Orientable embedding distributions by genus for certain type of non-planar graphs (II)

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Abstract– In this paper we give an explicit expression of the genus distributions of M_j^n, for $j = 1, 2, ..., 11$, which are introduced in the previous paper " Orientable embedding distributions by genus for certain type of non-planar graphs". For a connected graph $G = (V, E)$ with a cycle, let e be an edge on a cycle. By adding $2n$ vertices $u_1, u_2, u_3, \cdots, u_n, v_1, v_2, v_3, \cdots, v_n$ on e in sequence and connecting $u_k v_k$ for k, $1 \leq k \leq n$, a non-planar graph G_n is obtained for $n \geq 3$. Thus, the orientable embedding distribution of G_n by genus is obtained via the genus distributions of M_j^n.

Keywords– Embedding distribution; joint tree; surface; genus.

1. Introduction

Gross and Furst [2] introduced the embedding distributions of graphs by genus. There have been many studies on the explicit expressions of orientable embedding genus distribution for circular ladders and Mobius ladders [4], closed-end ladders and cobblestone paths [6], bouquets of circles [3] and Ringel ladders [1]. In a former

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paper [5], we have obtained linear equations of orientable embedding genus distributions for certain type of non-planar graphs. In this paper, we shall present the explicit expressions of these graphs, which solves a problem raised in [5]. An embedding of a graph considered is always assumed an orientable embedding.

For a graph G, let $g_i(G)$ be the number of embeddings with genus i. Then the embedding polynomial of G by genus is as follows:

$$f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i.$$

Given a positive integer n, let $y_1, y_2, \cdots, y_n$ denote n distinct letters. Let $Y_1 = y_{k_1}y_{k_2}y_{k_3}\cdots y_{k_r}$, $Y_2 = y_{k_{r+1}}y_{k_{r+2}}y_{k_{r+3}}\cdots y_{k_n}$, $Y_3 = y_{m_1}^- y_{m_2}^- y_{m_3}^- \cdots y_{m_s}^-$ and $Y_4 = y_{m_{s+1}}^- y_{m_{s+2}}^- y_{m_{s+3}}^- \cdots y_{m_n}^-$ where $n \geq k_1 > k_2 > k_3 > \cdots > k_r \geq 1$, $1 \leq k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \leq n$, $1 \leq m_1 < m_2 < m_3 < \cdots < m_s \leq n$, $n \geq m_{s+1} > m_{s+2} > m_{s+3} > \cdots > m_n \geq 1$ and $0 \leq r, s \leq n$, $k_p \neq k_q$, $m_p \neq m_q$ for $p \neq q$. Let $M_1^n = \{Y_1Y_2Y_3Y_4\}$, $M_2^n = \{Y_1Y_2Y_4Y_3\}$, $M_3^n = \{Y_1Y_3Y_2Y_4\}$, $M_4^n = \{aY_1Y_2a^-Y_3Y_4\}$, $M_5^n = \{aY_1Y_3a^-Y_2Y_4\}$, $M_6^n = \{aY_1Y_4a^-Y_2Y_3\}$, $M_7^n = \{aY_1a^-Y_2Y_4Y_3\}$, $M_8^n = \{Y_1Y_2aY_3a^-bY_4b^-\}$, $M_9^n = \{Y_1Y_3aY_2a^-bY_4b^-\}$, $M_{10}^n = \{Y_1Y_4aY_2a^-bY_3b^-\}$ and $M_{11}^n = \{Y_1aY_2a^-bY_3b^-cY_4c^-\}$. The linear equations are presented in Lemma 1 below.

Lemma 1 (Theorem 3.2 of [5]) Let $g_{i_j}(n)$ be the number of surfaces with genus i in M_j^n for $n \geq 0$, $i \geq 0$ and $1 \leq j \leq 12$. Let $f_{M_j^0}(x) = 1$. Then, for $n \geq 1$, $g_{i_j}(n) =$

$$4g_{i_7}(n-1), \text{ if } j = 1 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$2g_{i_2}(n-1) + 2g_{i_4}(n-1), \text{ if } j = 2 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$g_{i_3}(n-1) + g_{i_5}(n-1) + 2g_{i_7}(n-1), \text{ if } j = 3 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$4g_{(i-1)_2}(n-1), \text{ if } j = 4 \text{ and } 1 \leq i \leq \left[\frac{n+1}{2}\right];$$
$$2g_{(i-1)_2}(n-1) + 2g_{i_5}(n-1), \text{ if } j = 5 \text{ and } 0 \leq i \leq \left[\frac{n+1}{2}\right];$$

$2g_{(i-1)_3}(n-1)+2g_{i_{10}}(n-1)$, if $j=6$ and $1 \le i \le \left[\frac{n+1}{2}\right]$;
$g_{(i-1)_1}(n-1)+g_{(i-1)_2}(n-1)+g_{i_7}(n-1)+g_{i_8}(n-1)$,
if $j=7$ and $0 \le i \le \left[\frac{n+1}{2}\right]$;
$4g_{(i-1)_7}(n-1)$, if $j=8$ and $1 \le i \le \left[\frac{n}{2}\right]+1$;
$g_{(i-1)_5}(n-1)+2g_{(i-1)_7}(n-1)+g_{i_9}(n-1)$,
if $j=9$ and $0 \le i \le \left[\frac{n}{2}\right]+1$;
$g_{(i-1)_6}(n-1)+2g_{(i-1)_7}(n-1)+g_{i_{11}}(n-1)$,
if $j=10$ and $1 \le i \le \left[\frac{n}{2}\right]+1$;
$2g_{(i-1)_9}(n-1)+2g_{(i-1)_{10}}(n-1)$,
if $j=11$ and $1 \le i \le \left[\frac{n+1}{2}\right]+1$.

Given a graph G. Let e be an edge on a cycle of G. For a positive integer n, add $2n$ vertices $u_1, u_2, u_3, \cdots, u_n, v_1, v_2, v_3, \cdots, v_n$ on e in sequence. Obtain a new graph G_n from G by adding n new edges $u_l v_l$ $(1 \le l \le n)$.

Lemma 2 (Theorem 4.1 of [5]) $g_i(G_n)$ is a linear combination $g_{m_j}(n)$'s for $j=1,6$, $0 \le m \le i$ and $n \ge 1$.

In this paper, we solve the equations in Lemma 1. Hence the orientable embedding distribution of G_n by genus is obtained for each positive integer n by applying Lemma 2.

2. Main Theorem

Theorem 3 Let $C_n(i) = \binom{n-2-i}{i}$. Then,

$$g_{i_j}(n) = 2^{n+i}\frac{2n-3i}{n-i}C_{n+2}(i) \text{ for } j=2, 0 \le i \le \left[\frac{n}{2}\right] \text{ and } n \ge 1.$$

Proof. The conclusion will be verified by induction on n. It is easily seen that the conclusion holds for $n=1$ and 2 by Lemma 1.

Suppose the assertion holds for any integer k with $2 < k < n$. Let $E_n(i) = \prod_{k=0}^{i-2}(n-i-1-k)$. By Lemma 1 and induction hypothesis for n and $0 \le i \le \left[\frac{n}{2}\right]$,

$$\begin{aligned} g_{i_2}(n) &= 2g_{i_2}(n-1) + 2g_{i_4}(n-1) \\ &= 2g_{i_2}(n-1) + 8g_{(i-1)_2}(n-2) \\ &= 2^{n+i}\frac{2n-3i-2}{n-i-1}C_{n+1}(i) + 2^{n+i}\frac{2n-3i-1}{n-i-1}C_n(i-1) \\ &= 2^{n+i}\frac{E_n(i)}{i!(n-i-1)}\Big((n-2i)(2n-3i-2) + i(2n-3i-1)\Big) \\ &= 2^{n+i}\frac{2n-3i}{n-i}C_{n+2}(i). \end{aligned}$$

□

Theorem 4 Let $g_{0_6}(1) = 2$, $g_{1_6}(1) = 2$, $g_{0_6}(2) = 2$, $g_{1_6}(2) = 14$, $g_{0_7}(1) = 2$, $g_{1_7}(1) = 2$, $g_{0_9}(1) = 1$, $g_{1_9}(1) = 3$, $g_{0_{10}}(1) = 1$, $g_{1_{10}}(1) = 3$, $g_{1_{10}}(2) = 10$, $g_{2_{10}}(2) = 6$, $g_{1_{10}}(3) = 10$, $g_{2_{10}}(3) = 54$, $g_{1_{11}}(1) = 4$, $g_{1_{11}}(2) = 4$, $g_{2_{11}}(2) = 12$, $A_n(i) = \frac{2n-3i-2}{n-2i-1}$, $B_n(i) = \frac{n-i-1}{n-2i}$, $C_n(i) = \binom{n-2-i}{i}$ and $D_n(i) = \frac{n}{i}2^i$. Then, $g_{i_j}(n) =$

$$\begin{aligned}
&2^n + 4n - 2, \text{ if } j = 3, i = 0 \text{ and } n \ge 1; \\
&C_{n+2}(i+1)\Big(2^{3i+1}A_{n+2}(i+1) \\
&\qquad +(2^{n+i-1} - 2^{3i-2})\frac{(i+1)A_{n+2}(i)B_{n+2}(i+1)}{n-2i-1}\Big), \\
&\qquad\qquad \text{if } j = 3, 1 \le i \le \left[\frac{n}{2}\right] - 1 \text{ and } n \ge 2; \\
&C_{n+1}(i)\Big(2^{3i+1} + (2^{n+i-1} - 2^{3i-2})A_{n+2}(i)B_{n+2}(i+1)\Big), \\
&\qquad\qquad \text{if } j = 3, \left[\frac{n}{2}\right] - 1 < i \le \left[\frac{n-1}{2}\right] \text{ and } n \ge 2; \\
&(2^{n+i-1} - 2^{3i-2})A_{n+2}(i)C_{n+2}(i), \\
&\qquad\qquad \text{if } j = 3, \left[\frac{n-1}{2}\right] < i \le \left[\frac{n}{2}\right] \text{ and } n \ge 2; \\
&2^{n+i-1}\frac{2n-3i+2}{n-i+1}C_{n+3}(i), \text{ if } j = 5, 0 \le i \le \left[\frac{n+1}{2}\right] \text{ and } n \ge 1;
\end{aligned}$$

$2^n + 8n + 8,\ \text{if } j = 6, i = 1 \text{ and } n = 3, 4;$

$2^n + 8n,\ \text{if } j = 6, i = 1 \text{ and } n \geq 5;$

$(2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i),$

$\qquad \text{if } j = 6, 2 \leq i < \frac{n}{2} - 1 \text{ and } n \geq 5;$

$(2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^{n-1},$

$\qquad \text{if } j = 6, i = \frac{n}{2} - 1 \text{ and } n \geq 5;$

$(2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^n,$

$\qquad \text{if } j = 6, \frac{n}{2} - 1 < i \leq \frac{n-1}{2} \text{ and } n \geq 4;$

$(2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{\frac{3n}{2}+1} - 3 \cdot 2^{n-1},$

$\qquad \text{if } j = 6, \frac{n-1}{2} < i \leq \frac{n}{2} \text{ and } n \geq 4;$

$(2^n - 2^{2i-2})C_n(i-2)D_n(i-1),$

$\qquad \text{if } j = 6, \frac{n}{2} < i \leq \frac{n+1}{2} \text{ and } n \geq 3;$

$2,\ \text{if } j = 7, i = 0 \text{ and } n \geq 2;$

$2^{n+1} + 16n - 26,\ \text{if } j = 7, i = 1 \text{ and } n \geq 2;$

$C_{n+1}(i)\Big(2^{3i}A_{n+1}(i) + (2^{n+i-1} - 2^{3i-3})\frac{iA_{n+1}(i-1)B_{n+1}(i)}{n-2i}\Big),$

$\qquad \text{if } j = 7, 2 \leq i \leq \left[\frac{n-1}{2}\right] \text{ and } n \geq 2;$

$C_n(i-1)\Big(2^{3i} + (2^{n+i-1} - 2^{3i-3})A_{n+1}(i-1)B_{n+1}(i)\Big),$

$\qquad \text{if } j = 7, \left[\frac{n-1}{2}\right] < i \leq \left[\frac{n}{2}\right] \text{ and } n \geq 3;$

$(2^{n+i-1} - 2^{3i-3})A_{n+1}(i-1)C_{n+1}(i-1),$

$\qquad \text{if } j = 7, \left[\frac{n}{2}\right] < i \leq \left[\frac{n+1}{2}\right] \text{ and } n \geq 2;$

$1,\ \text{if } j = 9, i = 0 \text{ and } n \geq 2;$

$2^n + 4n - 3,\ \text{if } j = 9, i = 1 \text{ and } n \geq 2;$

$C_{n+2}(i)\Big(2^{3i-2}A_{n+2}(i) + (2^{n+i-2} - 2^{3i-5})\frac{iA_{n+2}(i-1)B_{n+2}(i)}{n-2i+1}\Big),$

$\qquad \text{if } j = 9, 2 \leq i \leq \left[\frac{n}{2}\right] \text{ and } n \geq 2;$

$C_{n+1}(i-1)\Big(2^{3i-2} + (2^{n+i-2} - 2^{3i-5})A_{n+2}(i-1)B_{n+2}(i)\Big),$

$\qquad \text{if } j = 9, \left[\frac{n}{2}\right] < i \leq \left[\frac{n+1}{2}\right] \text{ and } n \geq 3;$

$(2^{n+i-2} - 2^{3i-5})A_{n+2}(i-1)C_{n+2}(i-1),$

$\quad$ if $j = 9, \left[\frac{n+1}{2}\right] < i \leq \left[\frac{n}{2}\right] + 1$ and $n \geq 2$;

$6,$ if $j = 10, i = 1$ and $n \geq 4$;

$3 \cdot 2^n + 48n - 86,$ if $j = 10, i = 2$ and $n = 4, 5$;

$3 \cdot 2^n + 48n - 102,$ if $j = 10, i = 2$ and $n \geq 6$;

$$3C_n(i-1)\Big(2^{3i-2}B_{n+1}(i) + (2^{n+i-2} - 2^{3i-5})\frac{(i-1)B_n(i-1)B_{n+1}(i-1)}{n-2i+1}\Big),$$

$\quad$ if $j = 10, 3 \leq i < \frac{n-1}{2}$ and $n \geq 6$;

$$3C_n(i-1)\Big(2^{3i-2}B_{n+1}(i) + (2^{n+i-2} - 2^{3i-5})\frac{(i-1)B_n(i-1)B_{n+1}(i-1)}{n-2i+1}\Big) + 2^{n-1},$$

$\quad$ if $j = 10, i = \frac{n-1}{2}$ and $n \geq 7$;

$$3C_n(i-1)\Big(2^{3i-2}B_{n+1}(i) + (2^{n+i-2} - 2^{3i-5})\frac{(i-1)B_n(i-1)B_{n+1}(i-1)}{n-2i+1}\Big) + 2^n,$$

$\quad$ if $j = 10, \frac{n-1}{2} < i \leq \frac{n}{2}$ and $n \geq 6$;

$$C_{n-1}(i-2)\Big(2^{3i-2} + 3(2^{n+i-2} - 2^{3i-5})B_n(i-1)B_{n+1}(i-1)\Big) + 2^{\frac{3n+1}{2}} - 3 \cdot 2^{n-1},$$

$\quad$ if $j = 10, \frac{n}{2} < i \leq \frac{n+1}{2}$ and $n \geq 5$;

$3(2^{n+i-2} - 2^{3i-5})B_{n+1}(i-1)C_n(i-2),$

$\quad$ if $j = 10, \frac{n+1}{2} < i \leq \frac{n}{2} + 1$ and $n \geq 4$;

$2,$ if $j = 11, i = 1$ and $n \geq 3$;

$2^n + 8n + 6,$ if $j = 11, i = 2$ and $n = 3, 4$;

$2^n + 8n - 2,$ if $j = 11, i = 2$ and $n \geq 5$;

$(2^n - 2^{2i-4})C_n(i-3)D_n(i-2) + 2^{2i-2}C_n(i-2)D_n(i-1),$

$\quad$ if $j = 11$ and $3 \leq i < \frac{n}{2}$;

$(2^n - 2^{2i-4})C_n(i-3)D_n(i-2) + 2^{2i-2}C_n(i-2)D_n(i-1) + 2^{n-1},$

$\quad$ if $j = 11, i = \frac{n}{2}$ and $n \geq 5$;

$(2^n - 2^{2i-4})C_n(i-3)D_n(i-2) + 2^{2i-2}C_n(i-2)D_n(i-1) + 2^n,$
if $j = 11, \frac{n}{2} < i \le \frac{n+1}{2}$ and $n \ge 4$;
$(2^n - 2^{2i-4})C_n(i-3)D_n(i-2) + 2^{\frac{3n}{2}+1} - 3 \cdot 2^{n-1},$
if $j = 11, \frac{n+1}{2} < i \le \frac{n}{2} + 1$ and $n \ge 4$;
$(2^n - 2^{2i-4})C_n(i-3)D_n(i-2),$
if $j = 11, \frac{n}{2} + 1 < i \le \frac{n+1}{2} + 1$ and $n \ge 3$;
0, otherwise.

Proof. We shall prove the case for $j = 5$ by induction on n. The proofs for the other cases are similar and so are omitted. We can easily find $g_{0_5}(1) = 2$ and $g_{1_5}(1) = 2$ according to the formula. Then the conclusion holds for $n = 1$ by using Lemma 1.

Suppose that the conclusion holds for a positive integer $n-1$. By Lemma 1,

$$g_{i_5}(n) = 2g_{(i-1)_2}(n-1) + 2g_{i_5}(n-1), \text{ if } 0 \le i \le \left[\frac{n+1}{2}\right].$$

By Theorem 3 and induction hypothesis for n and $0 \le i \le \left[\frac{n+1}{2}\right]$,

$$\begin{aligned} g_{i_5}(n) &= 2^{n+i-1}\frac{2(n-1)-3(i-1)}{n-1-(i-1)}C_{n+1}(i-1) \\ &+2^{n+i-1}\frac{2(n-1)-3i+2}{n-i}C_{n+2}(i) \\ &= \frac{2^{n+i-1}}{n-i}\Big(\frac{2n-3i+1}{(i-1)!}E_{n+1}(i) \\ &+\frac{(2n-3i)(n-2i+1)}{i!}E_{n+1}(i)\Big) \\ &= 2^{n+i-1}\frac{C_{n+3}(i)}{(n-i)(n-i+1)}\Big(i(2n-3i+1) \\ &+(2n-3i)(n-2i+1)\Big) \\ &= 2^{n+i-1}\frac{2n-3i+2}{n-i+1}C_{n+3}(i). \end{aligned}$$

Hence the conclusion holds for $j = 5$ by induction on n. □

Corollary 5 Let $g_{0_1}(1) = 4$, $g_{1_1}(2) = 8$, $g_{1_4}(1) = 4$, $g_{1_8}(1) = 4$, $g_{2_8}(2) = 8$, $A_n(i) = \frac{2n-3i-2}{n-2i-1}$, $B_n(i) = \frac{n-i-1}{n-2i}$, $C_n(i) =$

$\binom{n-2-i}{i}$ and $D_n(i) = \frac{n}{i}2^i$. Then, $g_{i_j}(n) =$

$$
\begin{cases}
8, & \text{if } j=1, i=0 \text{ and } n \geq 2;\\
8, & \text{if } j=1, i=1 \text{ and } n = 2;\\
2^{n+2}+64n-168, & \text{if } j=1, i=1 \text{ and } n \geq 3;\\
C_n(i)\Big(2^{3i+2}A_n(i) + (2^{n+i}-2^{3i-1})\frac{iA_n(i-1)B_n(i)}{n-2i-1}\Big), & \text{if } j=1, 2 \leq i \leq \left[\frac{n}{2}\right]-1 \text{ and } n \geq 3;\\
C_{n-1}(i-1)\Big(2^{3i+2} + (2^{n+i}-2^{3i-1})iA_n(i-1)B_n(i)\Big), & \text{if } j=1, \left[\frac{n}{2}\right]-1 < i \leq \left[\frac{n-1}{2}\right] \text{ and } n \geq 4;\\
\Big(2^{n+i}-2^{3i-1}\Big)A_n(i-1)C_n(i-1), & \text{if } j=1, \left[\frac{n-1}{2}\right] < i \leq \left[\frac{n}{2}\right] \text{ and } n \geq 3;\\
2^{n+i}\frac{2n-3i+1}{n-i}C_{n+1}(i-1), & \text{if } j=4, 1 \leq i \leq \left[\frac{n+1}{2}\right] \text{ and } n \geq 2;\\
8, & \text{if } j=8, i=1 \text{ and } n \geq 2;\\
2^{n+2}+64n-168, & \text{if } j=8, i=2 \text{ and } n \geq 3;\\
C_n(i-1)\Big(2^{3i-1}A_n(i-1) + (2^{n+i-1}-2^{3i-4})\frac{(i-1)A_n(i-2)B_n(i-1)}{n-2i+1}\Big), & \text{if } j=8, 3 \leq i \leq \left[\frac{n}{2}\right] \text{ and } n \geq 3;\\
C_{n-1}(i-2)\Big(2^{3i-1} + (2^{n+i-1}-2^{3i-4})A_n(i-2)B_n(i-1)\Big), & \text{if } j=8, \left[\frac{n}{2}\right] < i \leq \left[\frac{n+1}{2}\right] \text{ and } n \geq 4;\\
\Big(2^{n+i-1}-2^{3i-4}\Big)A_n(i-2)C_n(i-2), & \text{if } j=8, \left[\frac{n+1}{2}\right] < i \leq \left[\frac{n}{2}\right]+1 \text{ and } n \geq 3;\\
0, & \text{otherwise.}
\end{cases}
$$

Proof. We shall present the proof for the case when $j=4$, $n \geq 2$ and $1 \leq i \leq \left[\frac{n+1}{2}\right]$. By Lemma 1, for $1 \leq i \leq \left[\frac{n+1}{2}\right]$,

$$g_{i_4}(n) = 4g_{(i-1)_2}(n-1).$$

By Theorem 4,

$$g_{i_4}(n) = 2^{n+i}\frac{2(n-1)-3(i-1)}{n-1-(i-1)}C_{n+1}(i-1)$$
$$= 2^{n+i}\frac{2n-3i+1}{n-i}C_{n+1}(i-1).$$

The proofs for the other cases can be similarly verified by using Lemma 1 and Theorem 4. □

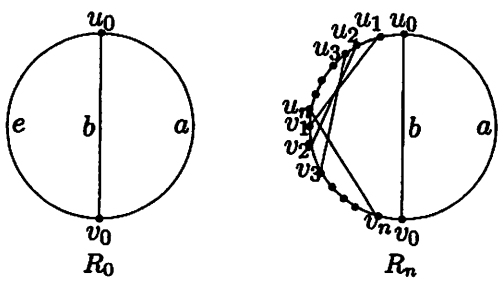

Figure: R_0 and R_n

Example The graph R_0 is given in Figure. Given a positive integer n, by adding $2n$ vertices $u_1, u_2, u_3, \cdots, u_n, v_1, v_2, v_3, \cdots, v_n$ on the edge u_0v_0 in sequence and connecting u_lv_l for $l, 1 \le l \le n$, the graph R_n is obtained. Let $g_i(n)$ denote the number of distinct embeddings for R_n with genus i. Let $K = \frac{(n-i)(n^2+n-3i^2+7i-4)}{(n-2i+2)(n-2i+3)}2^{n+i}$, $L = \frac{2n^2-2ni+2n-3i^2+i}{i}2^{3i}$ and $Q = \frac{(n-i)(2n^2-2ni+4n-3i^2+7i-4)}{(n-2i+2)(n-2i+3)}2^{3i-3}$. From [5] we know $g_i(n) = 2g_{i_6}(n) + 2g_{(i-1)_1}(n)$. Then $g_0(1) = 4$; $g_1(1) = 12$; $g_0(2) = 4$; $g_1(2) = 44$; $g_2(2) = 16$; $g_2(3) = 160$; $g_2(4) = 704$ and $g_i(n) =$

$2^{n+1} + 16n + 32$, if $i = 1, n = 3$ and 4;
$2^{n+1} + 16n + 16$, if $i = 1$ and $n \ge 5$;
$(n+2)2^{n+2} + 64n^2 - 80n - 272$, if $i = 2, n = 5$ and 6;
$(n+2)2^{n+2} + 64n^2 - 80n - 336$, if $i = 2$ and $n \ge 7$;
$\frac{1}{n-2i+1}(K+L-Q)C_n(i-1)$, if $3 \le i < \frac{n}{2} - 1$;
$\frac{1}{n-2i+1}(K+L-Q)C_n(i-1) + 2^n$, if $i = \frac{n}{2} - 1$ and $n \ge 8$;

$$\frac{1}{n-2i+1}(K+L-Q)C_n(i-1)+2^{n+1},$$
$$\text{if } \frac{n}{2}-1 < i \le \frac{n-1}{2} \text{ and } n \ge 7;$$
$$\frac{K+2^{3i}A_n(i-1)-Q}{n-2i+1}C_n(i-1)+2^{\frac{3n}{2}+2}-3\cdot 2^n,$$
$$\text{if } \frac{n-1}{2} < i \le \frac{n}{2} \text{ and } n \ge 6;$$
$$(2^{n+1}-2^{2i-1})C_n(i-2)D_n(i-1)$$
$$+C_{n-1}(i-2)\left(2^{3i}+(2^{n+i}-2^{3i-3})(i-1)A_n(i-2)B_n(i-1)\right),$$
$$\text{if } \frac{n}{2} < i \le \frac{n+1}{2} \text{ and } n \ge 5;$$
$$\left(2^{n+i}-2^{3i-3}\right)A_n(i-2)C_n(i-2), \text{ if } \frac{n+1}{2} < i \le \frac{n}{2}+1 \text{ and } n \ge 4;$$
0, otherwise.

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