Kite-designs intersecting in pairwise disjoint blocks

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Abstract

A kite-design of order n is a decomposition of the complete graph K_n into kites. Such systems exist precisely when $n \equiv 0, 1 \pmod 8$. Two kite systems $(X, \mathcal{K}_1)$ and $(X, \mathcal{K}_2)$ are said to intersect in m pairwise disjoint blocks if $|\mathcal{K}_1 \cap \mathcal{K}_2| = m$ and all blocks in $\mathcal{K}_1 \cap \mathcal{K}_2$ are pairwise disjoint. In this paper we determine all the possible values of m such that there are two kite-designs of order n intersecting in m pairwise disjoint blocks, for all $n \equiv 0, 1 \pmod 8$.

1 Introduction

A kite is a triangle with a tail consisting of a single edge. In what follows we will denote the following kite by $(a, b, c; d)$ or $(b, a, c; d)$ (See Figure 1).

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A kite-design of order n more usually known as a kite system is a pair $(X, \mathcal{K})$, where X is the vertex set of the complete graph K_n and $\mathcal{K}$ is an edge-disjoint decomposition of K_n into copies of kites. Following design terminology, we call these copies as blocks. Such systems exist precisely when $n \equiv 0, 1 \pmod 8$ [1]. One can even find such designs that are 2-colorable [4]. Necessary and sufficient conditions have also been found for embedding a complete kite system of order n in a complete kite system of order t [5].

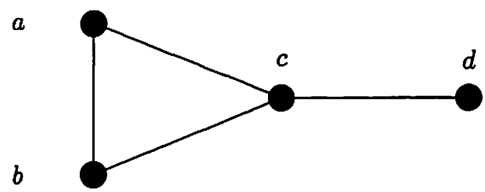

Figure 1: Kite $(a, b, c; d)$

Two kite-designs $(X, \mathcal{K}_1)$ and $(X, \mathcal{K}_2)$ are said to intersect in m blocks if $|\mathcal{K}_1 \cap \mathcal{K}_2| = m$. The intersection problem for kite-designs was completely solved by Billington and Kreher [2], showing that the intersection numbers for kite-design of order n are given by the set $\{0, 1, 2, \ldots, b-2, b\}$, where $b = n(n-1)/8$. If in addition, the blocks in $\mathcal{K}_1 \cap \mathcal{K}_2$ are pairwise disjoint (that is, for any $K, K' \in \mathcal{K}_1 \cap \mathcal{K}_2$, $K \neq K'$, we have $V(K) \cap V(K') = \emptyset$), then $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to intersect in m pairwise disjoint blocks. In 1975, a complete solution to the intersection problem for triple systems was given by C. C. Lindner and A. Rosa[7]. In 2004, Y. M. Chee gave the solution of the disjoint intersection problem for triple systems [3]. The purpose of this paper is to solve the disjoint intersection problem for kite-designs. In what follows we let $I(n)$ denote the set of integer k for which there exists two kite-designs of order n, $(X, \mathcal{K}_1)$ and $(X, \mathcal{K}_2)$, such that $\mathcal{K}_1$ and $\mathcal{K}_2$ are intersecting in k pairwise disjoint blocks. For $n \equiv 0, 1 \pmod 8$, let $J(n) = \{0, 1, 2, \ldots, \lfloor n/4 \rfloor\}$. In other words, $J(n)$ denotes the intersection numbers one expects to achieve with a kite-design of order n. From the results of Billington and Kreher [2], we have

Lemma 1.1 *[2] $0, 1 \in I(n)$, for all $n \equiv 0, 1$ (mod 8).*

We also modify this notation slightly and let $I(H)$ and $J(H)$ denote respectively the achievable and expected intersection numbers for kite-design of the graph H. Subsequently we shall need to decompose certain tripartite graphs into edge-disjoint copies of kites. Consider the following example.

Example 1.2 *[2]* $0 \in I(G)$, *where* $G = K_{2,2,2}$, $K_{4,4,4}$ *or* $K_{8,8,8}$.

For $K_{2,2,2}$, take the vertex sets $\{1,1'\} \cup \{2,2'\} \cup \{3,3'\}$. Then disjoint kite-decompositions are given by $\{(1,3,2;1'),(3,2',1';3'),(1,2',3';2)\}$ and $\{(1',3',2';1),(3',2,1;3),(1',2,3;2')\}$. Thus $0 \in I(K_{2,2,2})$. Now let the vertex set for $K_{8,8,8}$ be $\{A_1,A_2,A_3,A_4\} \cup \{B_1,B_2,B_3,B_4\} \cup \{C_1,C_2,C_3,C_4\}$, where each letter here is a set of two points. Let L be a latin square of order 4. Then we may take sixteen decompositions on the sixteen set $\{A_i,B_j,C_{L_{i,j}} \mid 1 \le i \le 4, 1 \le j \le 4\}$, yielding 48 blocks for a kite-design of $K_{8,8,8}$. Using $0 \in I(K_{2,2,2})$, we have $0 \in I(K_{8,8,8})$. Similar to $K_{4,4,4}$. ⋄

2 Small Cases

In this section, we give the solutions for the disjoint-blocks intersection problems for kite-design of order 8, 9, 16, 17, 24, 32, 40 and for graphs $K_{24} \setminus K_8$, $K_{40} \setminus K_8$, followed by an example which is necessary for the construction in next section.

Example 2.1 $I(8) = \{0,1\}$.

From Lemma 1.1, we have $\{0,1\} \subseteq I(8)$. Suppose $2 \in I(8)$. There is a kite-design of K_8 containing two kites of the form $(1,2,3;4),(5,6,7;8)$. The remaining 5 kites must come from the edges of $K_{4,4}$ and $\{\{1,4\},\{2,4\},\{5,8\},\{6,8\}\}$. However, since a kite contains a triangle, it cannot construct 5 triangles from the above edges. ⋄

Example 2.2 $I(9) = J(9)$.

Let $\mathcal{K} = \{(1,2,3;4),(5,6,7;8),(4,8,9;6),(2,5,8;1),(1,9,7;3),(3,8,6;1),(1,5,4;6),(4,7,2;6),(3,5,9;2)\}$. Then $\mathcal{K} \cap \pi\mathcal{K} = \{(1,2,3;4),(5,6,7;8)\}$, where $\pi = (1,2)(5,6)$. From the result and Lemma 1.1, we obtain $I(9) = J(9)$. ⋄

Example 2.3 $I(17) = J(17)$.

Let $\mathcal{K}$ = {(1, 2, 3; 4), (5, 6, 7; 8), (9, 10, 11; 12), (13, 14, 15; 16), (1, 5, 9; 4), (2, 10, 16; 14), (3, 8, 11; 14), (4, 5, 15; 1), (16, 12, 5; 3), (8, 1, 17; 14), (1, 13, 16; 11), (14, 10, 4; 1), (7, 10, 17; 11), (4, 7, 12; 9), (8, 14, 5; 2), (10, 13, 5; 17), (3, 10, 15; 17), (9, 14, 2; 6), (2, 11, 15; 7), (3, 12, 6; 15), (11, 13, 4; 16), (10, 1, 12; 15), (13, 6, 17; 12), (9, 15, 8; 4), (3, 16, 17; 9), (3, 14, 7; 9), (6, 9, 16; 7), (17, 2, 4; 6), (7, 2, 13; 8), (10, 6, 8; 16), (1, 14, 6; 11), (1, 7, 11; 5), (2, 8, 12; 14), (3, 9, 13; 12)}. Then

$$
\begin{aligned}
\mathcal{K}\cap\pi_1\mathcal{K} &= \{(1,2,3;4),(5,6,7;8),(9,10,11;12),(13,14,15;16)\}\\
\mathcal{K}\cap\pi_2\mathcal{K} &= \{(1,2,3;4),(5,6,7;8),(9,10,11;12)\}\\
\mathcal{K}\cap\pi_3\mathcal{K} &= \{(1,2,3;4),(5,6,7;8)\},
\end{aligned}
$$

where $\pi_1 = (1,2)(5,6)(9,10)(13,14)$, $\pi_2 = \pi_1(15,16)$ and $\pi_3 = \pi_2(11,12)$.◇

Example 2.4 $I(16) = J(16)$.

Let $\mathcal{K}$ = {(1, 2, 3; 4), (5, 6, 7; 8), (9, 10, 11; 12), (13, 14, 15; 16), (1, 5, 9; 12), (2, 10, 16; 7), (3, 8, 11; 14), (4, 5, 15; 7), (16, 12, 5; 11), (1, 13, 16; 4), (14, 10, 4; 9), (4, 7, 12; 13), (5, 8, 14; 12), (10, 13, 5; 3), (3, 10, 15; 6), (2, 9, 14; 16), (2, 11, 15; 1), (12, 6, 3; 16), (13, 4, 11; 6), (12, 10, 1; 4), (9, 15, 8; 13), (3, 14, 7; 10), (6, 9, 16; 8), (13, 7, 2; 5), (10, 6, 8; 1), (1, 14, 6; 13), (1, 7, 11; 16), (2, 8, 12; 15), (13, 3, 9; 7), (6, 2, 4; 8)}.Then

$$
\begin{aligned}
\mathcal{K}\cap\pi_1\mathcal{K} &= \{(1,2,3;4),(5,6,7;8),(9,10,11;12),(13,14,15;16)\}\\
\mathcal{K}\cap\pi_2\mathcal{K} &= \{(1,2,3;4),(5,6,7;8),(9,10,11;12)\}\\
\mathcal{K}\cap\pi_3\mathcal{K} &= \{(1,2,3;4),(5,6,7;8)\},
\end{aligned}
$$

where $\pi_1 = (1,2)(5,6)(9,10)(13,14)$, $\pi_2 = \pi_1(15,16)$ and $\pi_3 = \pi_2(11,12)$.◇

Example 2.5 $I(32) = J(32)$.

Let $\mathcal{K}$ = {(1, 2, 3; 4), (5, 6, 7; 8), (9, 10, 11; 12), (13, 14, 15; 16), (17, 18, 19; 20), (21, 22, 23; 24), (25, 26, 27; 28), (29, 30, 31; 32), (1, 4, 8; 9), (1, 6, 16; 4), (1, 9, 15; 27), (2, 5, 9; 28), (2, 7, 17; 4), (2, 10, 16; 29), (6, 10, 3; 15), (3, 8, 18; 31), (3, 11, 17; 8), (4, 7, 11; 30), (4, 9, 19; 28), (4, 12, 18; 2), (5, 8, 12; 28), (5, 10, 20; 7), (5, 13, 19; 8), (6, 9, 13; 25), (6, 11, 21; 30), (6, 14, 20; 4), (7, 10, 14; 1), (7, 12, 22; 2), (21, 7, 15; 28), (8, 11, 15; 2), (8, 13, 23; 32), (8, 16, 22; 6), (9, 12, 16; 17), (9, 14, 24; 25), (9, 17, 23; 7), (10, 13, 17; 26), (10, 15, 25; 9), (10, 18, 24; 22), (11, 14, 18; 30), (11, 16, 26; 7), (11, 19, 25; 12), (15, 19, 12; 13), (12, 17, 27; 7), (12, 20, 26; 6), (13, 16, 20; 8), (13, 18, 28; 8), (13, 21, 27; 4), (14, 17, 21; 5), (14, 19, 29; 8), (22, 28, 14; 27), (18, 22, 15; 4), (15, 20, 30; 32), (15, 23, 29; 9), (23, 16, 19; 6), (16, 21, 31; 15), (16, 24, 30; 28), (17, 20, 24; 3), (17, 22, 32; 20), (17, 25, 31; 8), (25, 18, 21; 19), (18, 23, 1; 13), (18, 26, 32; 11), (19, 22, 26; 28), (19, 24, 2; 14), (19, 27, 1; 31), (23, 27, 20; 21), (25, 3, 20; 31), (28, 2, 20; 29), (21, 24, 28; 7), (4, 21, 26; 10), (3, 21, 29; 10), (25, 29, 22; 31), (22, 27, 5; 14), (4, 22, 30; 17), (23, 26, 30; 7), (23, 28, 6; 8), (23, 31, 5; 17), (24, 27, 31; 10), (24, 29, 7; 9), (24, 32, 6; 27), (25, 28, 32; 2), (8, 25, 30; 9), (25, 1, 7; 19), (1, 26, 29; 13), (9, 26, 31; 12), (26, 2, 8; 24), (30, 2, 27; 16), (10, 27, 32; 16), (27, 3, 9; 22), (28, 31, 3; 22), (28, 1, 11; 20), (28, 4, 10; 22), (4, 29, 32; 19), (29, 2, 12; 23), (29, 5, 11; 27), (30, 1, 5; 18), (30, 3, 13; 32), (30, 6, 12; 24), (31, 2, 6; 18), (31, 4, 14; 16), (7, 13, 31; 19), (3, 7, 32; 1), (32, 5, 15; 26), (32, 8, 14; 26), (12, 1, 10; 23), (4, 13, 2; 21), (26, 3, 5; 28), (4, 5, 24; 11), (6, 15, 17; 1), (18, 7, 16; 28), (8, 10, 21; 1), (18, 20, 9; 32), (30, 10, 19; 3), (13, 22, 11; 31), (32, 12, 21; 9), (26, 13, 24; 15), (14, 23, 25; 2), (17, 28, 29; 6), (29, 18, 27; 8), (20, 22, 1; 24), (2, 11, 23; 3), (3, 12, 14; 30), (6, 25, 4; 23), (5, 25, 16; 3)}. Then

$$
\begin{array}{rcl}
\mathcal{K}\cap\pi_1\mathcal{K} & = & \{(4i+1,4i+2,4i+3;4i+4)\mid i=0,1,\ldots,7\} \\
\mathcal{K}\cap\pi_2\mathcal{K} & = & \{(4i+1,4i+2,4i+3;4i+4)\mid i=0,1,\ldots,6\} \\
\mathcal{K}\cap\pi_3\mathcal{K} & = & \{(4i+1,4i+2,4i+3;4i+4)\mid i=0,1,\ldots,5\} \\
\mathcal{K}\cap\pi_4\mathcal{K} & = & \{(4i+1,4i+2,4i+3;4i+4)\mid i=0,1,\ldots,4\} \\
\mathcal{K}\cap\pi_5\mathcal{K} & = & \{(4i+1,4i+2,4i+3;4i+4)\mid i=0,1,2,3\} \\
\mathcal{K}\cap\pi_6\mathcal{K} & = & \{(1,2,3;4),(5,6,7;8),(9,10,11;12)\} \\
\mathcal{K}\cap\pi_7\mathcal{K} & = & \{(1,2,3;4),(5,6,7;8)\} \\
\mathcal{K}\cap\pi_8\mathcal{K} & = & \{(1,2,3;4)\} \\
\mathcal{K}\cap\pi_9\mathcal{K} & = & \emptyset,
\end{array}
$$

where $\pi_1 = \prod_{i=0}^{7}(4i+1,4i+2)$, $\pi_2 = \pi_1(31,32)$, $\pi_3 = \pi_2(27,28)$, $\pi_4 = \pi_3(23,24)$, $\pi_5 = \pi_4(19,20)$, $\pi_6 = \pi_5(15,16)$, $\pi_7 = \pi_6(11,12)$, $\pi_8 = \pi_7(7,8)$ and $\pi_9 = \pi_8(3,4)$. Therefore, we have $I(32) = J(32)$. ◇

Example 2.6 $I(K_{24}\setminus K_8) = J(K_{24}\setminus K_8)$.

Let $\mathcal{K}$ = {(9, 10, 11; 12), (13, 14, 15; 16), (17, 18, 19; 20), (21, 22, 23; 24), (16, 1, 10; 7), (2, 11, 14; 1), (3, 11, 17; 1), (10, 12, 15; 3), (4, 12, 18; 7), (5, 13, 16; 24), (6, 13, 19; 1), (12, 14, 17; 24), (14, 20, 7; 15), (8, 15, 18; 10), (1, 15, 21; 3), (14, 16, 19; 10), (2, 16, 22; 4), (15, 17, 20; 1), (16, 17, 23; 3), (18, 21, 16; 20), (1, 18, 24; 5), (3, 19, 22; 17), (4, 19, 9; 2), (18, 20, 23; 5), (8, 20, 10; 5), (19, 21, 24; 4), (20, 21, 11; 8), (9, 20, 22; 24), (1, 22, 12; 5), (2, 23, 10; 22), (7, 23, 13; 22), (11, 6, 24; 12), (24, 14, 8; 23), (23, 9, 12; 3), (24, 9, 15; 11), (24, 10, 13; 12), (20, 3, 24; 2), (1, 9, 13; 11), (17, 2, 13; 8), (3, 10, 14; 22), (4, 15, 23; 14), (5, 9, 14; 4), (6, 10, 17; 4), (9, 17, 7; 22), (8, 12, 19; 23), (23, 1, 11; 22), (2, 12, 20; 5), (19, 2, 15; 6), (3, 13, 18; 2), (3, 9, 16; 8), (4, 10, 21; 2), (4, 11, 16; 7), (4, 13, 20; 6), (5, 11, 18; 22), (5, 15, 22; 8), (21, 5, 17; 8), (6, 9, 18; 14), (12, 16, 6; 23), (14, 21, 6; 22), (7, 11, 19; 5), (12, 21, 7; 24), (8, 9, 21; 13)}. Then

$$\begin{aligned}
\mathcal{K} \cap \pi_1\mathcal{K} &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2,3,4,5\} \\
\mathcal{K} \cap \pi_2\mathcal{K} &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2,3,4\} \\
\mathcal{K} \cap \pi_3\mathcal{K} &= \{(9,10,11;12), (13,14,15;16)\} \\
\mathcal{K} \cap \pi_4\mathcal{K} &= \{(13,14,15;16)\} \\
\mathcal{K} \cap \pi_5\mathcal{K} &= \emptyset,
\end{aligned}$$

where $\pi_1 = (1,2,3,4,5,6,7,8)(9,10)(13,14)(17,18)(21,22)$, $\pi_2 = \pi_1(23,24)$, $\pi_3 = \pi_2(19,20)$, $\pi_4 = \pi_3(11,12)$ and $\pi_5 = \pi_3(11,15)(12,16)$.◇

Example 2.7 $I(24) = J(24)$.

Combining the two kite-designs of K_8 and $K_{24} \setminus K_8$, we obtained a kite-design of order 24. From Example 2.1 and Example 2.6, we have $\{0,1,\ldots,5\} \subseteq I(24)$. Let $\mathcal{K}$ = {(1, 2, 3; 4), (5, 6, 7; 8), (9, 10, 11; 12), (13, 14, 15; 16), (17, 18, 19; 20), (21, 22, 23; 24), (10, 16, 1; 5), (14, 2, 11; 8), (11, 17, 3; 8), (10, 12, 15; 7), (4, 12, 18; 7), (16, 5, 13; 8), (6, 13, 19; 10), (14, 17, 12; 3), (7, 14, 20; 5), (15, 18, 8; 22), (15, 21, 1; 8), (14, 16, 19; 5), (2, 16, 22; 4), (15, 17, 20; 6), (17, 23, 16; 20), (21, 16, 18; 2), (1, 18, 24; 12), (3, 19, 22; 6), (4, 19, 9; 2), (18, 20, 23; 14), (8, 20, 10; 18), (24, 19, 21; 2), (20, 21, 11; 15), (9, 20, 22; 18), (12, 1, 22; 7), (2, 23, 10; 22), (7, 23, 13; 11), (6, 24, 11; 22), (8, 24, 14; 1), (23, 9, 12; 13), (24, 9, 15; 3), (24, 10, 13; 22), (24, 20, 3; 5), (1, 9, 13; 21), (13, 17, 2; 4), (14, 3, 10; 7), (15, 23, 4; 14), (9, 14, 5; 10), (10, 17, 6; 8), (7, 9, 17; 24), (8, 12, 19; 1), (11, 23, 1; 7), (2, 12, 20; 1), (19, 2, 15; 6), (13, 18, 3; 7), (3, 9, 16; 8), (10, 21, 4; 8), (11, 16, 4; 6), (13, 20, 4; 5), (11, 18, 5; 12), (5, 15, 22; 14), (21, 5, 17; 8), (6, 9, 18; 14), (6, 12, 16; 7), (14, 21, 6; 2), (11, 19, 7; 2), (7, 12, 21; 3), (9, 21, 8; 2), (1, 4, 17; 22), (2, 5, 24; 16), (23, 3, 6; 1), (4, 7, 24; 22), (5, 8, 23; 19)}. Then

$$\mathcal{K} \cap \pi\mathcal{K} = \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 0,1,\ldots,5\},$$

where $\pi = \prod_{i=0}^{5}(4i+1, 4i+2)$. Thus, we obtained $I(24) = J(24)$.◇

Example 2.8 $I(K_{40} \setminus K_8) = J(K_{40} \setminus K_8)$ *and* $I(40) = J(40)$.

Let $A = \{$(9, 10, 11; 12), (13, 14, 15; 16), (17, 18, 19; 20), (21, 22, 23; 24), (25, 26, 27; 28), (29, 30, 31; 32), (33, 34, 35; 36), (37, 38, 39; 40), (1, 14, 24; 38), (1, 15, 25; 5), (1, 16, 18; 32), (1, 20, 22; 29), (1, 21, 28; 19), (1, 36, 39; 2), (2, 16, 26; 36), (2, 17, 24; 36), (2, 19, 25; 32), (2, 20, 30; 36), (3, 11, 21; 5), (3, 12, 26; 1), (3, 13, 20; 40), (3, 14, 28; 40), (3, 36, 40; 2), (4, 11, 33; 5), (4, 12, 28; 6), (4, 13, 29; 8), (4, 17, 27; 8), (5, 11, 27; 36), (5, 14, 30; 32), (5, 15, 37; 1), (5, 16, 19; 40), (5, 22, 32; 15), (5, 39, 12; 20), (6, 10, 24; 8), (6, 12, 30; 37), (6, 13, 27; 33), (6, 15, 20; 31), (6, 17, 31; 15), (6, 19, 37; 3), (6, 21, 39; 16), (6, 22, 36; 12), (7, 9, 27; 37), (7, 10, 28; 5), (7, 11, 29; 5), (7, 13, 31; 1), (7, 22, 24; 11), (8, 11, 16; 27), (8, 12, 34; 36), (8, 17, 39; 28), (8, 18, 20; 11), (8, 22, 28; 11), (9, 12, 21; 31), (9, 13, 24; 32), (9, 31, 8; 38), (10, 13, 22; 38), (10, 15, 23; 28), (10, 17, 35; 1), (11, 1, 40; 5), (11, 14, 23; 40), (11, 15, 26; 4), (11, 18, 25; 4), (12, 16, 38; 40), (12, 22, 2; 38), (13, 16, 25; 39), (13, 17, 28; 36), (13, 18, 26; 6), (14, 17, 26; 5), (14, 18, 29; 6), (15, 18, 27; 1), (15, 19, 30; 1), (15, 40, 9; 35), (16, 3, 31; 2), (16, 17, 23; 1), (16, 21, 29; 36), (17, 20, 29; 1), (17, 21, 32; 40), (17, 22, 30; 3), (18, 21, 30; 7), (18, 22, 33; 40), (18, 23, 31; 36), (18, 24, 3; 22), (18, 34, 4; 10), (19, 1, 9; 39), (19, 12, 13; 35), (19, 22, 31; 33), (19, 23, 34; 1), (19, 27, 14; 39), (19, 39, 10; 2), (20, 21, 27; 29), (20, 23, 32; 4), (20, 25, 33; 1), (20, 34, 5; 31), (20, 38, 4; 40), (21, 7, 15; 28), (21, 24, 33; 26), (21, 25, 36; 7), (21, 26, 34; 6), (21, 35, 4; 22), (21, 37, 2; 33), (21, 40, 13; 39), (22, 9, 14; 12), (22, 25, 34; 3), (22, 27, 35; 5), (22, 39, 11; 32), (22, 40, 16; 24), (23, 2, 13; 38), (23, 4, 36; 2), (23, 5, 9; 2), (23, 25, 3; 27), (23, 26, 35; 2), (23, 27, 38; 1), (23, 29, 39; 4), (23, 30, 8; 10), (23, 37, 7; 25), (24, 25, 31; 4), (24, 26, 40; 31), (24, 27, 34; 2), (24, 28, 30; 40), (24, 29, 37; 8), (25, 6, 9; 3), (25, 10, 14; 21), (25, 28, 37; 4), (25, 29, 40; 6), (25, 30, 38; 6), (25, 35, 8; 40), (26, 29, 38; 5), (26, 30, 9; 16), (26, 31, 39; 7), (26, 32, 8; 15), (27, 31, 10; 12), (28, 2, 18; 12), (28, 29, 35; 37), (28, 31, 38; 7), (28, 32, 34; 40), (28, 33, 9; 4), (29, 2, 15; 17), (29, 3, 19; 36), (29, 32, 9; 34), (29, 33, 12; 32), (29, 34, 10; 40), (30, 4, 16; 28), (30, 33, 10; 16), (30, 34, 13; 8), (30, 35, 11; 6), (30, 39, 27; 12), (31, 34, 11; 2), (31, 37, 12; 23), (32, 2, 27; 40), (32, 6, 16; 35), (32, 7, 14; 2), (32, 33, 39; 3), (32, 36, 38; 3), (32, 37, 13; 5), (33, 3, 15; 22), (33, 7, 17; 11), (33, 8, 19; 32), (33, 36, 13; 1), (33, 37, 16; 20), (33, 38, 14; 16), (34, 7, 16; 36), (34, 37, 14; 20), (34, 38, 17; 5), (34, 39, 15; 4), (35, 7, 19; 26), (35, 38, 15; 36), (35, 39, 18; 6), (36, 9, 17; 3), (36, 10, 20; 39), (36, 37, 11; 13), (37, 9, 20; 28), (37, 10, 18; 7), (37, 22, 26; 28), (37, 40, 17; 1), (38, 9, 18; 40), (38, 10, 21; 8), (38, 11, 19; 21)}, $B = \{(3, 32, 35; 40), (4, 19, 24; 39), (7, 20, 26; 10), (7, 40, 12; 35), (12, 15, 24; 5), (17, 25, 12; 1), (18, 36, 5; 10), (20, 24, 35; 6), (23, 33, 6; 14), (31, 35, 14; 4), (32, 1, 10; 3), (36, 8, 14; 40)\}$ and $C = \{(1,2,3;4), (5,6,7;8), (1,4,7;2), (1,6,8;4), (3,32,35;6), (4,5,2;8), (4, 19, 24; 5), (7, 40, 12; 1), (10, 3, 5; 1), (12, 15, 24; 39), (14, 4, 6; 3), (14, 36, 8; 3), (17, 25, 12; 35), (18, 36, 5; 8), (20, 24, 35; 40), (20, 26, 7; 3), (23, 33, 6; 2), (31, 35, 14; 40), (32, 1, 10; 26)\}$. Let $\mathcal{K}_1 = A \cup B$ and $\mathcal{K}_2 = A \cup C$, then $\mathcal{K}_1$ is a kite-design of $K_{40} \setminus K_8$ and $\mathcal{K}_2$ is a kite-design of K_{40}. Thus we achieve the following intersection values.

$$
\begin{aligned}
\mathcal{K}_1 \cap \pi_1\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, \ldots, 9\} \\
\mathcal{K}_1 \cap \pi_2\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, \ldots, 8\} \\
\mathcal{K}_1 \cap \pi_3\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, \ldots, 7\} \\
\mathcal{K}_1 \cap \pi_4\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, \ldots, 6\} \\
\mathcal{K}_1 \cap \pi_5\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, 4, 5\} \\
\mathcal{K}_1 \cap \pi_6\mathcal{K}_1 &= \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 2, 3, 4\} \\
\mathcal{K}_1 \cap \pi_7\mathcal{K}_1 &= \{(9, 10, 11; 12), (13, 14, 15; 16)\} \\
\mathcal{K}_1 \cap \pi_8\mathcal{K}_1 &= \{(13, 14, 15; 16)\} \\
\mathcal{K}_1 \cap \pi_9\mathcal{K}_1 &= \emptyset,
\end{aligned}
$$

where $\pi_1 = (1,2,3,4,5,6,7,8)\prod_{i=2}^{9}(4i+1, 4i+2)$, $\pi_2 = \pi_1(39, 40)$, $\pi_3 = \pi_2(35, 36)$, $\pi_4 = \pi_3(31, 32)$, $\pi_5 = \pi_4(27, 28)$, $\pi_6 = \pi_5(23, 24)$, $\pi_7 = \pi_6(19, 20)$, $\pi_8 = \pi_7(15, 16)$ and $\pi_9 = \pi_8(11, 12)$. Therefore, we have $I(K_{40} \setminus K_8) = J(K_{40} \setminus K_8)$.

Combining the two kite-designs of K_8 and $K_{40} \setminus K_8$, we obtained a kite-design of order 40. From Example 2.1 and above results, we have $\{0, 1, \ldots, 9\} \subseteq I(40)$. Since $\mathcal{K}_2 \cap \pi\mathcal{K}_2 = \{(4i+1, 4i+2, 4i+3; 4i+4) \mid i = 0, 1, \ldots, 9\}$, where $\pi = \prod_{i=0}^{9}(4i+1, 4i+2)$, we have $I(40) = J(40)$.◇

3 General Construction

Take $2n \geq 6$ and $X = \{1, 2, \ldots, 2n\}$. Let $\mathcal{H}$ be a partition of X in sets of size 2 if $2n \equiv 0, 2 \pmod 6$ and of size 2 and 4 with one set of size 4 if $2n \equiv 4 \pmod 6$. The sets in $\mathcal{H}$ are called holes. Let $(X, \mathcal{H}, \mathcal{B})$ be a group divisible design (GDD) with groups $\mathcal{H}$ and blocks $\mathcal{B}$ of size 3 (see [6]).

$(8n+1)$-Construction

Let $V = \{\infty\} \cup (X \times \{1,2,3,4\})$ and define a collection $\mathcal{K}$ of kites on V as follows:

1. If $\mathcal{H}$ contains a hole h of size 4, take a kite-design of order 17 on $\{\infty\} \cup (h \times \{1,2,3,4\})$ and put these kites in $\mathcal{K}$.
2. For each hole h of size 2, take a kite-design of order 9 on $\{\infty\} \cup (h \times \{1,2,3,4\})$ and put these kites in $\mathcal{K}$.
3. For each block $\{a,b,c\}$ of the GDD, take a kite-design of $K_{4,4,4}$ on $\{(a,j) \mid 1 \le j \le 4\} \cup \{(b,j) \mid 1 \le j \le 4\} \cup \{(c,j) \mid 1 \le j \le 4\}$ and put these kites in $\mathcal{K}$.

$16n$-Construction

Let $V = X \times \{1,2,\dots,8\}$ and define a collection $\mathcal{K}$ of kites on V as follows:

1. If $\mathcal{H}$ contains a hole h of size 4, take a kite-design of order 32 on $h \times \{1,2,\dots,8\}$ and put these kites in $\mathcal{K}$.
2. For each hole h of size 2, take a kite-design of order 16 on $h \times \{1,2,\dots,8\}$ and put these kites in $\mathcal{K}$.
3. For each block $\{a,b,c\}$ of the GDD, take a kite-design of $K_{8,8,8}$ on $\{(a,j) \mid 1 \le j \le 8\} \cup \{(b,j) \mid 1 \le j \le 8\} \cup \{(c,j) \mid 1 \le j \le 8\}$ and put these kites in $\mathcal{K}$.

$(16n+8)$-Construction

Let $V = \{\infty_i \mid i = 1,2,\dots,8\} \cup (X \times \{1,2,\dots,8\})$ and define a collection $\mathcal{K}$ of kites on V as follows:

1. Take a hole h' of size 2 and a kite-design of order 24 on $\{\infty_i \mid i = 1,2,\dots,8\} \cup (h' \times \{1,2,\dots,8\})$ and put these kites in $\mathcal{K}$.
2. If $\mathcal{H}$ contains a hole h of size 4, take a kite-design of $K_{40} \backslash K_8$ on $\{\infty_i \mid i = 1,2,\dots,8\} \cup (h \times \{1,2,\dots,8\})$ with hole on $\{\infty_i \mid i = 1,2,\dots,8\}$ and put these kites in $\mathcal{K}$.

3. For each hole h of size 2, $h \neq h'$, take a kite-design of $K_{24} \setminus K_8$ on $\{\infty_i \mid i = 1, 2, \ldots, 8\} \cup (h \times \{1, 2, \ldots, 8\})$ with hole on $\{\infty_i \mid i = 1, 2, \ldots, 8\}$ and put these kites in $\mathcal{K}$.
4. For each block $\{a, b, c\}$ of the GDD, take a kite-design of $K_{8,8,8}$ on $\{(a, j) \mid 1 \leq j \leq 8\} \cup \{(b, j) \mid 1 \leq j \leq 8\} \cup \{(c, j) \mid 1 \leq j \leq 8\}$ and put these kites in $\mathcal{K}$.

4 Conclusion

Using the above constructions, Example 1.2 and the results in section 2, we obtain the following Main Theorem.

Main Theorem $I(n) = J(n)$, for $n \equiv 0, 1 \pmod 8$, $n \neq 8$ and $I(8) = \{0, 1\}$.

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