Two-color Rado number for $x + y + c = 4z$

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Abstract: For positive integers $c \geqslant 0$ and $k \geqslant 1$, let $n = R(c, k)$ be the least integer, provided it exists, such that every 2-coloring of the set $[1, n] = \{1, \ldots, n\}$ admits a monochromatic solution to the equation $x + y + c = 4z$ with $x, y, z \in [1, n]$. In this paper the precise value of $R(c, 4)$ is shown to be $\lceil (3c + 2)/8 \rceil$ for all even $c \geqslant 34$..

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $[a, b] = \{x \in \mathbb{N} : a \leqslant x \leqslant b\}$ for $a, b \in \mathbb{N}$. For $k, n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we call a function $\Delta : [1, n] \to [0, k - 1]$ a *k-coloring* of the set $[1, n]$, and $\Delta(i)$ the *color* of $i \in [1, n]$. Given a k-coloring of the set $[1, n]$, a solution to a given diophantine equation L among elements of same color is called *monochromatic* solution.

Let $k \in \mathbb{Z}^+$. In 1916, I. Schur [8] proved that if $n \in \mathbb{Z}^+$ is sufficiently large then for every *k-coloring* of the set $[1, n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_3$$

with $x_1, x_2, x_3 \in [1, n]$. An equation L having this property is referred to as *k-regular*, hence there is an integer n such that every k-coloring of the positive integers up to n contains a monochromatic solution to L. Later Rado [6] characterized all k-regular linear equations. The least value of such an n is called the *k-color Rado number* for the linear equations. The reader may consult the book [5] by B. M. Landman and A. Robertson for a survey of results on Rado numbers. Recently S.Guo and Z.W. Sun [2] determined the precise values of 2-color Rado number for the equation

$$a_1x_1 + \cdots + a_mx_m = x_0,$$

which confirmed a conjecture of B. Hopkins and D. Schaal[3].

In this paper we consider 2-coloring Rado number $n = R(c, k)$ for the family of equations of the form

$$L(c, k) : x_1 + x_2 + c = kx_3$$

for integers $c \geqslant 0$ and $k \geqslant 1$. If $R(c, k)$ does not exist, then the 2-color Rado number is defined to be infinite.

In 2004 S. Jones and D. Schaal [4] determined the 2-regular equations in the family $L(c, k)$.

Theorem 1. *For $c \geqslant 0$ and $k \geqslant 1$, $R(c, k)$ is finite if and only if k is odd or c is even.*

If $k = 4$ and c is even, S. Jones and D. Schaal showed that $\lceil (3c+2)/8 \rceil \leqslant R(c, 4) \leqslant \lceil (3c+2)/8 \rceil + 3$ where $\lceil \cdot \rceil$ denote the ceil function. Using a computer search, they gave the actual numbers $R(c, 4)$ for even c between 2 and 72(cf. [4] Table 2). In this paper we showed that the difference between $R(c, 4)$ and $\lceil (3c+2)/8 \rceil$ is zero for $c \geqslant 34$. (Note that $R(c, 4) > \lceil (3c+2)/8 \rceil$ for some $c \in [10, 28]$.) Thus the precise value of $R(c, 4)$ for all even c are determined.

Theorem 2. *For all even $c \geqslant 34$,*

$$R(c,4) = \lceil (3c+2)/8 \rceil = \begin{cases} (3c+2)/8 & if c \equiv 2 (\text{mod } 8) \\ (3c+4)/8 & if c \equiv 4 (\text{mod } 8) \\ (3c+6)/8 & if c \equiv 6 (\text{mod } 8) \\ (3c+8)/8 & if c \equiv 0 (\text{mod } 8) \end{cases}.$$

Based on the above result and some other evidences, the following conjecture is probably true.

Conjecture 1. *For $k \geqslant 3$ and c sufficiently large, if k is odd or c is even,*

$$R(c,k) = \lceil \frac{2\lceil (c+2)/k \rceil + c}{k} \rceil.$$

Here we give a lemma ([4], Lemma 1) which will be used in Section 2.

Lemma 1. *For all integers $c, b \geqslant 0$ and $k \geqslant 3$, we have $R(c,k) \leqslant R(c+b(2-k),k)+b$.*

2 Proof of Theorem 2

The inequality $R(c,4) \geqslant \lceil (3c+2)/8 \rceil$ was stated already in [4, 7]. It suffices to show that $R(c,4) \leqslant \lceil (3c+2)/8 \rceil$.

First we will prove it to be true for the case $c \equiv 2 (\mathrm{mod}\ 8)$ and $c \geqslant 34$. Assume that $c \equiv 2 + 8\delta (\mathrm{mod}\ 32)$ where $\delta \in \{0,1,2,3\}$. Let $n \geqslant (3c+2)/8$ be an integer and let $\Delta : [1,n] \to [0,1]$ be a 2-coloring of $[1,n]$. Without loss of generality, we may assume that

$$\Delta(1) = 0. \tag{1}$$

Suppose, for contradiction, that there doesn't exist any monochromatic solution to the equation $L(c,4)$.

Since $1+1+c = 4 \cdot \frac{c+2}{4}$, we have $\Delta(\frac{c+2}{4}) \neq \Delta(1) = 0$, and hence

$$\Delta(\frac{c+2}{4}) = 1. \tag{2}$$

Similarly, as $\frac{c+2}{4} + \frac{c+2}{4} + c = 4 \cdot \frac{3c+2}{8}$ and $\frac{c+6}{8} + \frac{3c+2}{8} + c = 4 \cdot \frac{3c+2}{8}$, we must have

$$\Delta(\frac{3c+2}{8}) = 0 \tag{3}$$

and

$$\Delta(\frac{c+6}{8}) = 1. \tag{4}$$

Observe that

$$1 \leqslant 9 - 2\delta \leqslant \frac{c+2}{4} - 4\delta \leqslant \frac{c+2}{4} + 4\delta \leqslant \frac{3c+2}{8}.$$

Combining (3) and

$$(\frac{c+2}{4}-4\delta)+(\frac{c+2}{4}+4\delta)+c=4\cdot\frac{3c+2}{8},$$

we have

$$\Delta(\frac{c+2}{4}-4\delta)=1 \text{ or } \Delta(\frac{c+2}{4}+4\delta)=1.$$

Below we distinguish two cases.

Case 3.1. $\Delta(\frac{c+2}{4}-4\delta)=1$.

Clearly,

$$(\frac{c+2}{4}-4\delta)+(\frac{c+2}{4})+c=4\cdot\frac{3c+2-8\delta}{8},$$

it follows that

$$\Delta(\frac{3c+2-8\delta}{8})=0 \tag{5}$$

by (2) and $\Delta(\frac{c+2}{4}-4\delta)=1$. Since $\Delta(\frac{3c+2}{8})=\Delta(\frac{3c+2-8\delta}{8})=0$ by (3) and

$$(\frac{3c+2-8\delta}{8})+(\frac{c+6+8\delta}{8})+c=4\cdot\frac{3c+2}{8},$$

we must have

$$\Delta(\frac{c+6+8\delta}{8})=1. \tag{6}$$

As $\Delta(\frac{c+2}{4})=1$ by (2), and

$$\frac{c+6+8\delta}{8}+\frac{c+2}{4}+c=4\cdot\frac{11c+10+8\delta}{32},$$

we have

$$\Delta(\frac{11c+10+8\delta}{32})=0. \tag{7}$$

Observe that

$$(1+2\delta)+\frac{3c+2-8\delta}{8}+c=4\cdot\frac{11c+10+8\delta}{32},$$

thus we have

$$\Delta(1+2\delta)=1.$$

by (5). Note that

$$(1+2\delta)+(1+2\delta)+c=4\cdot\frac{c+2+4\delta}{4},$$

then we have $\Delta(\frac{c+2+4\delta}{4}) \neq \Delta(1+2\delta)$, hence

$$\Delta(\frac{c+2+4\delta}{4}) = 0.$$

Recall $\Delta(1) = 0$ by (1). As $1 + (1+4\delta) + c = 4 \cdot \frac{c+2+4\delta}{4}$, we have

$$\Delta(1+4\delta) = 1.$$

Also, by $(1+4\delta) + (1+4\delta) + c = 4 \cdot \frac{c+2+8\delta}{4}$, we have

$$\Delta(\frac{c+2+8\delta}{4}) = 0.$$

If $\Delta(\frac{5c+6+8\delta}{16}) = 0$, then we establish that

$$(1, \frac{c+2+8\delta}{4}, \frac{5c+6+8\delta}{16})$$

is a monochromatic solution to $L(c,4)$ from

$$1 + \frac{c+2+8\delta}{4} + c = 4 \cdot \frac{5c+6+8\delta}{16}.$$

On the other hand, if $\Delta(\frac{5c+6+8\delta}{16}) = 1$, then we have

$$(\frac{c+6+8\delta}{8}, \frac{c+6+8\delta}{8}, \frac{5c+6+8\delta}{16})$$

is a monochromatic solution to $L(c,4)$ by (6) and

$$\frac{c+6+8\delta}{8} + \frac{c+6+8\delta}{8} + c = 4 \cdot \frac{5c+6+8\delta}{16}.$$

In either case, we get a monochromatic solution to $L(c,4)$, contradicting our assumption.

Note that all above numbers are positive integers not exceed $\frac{3c+2}{8}$ as $c \equiv 2 + 8\delta (\bmod 32)$ and $c \geqslant 34$.

Case 2. $\Delta(\frac{c+2}{4} + 4\delta) = 1$.

By

$$(1+8\delta) + (1+8\delta) + c = 4 \cdot \frac{c+2+16\delta}{4},$$

we have

$$\Delta(1+8\delta) = 0 \neq \Delta(\frac{c+2}{4} + 4\delta) = 1. \tag{8}$$

Since $\Delta(1) = 0$ by (1), and

$$1 + (1 + 8\delta) + c = 4 \cdot \frac{c + 2 + 8\delta}{4},$$

we obtain that $\Delta(\frac{c+2+8\delta}{4}) = 1$. Similarly, as

$$(1 + 4\delta) + (1 + 4\delta) + c = 4 \cdot \frac{c + 2 + 8\delta}{4}$$

and

$$1 + (1 + 4\delta) + c = 4 \cdot \frac{c + 2 + 4\delta}{4},$$

we must have

$$\Delta(\frac{c + 2 + 4\delta}{4}) = 1 \neq \Delta(1 + 4\delta) = 0 \tag{9}$$

Therefore we have

$$\Delta(1 + 2\delta) = 0 \tag{10}$$

by

$$(1 + 2\delta) + (1 + 2\delta) + c = 4 \cdot \frac{c + 2 + 4\delta}{4}.$$

Combining $\Delta(\frac{c+6}{8}) = 1$ by (4) and $\Delta(\frac{c+2+4\delta}{4}) = 1$ by (9) with

$$(\frac{c + 6}{8}) + (\frac{c + 2 + 4\delta}{4}) + c = 4 \cdot \frac{11c + 10 + 8\delta}{32},$$

we establish that

$$\Delta(\frac{11c + 10 + 8\delta}{32}) = 0. \tag{11}$$

Observe that

$$1 + 2\delta + \frac{3c + 2 - 8\delta}{8} + c = 4 \cdot \frac{11c + 10 + 8\delta}{32},$$

then we have

$$\Delta(\frac{3c + 2 - 8\delta}{8}) = 1 \tag{12}$$

by (10) and (11). Clearly,

$$\frac{c + 2 - 8\delta}{4} + \frac{c + 2 - 8\delta}{4} + c = 4 \cdot \frac{3c + 2 - 8\delta}{8},$$

it follows that

$$\Delta(\frac{c + 2 - 8\delta}{4}) = 0. \tag{13}$$

Since (11) and

$$\frac{c+2-8\delta}{4}+\frac{c+6+24\delta}{8}+c=4\cdot\frac{11c+10+8\delta}{32},$$

we have

$$\Delta(\frac{c+6+24\delta}{8})=1.$$

Finally, by

$$\frac{c+6+24\delta}{8}+\frac{c+6+24\delta}{8}+c=4\cdot\frac{5c+6+24\delta}{16},$$

one can find that

$$\Delta(\frac{5c+6+24\delta}{16})=0\neq\Delta(\frac{c+6+24\delta}{8}).$$

Recall $\Delta(1+8\delta)=0$ by (8) and $\Delta(\frac{c+2-8\delta}{4})=0$ by (13), we have a monochromatic solution

$$(1+8\delta,\frac{c+2-8\delta}{4},\frac{5c+6+24\delta}{16})$$

to $L(c,4)$ from

$$1+8\delta+\frac{c+2-8\delta}{4}+c=4\cdot\frac{5c+6+24\delta}{16},$$

which contradicts our assumption.

One can check that all above numbers are positive integers not exceed $\frac{3c+2}{8}$ as $c\equiv 2+8\delta(\text{mod } 32)$ and $c\geqslant 34$.

Combining the above cases, we obtain that $R(c,4)\leqslant\frac{3c+2}{8}$ for $c\equiv 2+8\delta(\text{mod } 32)$ and $c\geqslant 34$.

Below we assume that $c\not\equiv 2+8\delta(\text{mod}32)$. Let $i=\min\{j\geqslant 0: c-j\equiv 2+8\delta(\text{mod } 32),\delta\in\{0,1,2,3\}\}$. Clearly, i is even. Thus, by Lemma 1 with $k=4$ and $b=i/2$,

$$R(c,4)\leqslant R(c-i,4)+i/2.$$

Then

$$R(c,4)\leqslant\frac{3(c-i)+2}{8}+\frac{i}{2}=\frac{3c+i+2}{8}.$$

hence $R(c,4)\leqslant\lceil(3c+2)/8\rceil$. Now we complete the proof. □

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