A Family of Chromatically Unique 6-bridge Graphs

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Abstract

Let $P(G, \lambda)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique written χ−unique, if for any graph H, $G \sim H$ implies that G is isomorphic with H. In this paper we prove that the graph $\theta(a_1, a_2, ..., a_6)$ is χ−unique for exactly two distinct values of $a_1, a_2, ..., a_6$.

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1 Introduction

The graphs that we consider here are finite and simple. Let G be a graph and $\lambda \in \mathbb{N}$. A mapping $f : V(G) \longrightarrow \{1, 2, ..., \lambda\}$ is a λ−colourings of G if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. Two λ−colourings f and g of G are regarded as distinct if $f(x) \neq g(x)$ for some vertex x in G. The number of distinct λ−colourings of G is called the chromatic polynomial of G and denoted by $P(G, \lambda)$. Two graphs G and H are said to be chromatically equivalent, and we write $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi - unique$) if $G \cong H$ for any graph H such that $G \sim H$.

By subdivision we mean the operation of replacing an edge of a graph by a path. If a graph H can be derived from G by a sequence of subdivisions, we say H is a subdivision of G. For each positive integer h, the graph $G(h)$ obtained from G by replacing each edge of G with a path of length h is called the h-uniform subdivision of G.

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A chain in a graph G is a path in G every internal vertex of which has degree 2 in G. The operation that replaces a $u - v$ chain by a an edge uv is called chain-contraction. By contracting all maximal chains of a graph G, we arrive at multigraph M(G). Two graphs G and H are homeomorphic if $M(G) = M(H)$. If G is homeomorphic to H we also say G is a H-homeomorph.

For each integer $k \geq 2$, let θ_k be the multigraph with two vertices and k edges. Any subdivision of θ_k is called multi-bridge graph or k-bridge graph. We denote $\theta(a_1, a_2, ..., a_k)$ where $a_1, a_2, ..., a_k \in \mathbb{N}$ and $a_1 \leq a_2 \leq ... \leq a_k$ to be the graph obtained by replacing the edges of θ_k by paths of length $a_1, a_2, ..., a_k$ respectively. Li [8] proved that the graph $\theta(a_1, a_2, ..., a_5)$ is $\chi-$unique for exactly two distinct values of $a_1, a_2, ..., a_5$. In this paper we prove the chromatic uniqueness of a new family of 6-bridge graphs.

2 Auxiliary Results

In this section we cite some results use in the sequel.

A 2-bridge graph is simply a cycle, which is $\chi-$unique. Chao and Whitehead Jr. [2], showed that every 3-bridge graph $\theta(1, a_2, a_3)$ called a theta graph is $\chi-$unique. Loerinc [10] extended the above result to all 3-bridge graphs also called generalized $\theta - graph$. Chen et al. [3] proved that the 4-bridge graph $\theta(a_1, a_2, a_3, a_4)$ is $\chi-$unique if and only if for any $c \geq 2$, $(a_1, a_2, a_3, a_4) \neq (2, c, c+1, c+2)$. Bao and Chen [1] showed that every 5-bridge graph is $\chi-$unique if its shortest maximal chains of length greater than 3. The above result is a special case of general result due to Xu et al. [11].

Theorem 1 *([11]) For $k \geq 4$, $\theta(a_1, a_2, ..., a_k)$ is $\chi-$unique if $k - 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.*

Li and Wei [9] established that the 5-bridge graph $\theta(2, 2, 2, a, b)$ is $\chi-$unique if and only if $(a, b) \neq (3, 4)$. Ye [12] extended the above result to any k-bridge graph $\theta(2, 2, \cdots, 2, a, b)$ with $b \geq a \geq 3$ and $k \geq 5$. Xu et al. [11] showed that any h-uniform subdivision of θ_k is $\chi-$unique, as states in the following theorem:

Theorem 2 *([11]) For $k \geq 2$, the graph $\theta_k(h)$ is $\chi-$unique.*

The above result was proved independently by Dong [4], Koh and Teo [7], and Xu et al. [11]. Dong et al. [6] proved the following theorem.

Theorem 3 *([6]) If $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k < a_1 + a_2$, where $k \geq 3$, then the graph $\theta(a_1, a_2, ..., a_k)$ is $\chi-$unique.*

Let $k, a_1, a_2, ..., a_k \in \mathbb{N}$, and $G = \theta(a_1, a_2, ..., a_k)$. Then (see [5]):

$$P(G,\lambda) = \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}} \prod_{i=1}^{k}\Big((\lambda-1)^{a_i+1} + (-1)^{a_i+1}(\lambda-1)\Big) + \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k}\Big((\lambda-1)^{a_i} + (-1)^{a_i}(\lambda-1)\Big).$$

Let $\lambda = 1 - x$, then:

$$P(G,1-x) = \frac{(-1)^{a_1+a_2+...+a_k+1}}{(1-x)^{k-1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\Big) = \frac{(-1)^{e(G)+1}}{(1-x)^{e(G)-v(G)+1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\Big)$$

where $e(G) = \sum_{i=1}^{k} a_i$ and $v(G) = \sum_{i=1}^{k} a_i - k + 2$. Also they defined $Q(G,x)$ for any graph G and real number x as:

$$Q(G,x) = (-1)^{e(G)+1}(1-x)^{e(G)-v(G)+1}P(G,1-x),$$

and they got the following results:

Theorem 4 *([6]) For any $k, a_1, a_2, ..., a_k \in \mathbb{N}$,*

$$Q(\theta(a_1,a_2,...,a_k),x) = x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)$$

Theorem 5 *([6]) For any graphs G and H,*
(i) if $H \sim G$, then $Q(H,x) = Q(G,x)$;
(ii) if $Q(H,x) = Q(G,x)$ and $v(H) = v(G)$, then $H \sim G$.

Lemma 1 *([6]) Suppose that $\theta(a_1, a_2, ..., a_k) \sim \theta(b_1, b_2, ..., b_k)$, where $k \geq 3, 2 \leq a_1 \leq a_2 \leq ... \leq a_k$ and $2 \leq b_1 \leq b_2 \leq ... \leq b_k$. Then $a_i = b_i$ for all $i = 1, 2, ..., k$.*

Dong et al. [6] denote $g_e(G_1, G_2, ..., G_k)$ to be the collection of all edge-gluing of all $G_1, G_2, ..., G_k$, where $k \geq 2$ and $e(G_i) \geq 1$ for all i, and then they got the following Lemma:

Lemma 2 *([6]) Let $H \sim \theta(a_1, a_2, ..., a_k)$, where $k \geq 3$ and $a_i \geq 2$ for all i. Then one of the following is true:*

(i) $H \cong \theta(a_1, a_2, ..., a_k)$;
(ii) $H \in g_e(\theta(b_1, b_2, ..., b_t), C_{b_{t+1}+1}, ..., C_{b_k+1})$, *where* $3 \le t \le k-1$ *and* $b_i \ge 2$ *for all* $i = 1, 2, ..., k$.

Theorem 6 *([6]) Let* $k, t, b_1, b_2, ..., b_k \in \mathbb{N}$ *with* $3 \le t \le k-1$ *and* $b_i \ge 2$ *for all* $i = 1, 2, ..., k$. *If* $H \in g_e(\theta(b_1, b_2, ..., b_t), C_{b_{t+1}+1}, ..., C_{b_k+1})$, *then*

$$Q(H, x) = x\prod_{i=1}^{k}(x^{b_i} - 1) - \prod_{i=1}^{t}(x^{b_i} - x)\prod_{i=t+1}^{k}(x^{b_i} - 1).$$

It is well known (see [7]) that:

Lemma 3 *([7]) If* $G \sim H$, *then*
(i) $v(G) = v(H)$;
(ii) $e(G) = e(H)$;
(iii) $g(G) = g(H)$ *and*
(iv) G and H have the same number of shortest cycles.
where $v(G)$, $e(G)$ *and* $g(G)$ *denote number of vertices, number of edges and the girth of the graph G.*

3 Results

In this section we prove a new result on chromatic uniqueness of 6-bridge graphs.

Lemma 4 *Let* $H \in g_e(\theta(b_1, b_2, ..., b_t), C_{b_{t+1}+1}, ..., C_{b_k+1})$. *Then the maximum number of cycles of size g (the girth of H) is* $\binom{t}{2} + k - t$.

Proof. Note that the maximum number of cycles of order g in $\theta(b_1, b_2, ..., b_t)$ is $\binom{t}{2}$, and we can get another $k-t$ cycles of order g from cycles $C_{b_{t+1}+1}, C_{b_{t+2}+1}, ..., C_{b_k+1}$. We claim that H does not contain another cycles of order g except the possible $\binom{t}{2} + k - t$ cycles above. If $b_i + b_j = g$ for $1 \le i \le t$ and $t+1 \le j \le k$, then $b_j + 1 < g$ because $b_i \ge 2$, and this is not possiple. Similarly, we can show that $b_i + b_j > g$ for $t+1 \le i < j \le k$. Therefore, the maximum number of cycles of order g is $\binom{t}{2} + k - t$. ■

Lemma 5 *The 6-bridge graph* $\theta(a, a, a, a, a, b)$, *where* $2 \le a \le b$ *is* χ*−unique.*

Proof. Let $G = \theta(a, a, a, a, a, b)$, where $2 \le a \le b$. If $b < 2a$, then by Theorem 3, G is χ−unique. Suppose that $H \sim G$ and $b \ge 2a$. Then by Lemmas 1 and 2, we need only to consider three cases.

Case 1 $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$, where $2 \le b_1 \le b_2 \le b_3$, $2 \le b_4, b_5, b_6$ and $5a+b = b_1+b_2+\cdots+b_6$. By Lemma 3, $g(G) = g(H) = 2a$. By Lemma 4, the maximum number of cycles of order 2a in H is six. But G contains 10 cycles of order 2a. This is a contradiction by Lemma 3.

Case 2 $H \in g_e(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1})$, where $2 \le b_1 \le b_2 \le b_3 \le b_4$, $2 \le b_5, b_6$ and $5a+b = b_1+b_2+\cdots+b_6$. By Lemma 3, $g(G) = g(H) = 2a$. By Lemma 4, the maximum number of cycles of order 2a in H is eight. But G contains 10 cycles of order 2a. This is a contradiction by Lemma 3.

Case 3 $H \in g_e(\theta(b_1, b_2, b_3, b_4, b_5), C_{b_6+1})$, where $2 \le b_1 \le b_2 \le b_3 \le b_4 \le b_5$, $2 \le b_6$ and $5a+b = b_1+b_2+\cdots+b_6$. We have to consider two subcases:

Subcase 3.1 $b_6 + 1 = 2a$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 10 cycles of order 2a, H must have 10 cycles of the same order also. Therefore $b_i + b_j = 2a$, for $1 \le i < j \le 5$ and $(i, j) \ne (4, 5)$. Since $b_1 + b_i = 2a$ for i=2,3,4,5, we have $b_2 = b_3 = b_4 = b_5$. Since $b_2+b_3 = 2a$, $b_2 = b_3 = a$. Hence we have $b_i = a$, for each $i = 1, 2, ..., 5$. There are 11 cycles of size 2a in H and only 10 cycles of size 2a in G. This is a contradiction by Lemma 3.
Subcase 3.2 $b_6 + 1 \ne 2a$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 10 cycles of order 2a, H must have 10 cycles of the same order also. Therefore $b_i + b_j = 2a$, for $1 \le i < j \le 5$. Since $b_1 + b_i = 2a$, for $i = 2, 3, 4, 5$, we have $b_2 = b_3 = b_4 = b_5$. Since $b_2+b_3 = 2a$, we have $b_2 = b_3 = a$. Hence we have $b_i = a$, for each $i = 1, 2, ..., 5$. But $5a+b = b_1+b_2+\cdots+b_6$, give us $b_6 = b$. By Theorem 5, $Q(G, x) = Q(H, x)$. By using Theorems 4 and 6 and after cancel the same terms we get $x = 1$, which is impossible. ∎

Lemma 6 *The 6-bridge graph $\theta(a, a, a, a, b, b)$, $2 \le a \le b$ is χ−unique.*

Proof. Let $G = \theta(a, a, a, a, b, b)$ for $2 \le a \le b$. By Theorems 1 and 3, we can assume $2 \le a \le 4$ and $b \ge 2a$. Hence the number of cycles of order 2a in G is six. By Lemmas 1 and 2, we need only to consider three cases.

Case 1 $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3$, $2 \le b_4, b_5, b_6$ and $4a+2b = b_1+b_2+\cdots+b_6$. By Lemma 3, $g(G) = g(H) = 2a$. By Lemma 4, the maximum number of cycles of order 2a in H is six. This means $b_1 = b_2 = b_3 = a$ and $b_4 = b_5 = b_6 = 2a-1$. Now since $e(G) = e(H)$, we have $2b = 5a - 3$. Since b is a positive integer and $a \le 4$, we have $a = 3$. Hence $b = 6$, and we have

$$Q(G,x) = x + x^2 + x^3 - 3x^4 - 3x^5 - 4x^6 + 4x^8 + 4x^9 + 2x^{10} + 4x^{11} + 8x^{12} - 10x^{13} - 11x^{14} + x^{15} + 4x^{16} - 4x^{17} + 6x^{19} - x^{24}$$

$$Q(H,x) = x + x^2 - 3x^4 - 3x^6 - 3x^7 + 10x^9 + 3x^{11} + 3x^{12} - 12x^{14} - x^{16} - x^{17} + 6x^{19} - x^{24}$$

Clearly, $Q(G,x) \neq Q(H,x)$ which contradicts Theorem 5

Case 2 $H \in g_e(\theta(b_1,b_2,b_3,b_4), C_{b_5+1}, C_{b_6+1})$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4$, $2 \leq b_5, b_6$ and $4a + 2b = b_1 + b_2 + \cdots + b_6$. We consider three subcases.
Subcase 2.1 $a = 2$. By Lemma 3, $g(G) = g(H) = 4$ and G and H have the same number of cycles of order 4. Therefore $b_1 = b_2 = b_3 = b_4 = 2$, $b_5, b_6 \geq 4, b_5 + b_6 = 2b$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$Q_1(G,x) = 2x^6 + 2x^7 - 3x^8 + x^9 + 6x^{4+b} - 2x^{2+b} + 6x^{3+b} - 12x^{6+b} - 4x^{5+b} + 8x^{7+b} - 2x^{1+b}$$

$$Q_1(H,x) = -x^4 - x^{b_5+2} + 3x^{3+b_6} - x^{2+b_6} - 6x^{5+b_6} + 3x^5 + 3x^{3+b_5} + x^{b_5+8} - x^{1+b_6} - x^{1+b_5} + 4x^{4+b_6} - 6x^{5+b_5} + 4x^{4+b_5} + x^{b_6+8}$$

The term $-x^4$ in $Q_1(H,x)$ can not be cancelled in $Q_1(H,x)$. It must be cancelled in $Q_1(G,x)$. Since $b \geq 4$, the term $-x^4$ can not be cancelled in $Q_1(G,x)$ also. So the term $-x^4$ found in $Q_1(H,x)$ but it is not in $Q_1(G,x)$. Therefore this is not possible.
Subcase 2.2 $a = 3$. By Lemma 3, $g(G) = g(H) = 6$ and G and H have the same number of cycles of order 6. Therefore $b_1 = b_2 = b_3 = b_4 = 3$, and $b_5, b_6 \geq 6$, $b_5 + b_6 = 2b$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$Q_1(G,x) = -4x^6 + 6x^8 + 6x^9 - 4x^{10} - 4x^{11} + x^{13} - 2x^{2+b} - 2x^{1+b} - 12x^{7+b} + 8x^{6+b} - 12x^{8+b} + 8x^{10+b} - 2x^{3+b} + 8x^{5+b} + 6x^{4+b}$$

$$Q_1(H,x) = -x^4 - x^5 + 4x^7 - x^{12} - x^{2+b_6} - x^{1+b_6} + x^{12+b_5} - x^{3+b_5} - x^{3+b_6} + x^{12+b_6} - 6x^{7+b_6} + 4x^{5+b_6} + 4x^{4+b_6} + 4x^{5+b_5} - 6x^{7+b_5} + 4x^{4+b_5} - x^{2+b_5} - x^{1+b_5}$$

The term $-x^4$ in $Q_1(H,x)$ can not be cancelled in $Q_1(H,x)$. It must be cancelled in $Q_1(G,x)$. Since $b \geq 6$, the term $-x^4$ can not be cancelled in

$Q_1(G,x)$ also. So this term found in $Q_1(H,x)$ but it is not in $Q_1(G,x)$. Thus $Q_1(G,x) \neq Q_1(H,x)$ a contradiction by Theorem 5.

Subcase 2.3 $a = 4$. By Lemma 3, $g(G) = g(H) = 8$ and G and H have the same number of cycles of order 8. Therefore $b_1 = b_2 = b_3 = b_4 = 4$, $b_5, b_6 \geq 8$, $b_5 + b_6 = 2b$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$\begin{aligned} Q_1(G,x) =& x^4 - 4\,x^7 - 4\,x^8 + 6\,x^{10} + 6\,x^{11} - 4\,x^{13} - 4\,x^{14} + x^{17} + 8\,x^{6+b} \\ &+ 8\,x^{13+b} + 8\,x^{7+b} + 8\,x^{5+b} - 12\,x^{10+b} - 12\,x^{9+b} - 2\,x^{3+b} \\ &- 2\,x^{1+b} - 2\,x^{4+b} - 2\,x^{2+b} \end{aligned}$$

$$\begin{aligned} Q_1(H,x) =& - x^5 - x^{16} - x^{1+b_6} - x^{3+b_6} + x^{16+b_5} + 4\,x^{6+b_6} - 6\,x^{9+b_5} \\ &+ x^{16+b_6} + 4\,x^{5+b_5} + 4\,x^{5+b_6} + 4\,x^{6+b_5} - x^{2+b_5} - 6\,x^{9+b_6} \\ &- x^{1+b_5} - x^{2+b_6} - x^{3+b_5} \end{aligned}$$

Since $b \geq 8$, the term x^4 can not be cancelled in $Q_1(G,x)$. It must be cancelled in $Q_1(H,x)$. Also, since $b_5, b_6 \geq 8$, this term can not be cancelled in $Q_1(H,x)$. Thus $Q_1(G,x) \neq Q_1(H,x)$ a contradiction by Theorem 5.

Case 3 $H \in g_e(\theta(b_1, b_2, b_3, b_4, b_5), C_{b_6+1})$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$, $3 \leq b_6$ and $4a + 2b = b_1 + b_2 + \cdots + b_6$.

Claim. $b_6 \neq 2a - 1$. Let $b_6 = 2a - 1$. We have $g(G) = g(H) = 2a$. Since we have six cycles of order 2a in G, by Lemma 3 we have also the same number of cycles of order 2a in H. If $b_1 = b_2 = b_3 = a$, then the number of cycles of order 2a is four. If $b_1 = b_2 = b_3 = b_4 = a$, then the number of cycles of order 2a is seven. In Both cases, G and H have different numbers of shortest cycles. This is a contradiction by Lemma 3 and the claim is proved.

Now we need to consider three subcases:

Subcase 3.1 $a = 2$. We have $g(G) = g(H) = 4$. By Lemma 4, H must have six cycles of order 4. Since $b_6 \neq 3$ by the above Claim, we have $b_1 = b_2 = b_3 = b_4 = 2$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$\begin{aligned} Q_1(G,x) =& x^5 + 6\,x^{4+b} - 4\,x^8 + x^9 + 6\,x^7 + 8\,x^{7+b} - 2\,x^{2+b} + 6\,x^{3+b} \\ &- 4\,x^{5+b} - 2\,x^{1+b} - 12\,x^{6+b} \end{aligned}$$

$$Q_1(H,x) = 4x^6 - x^{2+b_5} + 3x^{3+b_5} + x^{8+b_5} - x^{1+b_5} - x^{b_6+1} + 3x^{b_6+3} + 3x^{b_6+4} - x^{b_6+2} + 4x^{b_6+7} - 6x^{5+b_5} - 2x^{b_6+5} + 4x^{4+b_5} - 6x^{b_6+6}$$

Note that $b_5 \geq 3$ (since $b_5 \geq b_4 = 2$ and $b_5 \neq 2$ because if $b_5 = 2$, then the number of shortest cycles of order 4 in G is six but this number is 10 in H which is a contradiction by Lemma 3) and $b_6 \geq 4$ (since $b_6 + 1 \geq 2a = 4$ and $b_6 \neq 2a-1 = 3$ by our claim). Since $b_5+b_6 = 2b$ and $b_5 \geq 3$, $b_6 \geq 4$, we have $b \geq 4$. The term x^5 found in $Q_1(G,x)$ but it is not found in $Q_1(H,x)$. To cancel this term, we must have $b = 4$. Since $b_5 + b_6 = 8$ and $b_5 \geq 3$, $b_6 \geq 4$, we have either (i) $b_5 = 3$ and $b_6 = 5$ or (ii) $b_5 = b_6 = 4$.

Subcase 3.1.1 $b_5 = 3$ and $b_6 = 5$. After cancelling the same term we obtain $Q_2(G,x) = Q_2(H,x)$ where

$$Q_2(G,x) = 9x^7 + 5x^8 - 10x^{10} + 13x^{11}$$

$$Q_2(H,x) = -x^4 + 8x^6 + 6x^9 + 4x^{12}$$

Clearly, $Q_2(G,x) \neq Q_2(H,x)$ which contradicts Theorem 5

Subcase 3.1.2 $b_5 = b_6 = 4$. After cancelling the same term we obtain $Q_3(G,x) = Q_3(H,x)$ where

$$Q_3(G,x) = 6x^7 - 6x^{10} + 4x^{11}$$

$$Q_3(H,x) = -x^5 + 4x^6 + 5x^8 - 5x^9 + x^{12}$$

Clearly, $Q_3(G,x) \neq Q_3(H,x)$ which contradicts Theorem 5

Subcase 3.2 $a = 3$. We have $g(G) = g(H) = 6$. By Lemma 3, H must have six cycles of order 6. Since $b_6 \neq 5$ by the above Claim, $b_1 = b_2 = b_3 = b_4 = 3$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$Q_1(G,x) = 6x^9 - 4x^{11} + x^{13} - 2x^{3+b} + 8x^{10+b} + 8x^{6+b} - 12x^{8+b} - 12x^{7+b} + 8x^{5+b} - 2x^{1+b} + 6x^{4+b} - 2x^{2+b}$$

$$Q_1(H,x) = -x^5 + 3x^{b_6+4} - 6x^{7+b_5} - 6x^{b_6+7} - x^{b_6+2} - x^{1+b_5} - x^{b_6+3} - x^{3+b_5} - x^{2+b_5} + 4x^{5+b_5} + 4x^7 - 6x^{b_6+8} + 4x^{b_6+6} + x^{12+b_5} + 4x^{b_6+10} + 4x^{4+b_5} + 4x^{b_6+5} - x^{b_6+1}$$

Note that $b_5 \geq 4$ (since $b_5 \geq b_4 = 3$ and $b_5 \neq 3$ because if $b_5 = 3$, then the number of shortest cycles of order 6 in G is six but this number is 10 in H which is a contradiction by Lemma 3) and $b_6 \geq 6$ (since $b_6 + 1 \geq 2a = 6$ and $b_6 \neq 2a - 1 = 5$ by our claim). Since $b_5 + b_6 = 2b$ and $b_5 \geq 4$, $b_6 \geq 6$, we have $b \geq 5$. Since $b_5 \geq 4$, $b_6 \geq 6$, the term $-x^5$ in $Q_1(H,x)$ can not be cancelled in $Q_1(H,x)$. It must be cancelled in $Q_1(G,x)$. Also, since $b \geq 5$, this term can not be cancelled in $Q_1(G,x)$. Thus $Q_1(G,x) \neq Q_1(H,x)$ a contradiction by Theorem 5.

Subcase 3.3 $a = 4$. We have $g(G) = g(H) = 8$. By Lemma 4, H must have six cycles of order 8. Since $b_6 \neq 7$ by the above Claim, $b_1 = b_2 = b_3 = b_4 = 4$. Since $G \sim H$, $Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$\begin{aligned} Q_1(G,x) = & -4x^8 + 6x^{11} - 4x^{14} + x^{17} - 2x^{1+b} - 2x^{3+b} - 2x^{4+b} - 2x^{2+b} \\ & + 8x^{6+b} - 12x^{10+b} + 8x^{7+b} - 12x^{9+b} + 8x^{13+b} + 8x^{5+b} \\ Q_1(H,x) = & -x^5 + 4x^{5+b_5} + 4x^{b_6+5} - 6x^{b_6+9} - x^{b_6+4} - x^{2+b_5} - 6x^{b_6+10} \\ & + 4x^{b_6+6} - x^{b_6+1} - x^{1+b_5} + x^{16+b_5} - x^{3+b_5} - 6x^{9+b_5} + 4x^{b_6+7} \\ & + 4x^{6+b_5} - x^{b_6+2} - x^{b_6+3} + 4x^{b_6+13} \end{aligned}$$

Note that $b_5 \geq 5$ (since $b_5 \geq b_4 = 4$ and $b_5 \neq 4$ because if $b_5 = 4$, then the number of shortest cycles of order 8 in G is six but this number is 10 in H which is a contradiction by Lemma 3) and $b_6 \geq 8$ (since $b_6 + 1 \geq 2a = 8$ and $b_6 \neq 2a - 1 = 7$ by our claim). Since $b_5 + b_6 = 2b$ and $b_5 \geq 5$, $b_6 \geq 8$, we have $b \geq 7$. Since $b_5 \geq 5$, $b_6 \geq 8$, the term $-x^5$ in $Q_1(H,x)$ can not be cancelled in $Q_1(H,x)$. It must be cancelled in $Q_1(G,x)$. Also, since $b \geq 7$, this term can not be cancelled in $Q_1(G,x)$. Thus $Q_1(G,x) \neq Q_1(H,x)$ a contradiction by Theorem 5. Hence G is $\chi-$unique ■

Lemma 7 *The 6-bridge graph $\theta(a,a,a,b,b,b)$, $2 \leq a \leq b$ is $\chi-$unique.*

Proof. The proof is similar to the proof of Lemma 6. ■

Lemma 8 *The 6-bridge graph $\theta(a,a,b,b,b,b)$, $2 \leq a \leq b$ is $\chi-$unique.*

Proof. The proof is similar to the proof of Lemma 6. ■

Lemma 9 *The 6-bridge graph $\theta(a,b,b,b,b,b)$, $2 \leq a \leq b$ is $\chi-$unique.*

Proof. Since $b < a + b$, from Theorem 3, the graph $\theta(a,b,b,b,b,b)$, $2 \leq a \leq b$ is $\chi-$unique. ■

By Lemmas 5 to 9, we have the following theorem.

Theorem 7 *A 6-bridge graph $\theta(a_1, a_2, ..., a_6)$ is $\chi-$unique if the positive integers $a_1, a_2, ..., a_6$ assume exactly two distinct values.*

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