $N^m(v)$) the set of vertices of distance at most m from v in G, excluding v itself. A vertex $v \in V(G)$ is *N^m-locally connected* if $N^m(v)$ induces a connected subgraph of G; and G is *N^m-locally connected* if every vertex $v \in V(G)$ is N^m-locally connected.

Conjecture 1.3 *(Li, [6]) Every 3-connected, N^2-locally connected claw-free graph is hamiltonian.*

The purpose of this note is to investigate whether Conjecture 1.3 is true. More precisely, we want to investigate whether such graphs can be hamiltonian, and if not, how close they are to being hamiltonian. We shall give a counterexample to Conjecture 1.3 in Section 2.

A graph is *eulerian* if it is connected and every vertex has even degree. Note that the graph K_1 is also eulerian. An eulerian subgraph C of G is called a *dominating eulerian subgraph* of G if $E(G-V(C)) = \emptyset$. A *factor* of a graph G is a spanning subgraph of G. Let H be a factor of G. Then H is an *even factor* of G if every vertex of H has even degree; H is a *connected factor* if H is a connected graph; H is an $[a, b]$-factor, for integers $a \leq b$, if for every vertex $v \in V(H)$, $a \leq d_H(v) \leq b$. A *connected even factor* is also called a *spanning eulerian subgraph*, and a graph that admits a connected even factor is a *supereulerian graph*, and a connected $[2, 2]$-factor of a graph G is a hamiltonian cycle of G.

Our main results of this note are the following two theorems, whose proofs will be given in Section 3.

Theorem 1.4 *Every connected N^2-locally connected claw-free graph with at least three vertices has a connected even $[2, 4]$-factor.*

Theorem 1.5 *Every connected N^3-locally connected claw-free graph with minimum degree at least three has a connected even $[2, 4]$-factor.*

Theorem 1.5 has an immediate consequence: every 3-edge-connected N^3-locally connected claw-free graph has a connected even $[2, 4]$-factor, which also shows that the graph satisfying the condition of Conjecture 1.3 has a connected even $[2, 4]$-factor.

2 Counterexamples

In this section, our main aim is to give a counterexample to Conjecture 1.3.

The *line graph* of a graph G, denote by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The following result by Harary and Nash-Williams [4] is well known.

Theorem 2.1 *(Harary and Nash-Williams [4]) Let G be a graph with at least three edges. Then the line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

We now give a counterexample to Conjecture 1.3 as follows: Let P_{10} denote the Petersen graph. For an integer $k > 0$ and let $P_{10}(k)$ denote the graph obtained from P_{10} by adding k pendant edges at each vertex of P_{10}. Let $G(k) = L(P_{10}(k))$ be the line graph of $P_{10}(k)$. For an example when $k = 3$ see Figure 2.

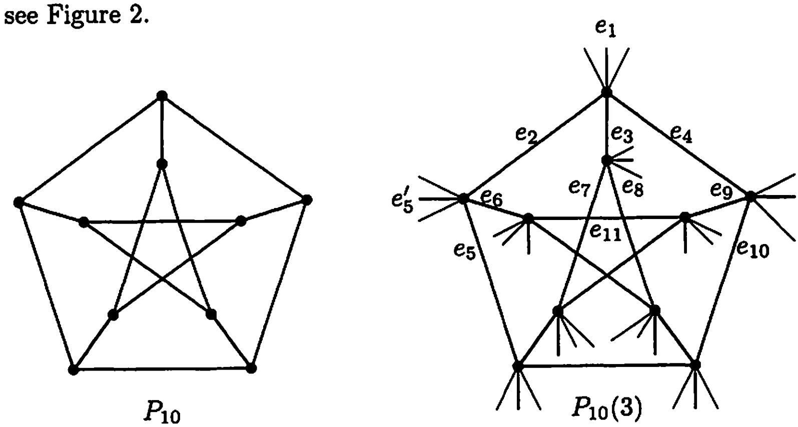

Figure 2 The Petersen graph P_{10} and $P_{10}(3)$

By Theorem 2.1, $G(k)$ does not have a Hamilton cycle. On the other hand, we shall check that each $G(k)$ is claw-free 3-connected (hence 3-edge-connected) and N^2-locally connected. Since line graphs are claw-free, $G(k)$ must also be claw-free. Since $P_{10}(k)$ does not have an edge cut with size less than 3 whose deleting will produces two components containing edges, the connectivity of $G(k)$ is at least 3. It remains to check that each $G(k)$ is N^2-locally connected.

We shall use the notation in Figure 2 to show that $G(k)$ is locally N^2-connected, by the symmetry of the Petersen graph, it suffices to show that both vertices e_1 and e_2 are locally N^2-connected in $G(k)$.

Let v_1, v_2 and v_3 denote the vertices in $P_{10}(k)$ that are incident with both e_1 and e_2, both e_2 and e_6, and both e_3 and e_7, respectively. For each vertex $v \in V(P_{10}(k))$, let $K(v)$ denote the complete graph in $G(k)$ induced by the edge incident with v in $P_{10}(k)$.

Since e_1 is a pendant edge in $P_{10}(k)$, it lies in a complete subgraph $K(v_1)$ of $G(k)$ containing e_2, e_3, e_4. Any vertices that are of distance 2 from e_1 in $G(k)$ must be a vertex adjacent to one of e_2, e_3 and e_4. Therefore, e_1 is a locally N^2-connected vertex in $G(k)$, see Figure 3.

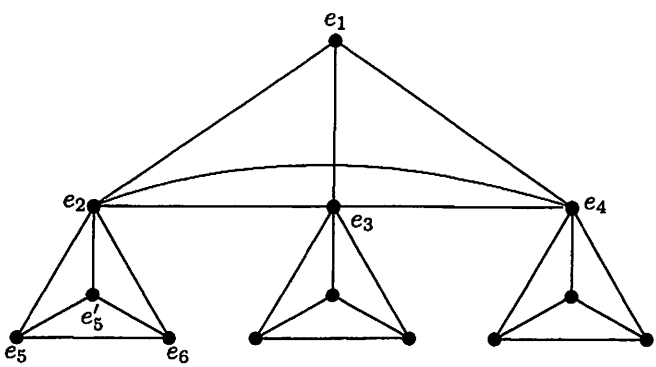

Figure 3 An illustration of the proof that e_1 is locally N^2-connected in $G(k)$

In $G(k)$, e_2 is in the intersection of two complete subgraphs $K(v_2)$ and $K(v_3)$. Every vertex of distance 2 in $G(k)$ to e_2 must be adjacent to a vertex in $\{e_3, e_4, e_5, e_6\}$. To show that e_2 is locally N^2-connected in $G(k)$, it suffices to show that $G(k)$ has a path P connecting the complete subgraph $K(v_2) - e_2$ and the complete subgraph $K(v_3) - e_2$, such that all vertices in $V(P)$ are of distance 2 from e_2 in $G(k)$. Note that $P = e_6, e_{11}, e_9, e_4$ is such a path in $G(k)$, and so e_2 is locally N^2-connected.

3 Proofs of main results

We start with the following lemma which is a special case of a result in [8].

Lemma 3.1 *Let G be a $K_{1,3}$-free graph. If G has a connected even factor H, then G has a connected $[2,4]$-factor.*

Proof. Let G be a supereulerian $K_{1,3}$-free graph and F a connected even factor of G with the maximum degree $\Delta(F)$, and $n(G, F, \Delta)$ the number of vertices in F with maximum degree $\Delta(F) = \Delta$. Without loss of generality assume that F is the one with $n(G, F, \Delta)$ minimum among all such connected even factors. Then we claim that $\Delta(F) \leq 4$, i.e., F is a connected even $[2,4]$-factor. Suppose, otherwise, $\Delta(F) \geq 6$. Let w be a vertex of degree $d_F(w) = \Delta(F)$. Then F has at least 3 edge-disjoint cycles C_1, C_2, C_3 with a common vertex w, since F is an even factor. Let $U = \{u_1, v_1, u_2, v_2, u_3, v_3\} \subseteq N_F(w)$ such that $\{wu_i, wv_i\} \subseteq E(C_i)$ for each i $(1 \leq i \leq 3)$. Then we have the following fact.

Claim 1. *For any pair of vertices $x_i \in \{u_i, v_i\}$ and $x_j \in \{u_j, v_j\}$, we have $x_ix_j \notin E(G)$ except that $\{x_ix_j, x_jw, wx_i\}$ is an edge cut set of F, and in the exceptional case, exactly one of $\{x_ix_j, x_jw\}$ and $\{x_ix_j, x_iw\}$ is an edge cut set of F.*

Proof of Claim 1. Suppose, otherwise, there exists a pair of vertices $x_i \in \{u_i, v_i\}$ and $x_j \in \{u_j, v_j\}$ such that either $x_ix_j \in E(G) \setminus E(F)$ or $\{x_ix_j, x_jw, wx_i\}$ is not an edge cut set of F and then $x_ix_j \in E(F)$. Thus

$$F' = F + (\{x_ix_j\} \cap (E(G) \setminus E(F))) - (\{x_ix_j, x_jw, wx_i\} \cap E(F))$$

is a connected even factor and $n(G, F', \Delta) = n(G, F, \Delta) - 1$, a contradiction.

By the choice of x_i and x_j, we deduce that if $\{x_ix_j, x_iw, x_jw\}$ is an edge cut set of F then exactly one of $\{x_ix_j, x_jw\}$ and $\{x_ix_j, x_iw\}$ is an edge cut set of F. This completes the proof of Claim 1.

Claim 2. *There exist three vertices in U as a vertex set X with*

$$E(G[X]) = \emptyset. \tag{1}$$

Proof of Claim 2. If $\{y_iy_j, y_jw, wy_i\}$ is not an edge cut set of F for any pair of vertices $y_i \in \{u_i, v_i\}$ and $y_j \in \{u_j, v_j\}$, then by Claim 1, $y_iy_j \notin E(G)$ for any $\{i,j\} \subseteq \{1, 2, \cdots, k\}$. Then any vertex set $X = \{x_1, x_2, x_3\}$ in U with $x_i \in \{u_i, v_i\}$ satisfies (1).

It remains the case that there exists a pair of vertices $y_i \in \{u_i, v_i\}$ and $y_j \in \{u_j, v_j\}$, say, $y_i = u_1$ and $y_j = u_2$, such that $\{u_1u_2, u_2w, wu_1\}$ is an edge cut set of F, then, by Claim 1, we can assume, without loss of generality, that $\{u_1u_2, u_1w\}$ is an edge cut set of F. Now let $C_1' = u_2u_1wu_2$ and $C_2' = G[(E(C_1) \cup E(C_2)) \setminus \{u_1u_2, u_2w, wu_1\}]$. If $\{z_1z_2, z_2w, wz_1\}$ is not an edges cut set of F for any pair of vertices $z_1 \in \{v_1, v_2\}$ and $z_3 \in \{u_3, v_3\}$, then, by Claim 1, $z_1z_3 \notin E(G)$. We choose a vertex set $X = \{z_1, z_3, u_1\}$ with (1), where $z_1 \in \{v_1, v_2\}$ and $z_3 \in \{u_3, v_3\}$. In the remained case that there exists a pair of vertices $z_1 \in \{v_1, v_2\}$ and $z_3 \in \{u_3, v_3\}$ such that $\{z_1z_3, z_3w, wz_1\}$ is an edge cut set of F, then, by Claim 1, we can assume, without loss of generality, that $\{z_1z_3, z_1w\}$ is an edge cut set of F. Now let $C_2'' = z_1z_3wz_1$ and $C_3' = G[(E(C_2') \cup E(C_3)) \setminus \{z_1z_3, z_3w, wz_1\}]$. Then $X = \{u_1, z_1, x_3\}$ where $x_3 \in N(w) \cap V(C_3')$, is a vertex set with (1), which completes the proof of Claim 2.

By Claim 2, there exists a set of three vertices $X \subseteq U$, such that the induced subgraph $G[\{w\} \bigcup X] \cong K_{1,3}$ in G which contradicts the assumption of Lemma 3.1. This completes the proof of Lemma 3.1. □

We need the following results. Theorem 3.2 was first proved by Paulraja, and later improved by Catlin and Lai.

Theorem 3.2 *(Paulraja, [10], and Catlin and Lai, [3]) Let G be a con-*

nected claw-free graph. If every edge of G lies in a cycle of length at most 5, then G is supereulerian.

Theorem 3.3 *(Catlin and Lai, [3]) Let G be a 2-connected claw-free graph with $\delta(G) \geq 3$. If every edge of G lies in a cycle of length at most 7, then G is supereulerian.*

Theorem 3.4 below shows a property of a connected N^k-locally connected graph.

Theorem 3.4 *Let k be an positive integer and G be an connected N^k-locally connected graph containing at least two edges. Then every edge lies in a cycle of length at most $2k+1$.*

Proof. By contradiction, suppose that there is an edge $e = uv$ in G which does not lie in a cycle of length at most $2k+1$. Let $N_i(x)$ denote the set of vertices that are of distance i from x in G for any vertex $x \in V(G)$, and $N_i(x \setminus y)$ the set of vertices in $N_i(x)$ that have a path, of length i from x, not containing y, for any edge $xy \in E(G)$.

If $N_i(u \setminus v) \cap N_j(v \setminus u) \neq \emptyset$ for some $1 \leq i, j \leq k$, then there is a cycle of length $i + j + 1 \leq 2k + 1$ containing e, a contradiction. Thus $N_i(u \setminus v) \cap N_j(v \setminus u) = \emptyset$ for any $1 \leq i, j \leq k$, which contradicts the fact that u and v are N^k-locally connected. Therefore, we complete the proof of Theorem 3.4. □

Now we can give the proof of our main results.

Proof of Theorem 1.4. It follows from Lemma 3.1, Theorems 3.2 and 3.4.□

Proof of Theorem 1.5. If there is a cut vertex v in G, then v is not N^3-locally connected vertex. It implies the fact that every connected N^3-

locally connected graph is 2-connected. Now the proof of Theorem 1.5 follows from Lemma 3.1, Theorems 3.3 and 3.4.□

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