A note on the eigenvalues of graphs *

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Abstract

In this note, we present some upper bounds for the kth largest eigenvalue of the adjacency matrix as well as the Laplacian matrix of graphs. Special attention is paid to the Laplacian matrix of trees.

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1 Introduction

In this note, we only consider undirected simple graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with n vertices and m edges. The degree of a vertex $v \in V$ is denoted by d_v, and the degree sequence of G is written in non-increasing order by $d_1 \geq d_2 \geq \cdots \geq d_n$. The adjacency matrix of G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if two vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The degree diagonal matrix of G is denoted by $D(G) = diag(d_1, d_2, \cdots, d_n)$, then $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The eigenvalues $\lambda_i(G), \mu_i(G)$, or just $\lambda_i, \mu_i, 1 \leq i \leq n$, of $A(G), L(G)$ are ordered by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

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$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,$$

respectively. For any $n \times n$ matrix M, we also use $\lambda_i(M)$ to denote its eigenvalues.

For the adjacency matrix $A(G)$ of a connected graph G, from the Perron-Frobenius theorem, there is a unique positive eigenvector corresponding to λ_1 whose entries sum to 1, we call this vector the *Perron vector.* For more results on graph theory, spectral radius of graphs, Laplacian eigenvalues of graphs, we refer the reader to see [1], [3], [12] and the references therein.

Up to now, many researchers studied the kth largest eigenvalues of $A(G)$, see [2], [8], [14], [15], [18] for details. But for the kth largest eigenvalue of $L(G)$, there are only few results. Guo [7] and Zhang, Li [17] studied μ_2 for trees. Zhang and Li [16] investigated μ_k for arbitrary graphs. In this note, we present a new bound for λ_k and give some applications further.

2 Lemmas and results

Lemma 2.1 *[6] Let T be a tree and L_T be its line graph, then*

$$\mu_k(T) = 2 + \lambda_k(L_T), \quad for \quad k = 1, 2, \cdots, n.$$

Lemma 2.2 *[14] Let G be a graph on n vertices, then*

$$\lambda_2(G) \leq \begin{cases} \frac{n-3}{2}, & if\ n\ is\ odd; \\ \frac{n}{2} - 1, & if\ n\ is\ even. \end{cases}$$

By using Lemma 2.2, Zhang [17] obtained an upper bound for μ_2 of a tree.

Theorem 2.3 *[17] Let T be a tree of order n, then*

$$\mu_2(T) \leq \lceil \frac{n}{2} \rceil.$$

The bound is sharp for n is even.

Lemma 2.4 *[15] Let G be a connected graph on n vertices, then*

$$\lambda_k(G) \leq \lceil \frac{n}{k} \rceil - 1 \quad for \quad k \leq \frac{n}{2}.$$

Theorem 2.5 *Let T be a tree of order n, then*

$$\mu_k(T) \leq \lceil \frac{n-1}{k} \rceil + 1 \quad for \quad k \leq \frac{n-1}{2}.$$

Proof. Consider the line graph of T, by Lemma 2.1 and Lemma 2.4, we can get the result. ■

Lemma 2.6 *(Courant-Fischer) For a real symmetric $n \times n$ matrix M with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we have*

$$\lambda_k = \min_{w_1, w_2, \cdots, w_{n-k} \in R^n} \max_{\substack{x \perp w_1, w_2, \cdots, w_{n-k}; \\ x \neq 0}} \frac{x^t M x}{x^t x}.$$

Next, we present the main result of this paper.

Theorem 2.7 *Let G be a simple graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\lambda_k(G) \leq d_k$. The equality holds if and only if G has at least k connected components, and each of which is d_k regular.*

Proof. The proof the theorem borrows ideas from the proof of the well known Weyl's inequality (see, for example [10]). We present the details for completeness. For any $k = 1, 2, \cdots, n-1$, by Lemma 2.6, we have

$$\begin{aligned}
\lambda_k(D) &= \lambda_k(D - A + A) \\
&= \min_{w_1, w_2, \cdots, w_{n-k} \in R^n} \max_{\substack{x \perp w_1, w_2, \cdots, w_{n-k}; \\ x \neq 0}} \frac{x^t (D - A + A) x}{x^t x} \\
&= \min_{w_1, w_2, \cdots, w_{n-k} \in R^n} \max_{\substack{x \perp w_1, w_2, \cdots, w_{n-k}; \\ x \neq 0}} \left(\frac{x^t A x}{x^t x} + \frac{x^t (D - A) x}{x^t x} \right) \\
&\geq \min_{w_1, w_2, \cdots, w_{n-k} \in R^n} \max_{\substack{x \perp w_1, w_2, \cdots, w_{n-k}; \\ x \neq 0}} \left(\frac{x^t A x}{x^t x} + \lambda_n(D - A) \right) \\
&= \lambda_k(A) + \lambda_n(D - A)
\end{aligned}$$

It is well known that $\lambda_n(D - A) = 0$, so we can get the result.

If the equality holds, that is $\lambda_k(G) = d_k$, then the inequality in the above should be equality. So the all one vector $\mathbf{1}$ as the eigenvector of $\lambda_n(D - A) = \mu_n(L(G)) = 0$ is also the eigenvector of

$\lambda_k(D(G))$ and $\lambda_k(A(G))$ for $1 \leq k \leq n-1$. So from $A(G)\mathbf{1} = d_k\mathbf{1}$, G must be $d = d_k$ regular.

If $k = 1$, then it is well known that G is $d = d_1$ regular.

Now, we consider the case $k \geq 2$. Note that for a symmetric real matrix, the eigenvectors of different eigenvalues are orthogonal. If G is connected, from the Perron-Frobenius theorem, we know the Perron vector is positive, which can not be orthogonal to $\mathbf{1}$, a contradiction. Hence G must be disconnected. If G has l connected components each of which is d regular and $l < k$, then the eigenvector of the $(l+1)$th largest eigenvalue must have negative entries. Since $l + 1 \leq k$, the eigenvector corresponding to λ_k must have negative entries since it must be orthogonal to $\mathbf{1}$, this contradicts to the fact that $\mathbf{1}$ is the eigenvector of λ_k. Hence G has at least k components and each of which is d regular. Conversely, it is easy to check the result holds. ■

Corollary 2.8 *Let G be a simple graph of order n with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\lambda_1(G) \leq d_1$, $\lambda_2(G) \leq d_2$.*

In the following, for a graph G, when we say the *degree of an edge* $e = uv \in E(G)$, we mean it is the degree sum of the endpoints of e.

Lemma 2.9 *[11] Let G be a connected graph, then the line graph L_G of G is regular if and only if G is regular or semiregular bipartite.*

Proof. We present the proof here for completeness. Suppose L_G is regular, then for any edge $e = uv \in E(G)$, the degree of e in L_G is $d_u + d_v - 2$. For any two adjacent edges $e_1 = uv$, $e_2 = uw$ in $E(G)$, we have $d_u + d_v - 2 = d_u + d_w - 2$. Hence $d_v = d_w$. If G contains an odd cycle, since G is connected, so all vertices in G have the same degree. If G does not contain any odd cycle, then G is bipartite and the vertices in the same partition set have equal degree. So we get the result. Conversely, if G is regular or semiregular bipartite, it is easy to see L_G is regular. ■

Next, we present some applications of Theorem 2.7.

Theorem 2.10 *Let T be a forest, then*

$$\mu_k(T) \leq d_u + d_v,$$

where $uv \in E(T)$ *and* $d_u + d_v$ *is the* k*th largest degree of edges of* T*. The equality holds if and only if* T *has at least* k *components each of which is a star on* $d_u + d_v - 1$ *vertices.*

Proof. Consider its line graph L_T of T, for an edge $e = uv \in E(T)$, the degree of e as a vertex in L_T is $d_u + d_v - 2$. Together with Lemma 2.1 and Theorem 2.7, we get the result. The equality case follows from Theorem 2.7 and Lemma 2.9, since the star is the only connected semiregular bipartite graph for trees. ■

Corollary 2.11 *Let* T *be a tree of order* n*, then*

$$\mu_1(T) \leq \max_{uv \in E(T)} \{d_u + d_v\}.$$

As another application of Theorem 2.7, we consider the following problem ([5], [13]): *Does it hold for every graph that* $\lambda_1 + \lambda_2 \leq n$*?*

From Theorem 2.7, we have

Theorem 2.12 *Let* G *be a graph of order* n *with degree sequence* $d_1 \geq d_2 \geq \cdots \geq d_n$*, then* $\lambda_1 + \lambda_2 \leq d_1 + d_2$*.*

At last, we present a rough upper bound about the above poblem.

Theorem 2.13 *Let* G *be a graph of order* n*, then* $\lambda_1 + \lambda_2 \leq \frac{3}{\sqrt{6}} n \approx 1.2247n$*.*

Proof. If $\lambda_2 \leq 0$, then the result is obvious, so we can assume that $\lambda_2 > 0$. By the result $\lambda_1 \leq \sqrt{2m - n + 1}$ (see [9]), we have $\lambda_1 \leq \sqrt{2m - n + 1} \leq \sqrt{2m}$. On the other hand, Weyl's inequalities imply that $\lambda_2(G) + \lambda_n(\bar{G}) \leq -1$. So $\lambda_2(G)^2 < \lambda_n(\overline{G})^2$, where $\overline{G}$ is the complement of G. Hence, by the result $\lambda_i \leq \sqrt{m}$, for $2 \leq i \leq n$, (see [4], page 206, Corollary 2.14), we have

$$\lambda_2(G)^2 < \lambda_n(\overline{G})^2 \leq \frac{n(n-1)}{2} - m < \frac{n^2}{2} - m.$$

Therefore

$$\lambda_1 + \lambda_2 \leq \sqrt{2m} + \sqrt{\frac{n^2}{2} - m}.$$

Let $f(x) = \sqrt{2x} + \sqrt{\frac{n^2}{2} - x}$, a simple calculation shows that $f(x)$ attains its maximum when $x = \frac{n^2}{3}$, and the maximum value is $\frac{3}{\sqrt{6}}n$, which implies the result. ■

Note. In [13], the author obtained an upper bound $\frac{2}{\sqrt{3}}n$. Unfortunately, its proof is wrong, since he said that "the value $\sqrt{2m - x^2} + x$ is increasing in x for $x \leq m$", which should be $x \leq \sqrt{m}$.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, New York, Macmilan Press, 1976.

[2] R.C. Brigham, R.D. Dutton, Bounds on graph spectra, J. Comb. Theory B **37** (1984) 228-234.

[3] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[4] O. Favaron, M. Mahéo, and J.-F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti - II), Discrete Math. **111** (1993) 197-220.

[5] D. Gernert, personal communication.

[6] C.D. Godsil, I. Gutman, Wiener index, graph spectrum, line graph, ACH Models Chem. **136** (1999) 503-510.

[7] J.-M. Guo, On the second largest Laplacian eigenvalue of trees, Linear Algebra Appl. **404** (2005) 251-261.

[8] J.-M. Guo, A note on the kth eigenvalues of trees, Linear Algebra Appl. **413** (2006) 148-154.

[9] Y. Hong, Bounds on spectral radius of graphs, Linear Algebra Appl. **108** (1988) 135-139.

[10] R.A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[11] Y. Hou, J. Li, Y. Pan, On the Laplacian eigenvalues of signed graphs, Linear and Multilinear Algebra **51** (2003) 21-30.

[12] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. **197-198** (1994) 143-176.

[13] V. Nikiforov, Linear combinations of graph eigenvalues, Electron. J. Linear Algebra **15** (2006) 329-336.

[14] D.L. Powers, Graph partitioning by eigenvectors, Linear Algebra Appl. **101** (1988) 121-133.

[15] D.L. Powers, Bounds on graph eigenvalues, Linear Algebra Appl. **117** (1989) 1-6.

[16] X.D. Zhang, J.S. Li, On the kth largest eigenvalue of the Laplacian matrix of a graph, Acta. Math. Appl. Sinica. **17** (2001) no.2 1-8.

[17] X.D. Zhang, J.S. Li, The two largest eigenvalue of Laplacian matrices of trees, J.China Univ. of Science and Tech. **28** (1998) no.5 513-518, (in Chinese).

[18] H. Yuan, The kth largest eigenvalues of a tree, Linear Algebra Appl. **73** (1986) 151-155.