Real zeros and the signless r-associated Stirling numbers of the first kind

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Abstract

The signless r-associated Stirling numbers of the first kind $d_r(n,k)$ counts the number of permutations of the set $\{1,2,\ldots,n\}$ that have exactly k cycles, each of which is of length greater than or equal to r, where r is a fixed positive integer. F. Brenti obtained that the generating polynomials of the numbers $d_r(n,k)$ have only real zeros. Here we consider the location of zeros of these polynomials.

Keywords: Stirling numbers; Real zeros

1 Introduction

Let $[n]$ denote the set $\{1,2,\ldots,n\}$, where n is a positive integer. Let $\mathcal{S}_n$ be the symmetric group on $[n]$ and $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. As usual, we denote by $\operatorname{cyc}(\pi)$ the number of *cycles* of π. For example, the permutation $\pi = 315426 \in S_6$ has the cycle decomposition $\pi = (1,3,5,2)(4)(6)$, so $\operatorname{cyc}(\pi) = 3$. A *fixed point* in π is an index i such that $\pi(i) = i$. We say that π is a *derangement* if π has no fixed points. For a positive integer r, let $D_r(n,k)$ denote the set of all $\pi \in \mathcal{S}_n$ that have exactly k cycles, each of which is of length greater than or equal to r. The *signless r-associated Stirling numbers of the first kind* $d_r(n,k)$ counts the number of

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permutations in $D_r(n,k)$. The numbers $d_r(n,k)$ are often called $r-1$-*derangements* (see [4, Section 3.2] for details). The set $D_r(n+1,k)$ may be seen as a disjoint union of the sets P_1 and P_2, where P_1 represents the subset of $D_r(n+1,k)$ in which the element $n+1$ belongs to a cycle with more than r elements, and P_2 represents the subset of $D_r(n+1,k)$ in which the element $n+1$ belongs to a cycle with exactly r elements. Hence the numbers $d_r(n,k)$ satisfy the recurrence

$$d_r(n+1,k) = nd_r(n,k) + (n)_{r-1}d_r(n-r+1,k-1) \quad \text{for } n \geq kr, \qquad (1)$$

with the initial conditions $d_r(n,k) = 0$ if $n < kr$ and $d_r(kr,k) = (kr)!/(k!r^k)$, where $(n)_{r-1}$ is the falling factorial, i.e., $(n)_0 = 1$ and $(n)_{r-1} = n(n-1)\cdots(n-r+2)$ for $r \geq 2$.

Polynomials with only real zeros arise often in combinatorics, algebra, analysis, geometry, probability and statistics (see Stanley [8] for details). Let RZ denote the set of real polynomials with only real zeros. Furthermore, denote by RZ(I) the set of such polynomials all of whose zeros are in the interval I.

Let

$$D_n^{(r)}(x) = \sum_{k\geq 0} d_r(n,k)x^k \quad \text{for } n \geq 0. \qquad (2)$$

It is evident that $D_n^{(1)}(1) = nD_n^{(n)}(1) = n!$. Note that $d_1(n,k)$ is just the signless Stirling numbers of the first kind $c(n,k)$. It is well-known [9, Proposition 1.3.4] that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

Hence the polynomial $D_n^{(1)}(x) \in \text{RZ}[1-n,0]$. Moreover, $D_n^{(1)}(x)$ has only simple real zeros and the zeros of $D_n^{(1)}(x)$ and $D_{n+1}^{(1)}(x)$ are interlaced. In the case of $r=2$, E. R. Canfield [3] obtained that the polynomials $D_n^{(2)}(x) \in \text{RZ}(-\infty,0]$.

Remark 1. *According to Comtet [5, p. 295], it follows from the results of Tricomi [10] that the polynomials $D_n^{(2)}(x)$ have only real nonpositive simple zeros.*

Using the theory of total positivity [7, Chap. 8, Corollary 3.1], F. Brenti [2, Corollary 5.9] proved that the polynomial $D_n^{(r)}(x) \in \mathrm{RZ}(-\infty, 0]$. Recently, M. Bóna [1] showed the following result.

Theorem 2. *[1, Theorem 3.6] For every positive integer t and every $\epsilon > 0$, there exists a positive integer N such that if $n > N$, then $D_n^{(r)}(x)$ has a zero x_t satisfying the inequality $\mid t + x_t \mid < \epsilon$.*

The object of this note is to provide a characterization for the location of zeros of the polynomials $D_n^{(r)}(x)$.

2 Real Zeros

Using (1), the polynomials $D_n^{(r)}(x)$ satisfy the recurrence

$$D_{n+1}^{(r)}(x) = nD_n^{(r)}(x) + (n)_{r-1}xD_{n-r+1}^{(r)}(x) \quad \text{for } n \geq r. \tag{3}$$

By the *pigeon-hole principle*, we have $\deg(D_n^{(r)}(x)) = \lfloor n/r \rfloor$ and $x = 0$ is a simple zero of the polynomials $D_n^{(r)}(x)$ for $n \geq r$. Let $\operatorname{sgn}(c)$ denote the sign function of a real number c. We can now present the main result of this note.

Theorem 3. *The polynomials $D_n^{(r)}(x)$ have only real nonpositive simple zeros. Moreover, the zeros of $D_n^{(r)}(x)$ and $D_{n+1}^{(r)}(x)$ are interlacing in the following way. Let $n = qr+i$ and let $D_{q,i}(x) = D_n^{(r)}(x)$, where $0 \leq i \leq r-1$. Let $z_{q,i;1} < z_{q,i;2} < \cdots < z_{q,i;q} = 0$ be the zeros of $D_{q,i}(x)$. Then*

(a) $z_{q,i;j} < z_{q,i+1;j}$ *for* $0 \leq i \leq r-2$ *and* $1 \leq j \leq q-1$;

(b) $z_{q,i;j} < z_{q-1,i;j}$ *for* $0 \leq i \leq r-1$ *and* $1 \leq j \leq q-1$;

(c) $z_{q-1,r-1;j} < z_{q,0;j+1}$ *for* $1 \leq j \leq q-2$.

Proof. By definition (2) we have $D_{0,0}(x) = 1$ and $D_{0,i}(x) = 0$ for $1 \leq i \leq r-1$. Hence we may assume that $q \geq 1$. It follows from (3) that

$$D_{q,0}(x) = a_{q,0}D_{q-1,r-1}(x) + b_{q,0}xD_{q-1,0}(x), \tag{4}$$

$$D_{q,i}(x) = a_{q,i}D_{q,i-1}(x) + b_{q,i}xD_{q-1,i}(x) \quad \text{for } 1 \leq i \leq r-1, \tag{5}$$

where $a_{q,i} = qr + i - 1$ and $b_{q,i} = (qr + i - 1)_{r-1}$. We proceed by induction on q. It follows from (4) and (5) that $D_{1,i}(x) = \prod_{k=1}^{i} a_{1,k} b_{1,0} x$. So it suffices to consider the $q \geq 2$ case.

First consider the $q = 2$ case. We will divide the proof into three steps.

Step 1 : If $i = 0$, then $D_{2,0}(x) = \prod_{k=1}^{r-1} a_{1,k} a_{2,0} b_{1,0} x + b_{2,0} b_{1,0} x^2$. Note that $a_{q,i} > 0$ and $b_{q,i} > 0$. We get $z_{2,0;1} < z_{1,0;1} = 0$.

Step 2 : If $i = 1$, then $D_{2,1}(x) = a_{2,1} D_{2,0}(x) + a_{1,1} b_{2,1} b_{1,0} x^2$. Note that $\operatorname{sgn}(D_{2,1}(z_{2,0;1})) = +1$. By Weierstrass Intermediate Value Theorem, we have $z_{2,0;1} < z_{2,1;1} < z_{1,1;1} = 0$.

Step 3 : If $1 \leq i \leq r - 2$, then

$$D_{2,i+1}(x) = a_{2,i+1} D_{2,i}(x) + \prod_{k=1}^{i+1} a_{1,k} b_{2,i+1} b_{1,0} x^2.$$

Assume now that $z_{2,i-1;1} < z_{2,i;1} < z_{1,i;1} = 0$. Note that

$$\operatorname{sgn}(D_{2,i+1}(z_{2,i;1})) = +1.$$

Therefore $D_{2,i+1}(x)$ has precisely one zero in the interval $(z_{2,i;1}, 0)$, i.e., $z_{2,i;1} < z_{2,i+1;1} < z_{1,i+1;1} = 0$. Hence the result holds for $q = 2$. In the same way, it is not difficult to verify that the result holds for $D_{3,i}(x)$, i.e., $z_{2,r-1;1} < z_{3,0;2}$, $z_{3,i;j} < z_{3,i+1;j}$ and $z_{3,i;j} < z_{2,i;j}$ for $1 \leq j \leq 2$.

Next assume that the result holds for all positive integers up to q, and let us prove the result holds for $q + 1$. We also divide our proof into three steps.

Step 4 : If $i = 0$, then $D_{q+1,0}(x) = a_{q+1,0} D_{q,r-1}(x) + b_{q+1,0} x D_{q,0}(x)$. By the assumption, we have $z_{q,r-1;j} < z_{q-1,r-1;j} < z_{q,0;j+1}$ for $1 \leq j \leq q - 2$. Note that

$$\operatorname{sgn}\ (D_{q+1,0}(z_{q,r-1;j})) = (-1)^{q+j+1},$$
$$\operatorname{sgn}\ (D_{q+1,0}(z_{q,0;j+1})) = (-1)^{q+j}.$$

Therefore $D_{q+1,0}(x)$ has a zero in each of $q - 2$ intervals $(z_{q,r-1;j}, z_{q,0;j+1})$. Also, $\operatorname{sgn}(D_{q+1,0}(z_{q,0;1})) = (-1)^q$ and $\operatorname{sgn}(D_{q+1,0}(-\infty)) = (-1)^{q+1}$. Thus $D_{q+1,0}(x)$ has a zero in the interval $(-\infty, z_{q,0;1})$. Moreover, $D_{q+1,0}(x)$ has a zero in the interval $(z_{q,r-1;q-1}, 0)$ since $\operatorname{sgn}(D_{q+1,0}(z_{q,r-1;q-1})) = +1$. Hence $D_{q+1,0}(x)$ has only simple real zeros. In particular,

$$z_{q+1,0;1} < z_{q,0;1},\ z_{q,r-1;j} < z_{q+1,0;j+1} < z_{q,0;j+1} \quad \text{for } 1 \leq j \leq q - 1. \quad (6)$$

Step 5 : If $i = 1$, then $D_{q+1,1}(x) = a_{q+1,1}D_{q+1,0}(x) + b_{q+1,1}xD_{q,1}(x)$. For $1 \le j \le q-1$, it follows from (6) that sgn $(D_{q+1,1}(z_{q+1,0;j})) = (-1)^{q+j}$ and sgn $(D_{q+1,1}(z_{q,1;j})) = (-1)^{q+j+1}$. Also, sgn $(D_{q+1,1}(z_{q+1,0;q})) = +1$. Thus $D_{q+1,1}(x)$ has precisely one zero in each of q intervals $(z_{q+1,0;j}, z_{q,1;j})$ for $1 \le j \le q$. Hence the result holds for $D_{q+1,1}(x)$.

Step 6 : If $1 \le i \le r-2$, then $D_{q+1,i+1}(x) = a_{q+1,i+1}D_{q+1,i}(x) + b_{q+1,i+1}xD_{q,i+1}(x)$. Assume that

$$z_{q+1,i;j} < z_{q,i;j} < z_{q,i+1;j}, z_{q+1,0;j} < z_{q+1,1;j} < \cdots < z_{q+1,i;j} \quad \text{for } 1 \le j \le q.$$

We have

$$\text{sgn } (D_{q+1,i+1}(z_{q+1,i;j})) = (-1)^{q+j},$$

$$\text{sgn } (D_{q+1,i+1}(z_{q,i+1;j})) = (-1)^{q+j+1}.$$

Thus $D_{q+1,i+1}(x)$ has precisely one zero in each of q intervals $(z_{q+1,i;j}, z_{q,i+1;j})$. Hence $z_{q+1,i;j} < z_{q+1;i+1,j} < z_{q,i+1;j}$ and $z_{q,r-1;j} < z_{q+1,0;j+1} < z_{q,0;j+1}$. This completes the proof. □

Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial with $\deg(f) \ge 2$. If all the zeros of $f(x)$ are real, a result due to Newton [6] implies that the coefficients of $f(x)$ satisfy the following concavity condition:

$$a_k^2 \ge a_{k-1}a_{k+1}\frac{k+1}{k}\frac{n-k+1}{n-k} \quad \text{for } 1 \le k \le n-1. \tag{7}$$

If the zeros of $f(x)$ are not all equal, these inequalities are strict. By Theorem 3 and the Newton's inequality (7), we have the following corollary.

Corollary 4. *Let* $n = qr + i$, *where* $0 \le i \le r-1$. *Then*

$$d_r(n,k)^2 > d_r(n,k-1)d_r(n,k+1)\frac{k+1}{k}\frac{q-k+1}{q-k} \quad \textit{for } 1 \le k \le q-1.$$

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