Upper Domination Number and Domination Number in a Tree

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Abstract

For a graph $G = (V(G), E(G))$, a set $S \subseteq V(G)$ is called a dominating set if $N_G[S] = V(G)$. A dominating set S is said to be minimal if no proper subset $S' \subset S$ is a dominating set. Let $\gamma(G)$ (called the domination number) and $\Gamma(G)$ (called the upper domination number) be the minimum cardinality and the maximum cardinality of a minimal dominating set of G, respectively. For a tree T of order $n \geq 2$, it is obvious that $1 = \gamma(K_{1,n-1}) \leq \gamma(T) \leq \Gamma(T) \leq \Gamma(K_{1,n-1}) = n-1$. Let $t(n) = \min_{|T|=n}(\Gamma(T) - \gamma(T))$. In this paper, we determine $t(n)$ for all natural numbers n. We also characterize trees T with $\Gamma(T) - \gamma(T) = t(n)$.

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, we refer to $V(G)$ and $E(G)$ as the vertex set and the edge set, respectively. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. The *(open) neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G, and the *close neighborhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \cup_{x \in A} N_G(x)$ and $N_G[A] = \cup_{x \in A} N_G[x]$. The *degree* $d_G(x)$ of a vertex x is the cardinality of $N_G(x)$. A vertex x is said to be a *leaf* if $d_G(x) = 1$. Two distinct vertices u and v are called *duplicated* if $N_G(u) = N_G(v)$. In a graph G, an *x-y path* is a sequence $x = v_0, v_1, \cdots, v_r = y$ of distinct vertices with $v_{i-1}v_i \in E(G)$ for $1 \leq i \leq r$; and r is called the *length* of this path. For a connected graph G, we define the *distance $d_G(x, y)$ between x and y* as the minimum length of an x-y path, and the *diameter* of a graph G is $\mathrm{diam}(G) = \max_{x,y} d(x,y)$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. Suppose that u and v are duplicated vertices in a tree T, then they are both leaves. For notation and terminology in graphs we follow [1] in general.

A *dominating set* of a graph G is a subset $S \subseteq V(G)$ such that $N_G[S] = V(G)$. A dominating set S is said to be *minimal* if no proper subset $S' \subset S$ is a dominating set. Let $\gamma(G)$ and $\Gamma(G)$ be the minimum cardinality and the maximum cardinality of a minimal dominating set of G. The numbers $\gamma(G)$ and $\Gamma(G)$ are called the *domination number* and the *upper domination number*, respectively. A minimal dominating set of cardinality $\gamma(G)$ ($\Gamma(G)$) in graph G is said to be a *γ-set* (*Γ-set*). Domination and its variations in graphs are well studied, an estimated thousand papers have been written on this topic (see [3],[4]). For a tree T of order $n \geq 2$, it is obvious that $1 = \gamma(K_{1,n-1}) \leq \gamma(T) \leq \Gamma(T) \leq \Gamma(K_{1,n-1}) = n-1$. Our aim is to find trees T such that $\Gamma(T) - \gamma(T)$ is as small as possible. For easy presentation, let $t(n) = \min_{|T|=n}(\Gamma(T) - \gamma(T))$ for $n \geq 1$. In this paper, we determine $t(n)$ for all natural numbers n. We also characterize trees T with $\Gamma(T) - \gamma(T) = t(n)$.

2 Characterization

A vertex v of G is a *support vertex* if it is adjacent to a leaf in G. We denote the set $L(G)$ the collection of all leaves of G, and the set $U(G)$ the collection of all support vertices of G. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ by removing all vertices in A and all edges incident to these vertices. For a subset $B \subseteq E(G)$, the *edge deletion of B from G* is the graph $G - B$ by removing all edges of B.

An *independent set* in a graph G is a subset $I \subseteq V(G)$ of pairwise non-adjacent vertices. A *maximal independent set* of G is an independent set I that is not a proper subset of any other independent set. The *independence number* $\alpha(G)$ of G is the maximum cardinality of a maximal independent set of G. A maximal independent set of cardinality $\alpha(G)$ in graph G is said to be an *α-set.* Let $i(G)$ be the minimum cardinality of a maximal independent set of G. Since each maximal independent set in a graph G is a minimal dominating set of G, we have the following lemma.

Lemma 1 ([3]) *For any graph G, we have $\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$.*

For a graph G, let $\widehat{G}$ be the graph with vertex set $V(\widehat{G}) = V(G) \cup \{\hat{x} : x \in V(G)\}$ and edge set $E(\widehat{G}) = E(G) \cup \{x\hat{x} : x \in V(G)\}$. And let $\widehat{G}_{+1}$ be the graph with vertex set $V(\widehat{G}_{+1}) = V(\widehat{G}) \cup \{x'\}$ and edge set $E(\widehat{G}_{+1}) = E(\widehat{G}) \cup \{xx'\}$ for some vertex $x \in V(G)$. Notice that there are exactly two duplicated leaves $\hat{x}$ and x' in the graph $\widehat{G}_{+1}$.

Theorem 1 ([5]) *If T is a tree of order $n \geq 2$, then $\gamma(T) = \alpha(T)$ if and only if $T = \widehat{G}$ for some tree G of order $n/2$.*

If $T = \widehat{G}$ for some tree G of order $n/2$, then $U(T) = \{x : x \in V(G)\}$ is a γ-set and $L(T) = \{\hat{x} : x \in V(G)\}$ is an α-set of T. Note that $\gamma(T) = |U(T)| = |L(T)| = \alpha(T) = n/2$.

The following theorem is proved by Cockayne, Favaron, Payan, and Thomason in 1981.

Theorem 2 ([2]) *If G is a bipartite graph, then $\alpha(G) = \Gamma(G)$.*

As trees are bipartite graphs, Theorem 2 gives that $\alpha(T) = \Gamma(T)$ for any tree T. This means that $t(n) = \min_{|T|=n}(\Gamma(T) - \gamma(T)) = \min_{|T|=n}(\alpha(T) - \gamma(T))$ for all trees T of order $n \geq 1$. Theorem 1 indicates that $t(n) = 0$ for all even n and $t(n) \geq 1$ for all odd $n \geq 3$. In this paper, we want to prove that $t(n) = 1$ for all odd $n \geq 3$. Additionally, we also characterize trees of odd order n such that $t(n) = 1$.

Lemma 2 *If T is a tree of order $n \geq 3$, then there exists an α-set I of T such that $L(T) \subseteq I$.*

Proof. Let I_1 be an α-set of T. If $L(T) \subseteq I_1$, then we are done. So we assume that there exists a leaf x such that $x \notin I_1$. This implies $y \in I_1$, where y is the neighbor of the leaf x in T. So $I_2 = (I_1 - \{y\}) \cup \{x\}$ is an α-set of T such that $x \in I_2$. Continue this procedure, we can get an α-set I such that $L(T) \subseteq I$. □

Lemma 3 *If T is a tree of order $n \geq 3$, then there exists a γ-set S of T such that $U(T) \subseteq S$.*

Proof. Let S_1 be a γ-set of T. If $U(T) \subseteq S_1$, then we are done. So we assume that there exists a support vertex y such that $y \notin S_1$. This implies that $x \in S_1$ for each leaf x adjacent to y. Let $S_2 = (S_1 - \{x\}) \cup \{y\}$. Then S_2 is a dominating set T with cardinality $|S_2| = |S_1| = \gamma(T)$. Thus S_2 is a γ-set with $y \in S_2$. Continue this procedure, we get a γ-set S such that $U(T) \subseteq S$. □

Lemma 4 *Let T be a tree of order $n \geq 2$. Then $\alpha(T) \geq \frac{n}{2}$ and $\gamma(T) \leq \frac{n}{2}$.*

Proof. Since T is a bipartite graph, it is possible to partition $V(T)$ into V_1 and V_2 such that every V_i is an independent set of T. Every V_i is an independent set and also a dominating set of T. These imply that $\alpha(T) = \Gamma(T) \geq \max\{|V_1|, |V_2|\} \geq \frac{n}{2}$ and $\gamma(T) \leq \min\{|V_1|, |V_2|\} \leq \frac{n}{2}$. □

Suppose that x_1 and x_2 are duplicated leaves in a tree T, and y is the common neighbor of them. Let $T' = T - \{x_2\}$. Note that T' is also a tree. And we have the following properties:

(i) $y \in S$ for each γ-set S of T;

(ii) $x_1, x_2 \in I$ for each α-set of T;

(iii) $\gamma(T) = \gamma(T')$ and $\alpha(T) = \alpha(T') + 1$.

Lemma 5 *Suppose that T is a tree of order $n \geq 3$. Let T' be a maximal subgraph of T such that T' has no duplicated leaves. Then $\alpha(T) \geq \gamma(T) + m$, where $m = |T| - |T'|$ is the number of vertices removing from T.*

Proof. Let X be the set of vertices removing from T, say $T' = T - X$ and $|X| = m$. Since $\alpha(T') \geq \gamma(T')$, by Lemmas 2 and 3, there exist an α-set I' and a γ-set S' of subtree T' with $|I'| \geq |S'|$ such that $L(T') \subseteq I'$ and $U(T') \subseteq S'$. However, S' is also a γ-set of T. So, $I = I' \cup X$ is an independent set of T because of $X \subset L(T)$. Thus

$$\alpha(T) \geq |I| = |I'| + m \geq |S'| + m = \gamma(T) + m.$$

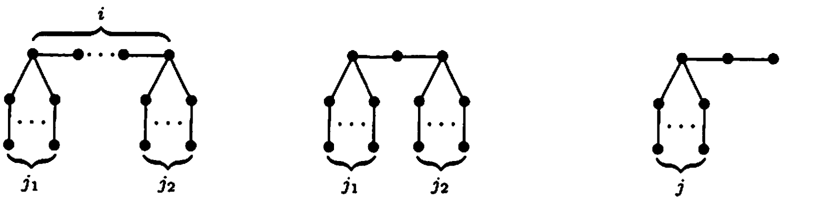
□

By Lemma 5, the number m is at most 1 whenever $\alpha(T) - \gamma(T) = 1$. That is, there is at most one pair of duplicated leaves in T whenever $\alpha(T) - \gamma(T) = 1$.

3 Main Theorem

For $i \geq 1$ and $j \geq 0$, we define a *baton* $B(i, j)$ as follows. Start with a basic path P with i vertices and attach j paths of length two to the endpoints of P; see Figure 1. Note that $B(i, j)$ is a tree with $i + 2j$ vertices.

Theorem 3 *Let T be a tree of odd order $n = 2k + 1 \geq 3$. Then $\alpha(T) = \gamma(T) + 1$ if and only if $T = \widehat{G}_{+1}$ for some tree G of order k or $T \cong B(3, k-1)$.*



$j = j_1 + j_2$ $\quad$ $j = j_1 + j_2$

$B(i, j)$ $\quad$ $B(3, j)$ $\quad$ $B(3, j) = B(1, j+1)$

Figure 1: Batons

Proof. The theorem is true for $n = 3$. And the sufficiency is clear, so we only need to prove the necessity. Suppose that T is a tree of odd order $n = 2k+1 \geq 5$ such that $\alpha(T) = \gamma(T) + 1$. Then we consider the following cases.

Case 1. T have some duplicated leaves.
By Lemma 5, then T has exactly one pair of duplicated leaves, say x_1 and x_2. Let $T' = T - \{x_2\}$. By Lemma 2 and Lemma 3, we have $\alpha(T') = \alpha(T) - 1 = (\gamma(T) + 1) - 1 = \gamma(T) = \gamma(T')$. So T' is a tree of even order $n' = 2k \geq 4$ such that $\alpha(T') = \gamma(T')$. By Theorem 1, $T' = \widehat{G}$ for some tree G of order k, this implies that $T = \widehat{G}_{+1}$ for some tree G of order k.

Case 2. T has no duplicated leaves.
Let $A = U(T) \cup L(T)$. Since T has no duplicated leaves, $|U(T)| = |L(T)| = |A|/2$. As T is odd, $T - A$ is not empty, say it has r components $T_1, T_2, \cdots, T_r$. Let $T_i' = T_i - L(T_i)$, where the degree zero vertex in a tree of one vertex is considered as a leaf. Let $n_i = |T_i|$ and $l_i = |L(T_i)|$ for $1 \leq i \leq r$. Choose a maximum independent set I_i of T_i, and a minimum dominating set D_i of T_i'. The set D_i is empty whenever $|T_i| \leq 2$. Since no vertex of T_i is adjacent to a vertex in $L(T)$, we have that $L(T) \cup \cup_i I_i$ is an independent set of T of size at least $|L(T)| + \sum_i \lceil \frac{n_i}{2} \rceil$. Since every vertex of $L(T_i)$ is adjacent to some vertex in $U(T)$, we have that $U(T) \cup_i \cup_i D_i$ is a dominating set of T of size at most $|L(T)| + \sum_i \lfloor \frac{n_i - l_i}{2} \rfloor^*$, where $\lfloor \frac{a}{2} \rfloor^* = \lfloor \frac{a}{2} \rfloor$ except that $\lfloor \frac{1}{2} \rfloor^* = 1$. Then $\lceil \frac{n_i}{2} \rceil - \lfloor \frac{n_i - l_i}{2} \rfloor^* \leq 1$ with equality if and only if $n_i \leq 3$ or n_i is even with $l_i = 2$. $\alpha(T) = \gamma(T) + 1$ then gives that $r = 1$ and so n_1 is odd, or $T' = P_1$ or P_3. This gives that T is $B(1, k) = B(3, k-1)$.
□

Theorem 4 *Let* $t(n) = \min_{|T|=n}(\Gamma(T) - \gamma(T))$ *for* $n \geq 1$. *Then*

$$t(n)=\begin{cases} 0, & \textit{if } n = 1 \textit{ or } 2k, \\ 1, & \textit{if } n = 2k+1 \geq 3. \end{cases}$$

Furthermore, T is a tree with $\Gamma(T) - \gamma(T) = t(n)$ if and only if

$$T=\begin{cases} \widehat{G}, & \textit{if } n = 2k, \\ \widehat{G}_{+1}, P_1 \textit{ or } B(3, k-1), & \textit{if } n = 2k+1 \geq 1, \end{cases}$$

for some tree G of order k.

References

[1] G. Chartrand and L. Lesniak, Graphs and digraphs.

[2] E.J. Cockayne, O. Favaron, C. Payan, and A.G. Thomason. Contributions to the theory of domination, independence and irredundance in graphs. Discrete Math., 33:249-259, 1981.

[3] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker Inc., New York, 1998.

[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs:Advanced Topics, Marcel Dekker Inc., New York, 1998.

[5] M.J. Jou, Dominating sets and independent sets in a tree, to appear in Ars Combinatoria.