On the Ramsey Numbers $R(C_m, B_n)$†

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Abstract. For given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is defined to be the least positive integer n such that every graph G on n vertices, either G contains a copy of G_1 or the complement of G contains a copy of G_2. In this note, we show that $R(C_m, B_n) = 2m - 1$ for $m \geq 2n - 1 \geq 7$. With the help of computers, we obtain the exact values of 14 small cycle-book Ramsey numbers.

1 Introduction

In this note, only graphs without multiple edges or loops are considered. For a graph G, the complement of G is denoted by $\overline{G}$. A *cycle* on i vertices is denoted by C_i. A *star* graph, denoted by S_i, is an acyclic connected graph on n vertices with one vertex of degree $i-1$. A *book* graph, denoted by B_i, has $i+2$ vertices and is the result of a single vertex being connected to every vertex of a star S_{i+1}. The cardinality of a set S is denoted by $|S|$.

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We denote by $c(G)$ and $g(G)$ the lengths of the longest and shortest cycles of G, respectively. A graph of order n is pancyclic if it has cycles of every length l, $3 \leq l \leq n$ and weakly pancyclic if it has cycles of every length l, $g(G) \leq l \leq c(G)$.

For graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is defined to be the least positive integer n such that any graph G on n vertices either G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2. A graph is called a (G_1, G_2)-graph if neither G contains G_1 nor $\overline{G}$ contains G_2. A (G_1, G_2)-graph on n vertices is denoted by $(G_1, G_2; n)$-graph. The set of all nonisomorphic $(G_1, G_2; n)$-graphs is denoted by $\mathcal{R}(G_1, G_2)$ or $\mathcal{R}(G_1, G_2; n)$.

In [6], it was proved that

Theorem 1 *[6] If $m \geq 7$ is an odd integer and $n \geq 4m - 13$, then $R(C_m, B_n) = 2n + 3$; if $n \geq 1$ and $m \geq 2n + 2$, then $R(C_m, B_n) = 2m - 1$.*

In [11], some small Ramsey numbers $R(C_m, B_n)$ were obtained. By using the algorithm described in [11], it is hard to obtain more Ramsey numbers $R(C_m, B_n)$. In this note, we improve the result of Theorem 1 slightly.

Theorem 2 *$R(C_m, B_n) = 2m - 1$ for $m \geq 2n - 1 \geq 7$.*

With the help of computers, we also obtain some small cycle-book Ramsey numbers.

2 Proof of Theorem 2

In order to prove Theorem 2, we need the following Lemmas. Note many related results were gathered in [3].

Lemma 1 (Burr [4]) *For a connected graph G, if $|V(G)| \geq s(F)$, then $R(F, G) \geq (\chi(F) - 1)(|V(G)| - 1) + s(F)$, if $|V(G)| \geq s(F)$, where $\chi(F)$ is the chromatic number of F and $s(F)$ the minimum number of vertices in some color class under all vertex colorings by $\chi(F)$ colors.*

Lemma 2 (Bondy [1]) . *Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G \cong K_{n/2,n/2}$.*

Lemma 3 (Brandt [2]) *Every nonbipartite graph G of order n with $\delta(G) \geq (n + 2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 4 (Dirac [5]) *Let G be a connected graph of order $n \geq 3$ with $\delta = \delta(G) \geq 2$. Then $c(G) \geq \delta + 1$ and if G is 2-connected, then $c(G) \geq \min\{2\delta, n\}$.*

Lemma 5 (Lawrence [7])
$R(C_m, K_{1,n}) = \begin{cases} m, & \text{for } m \geq 2n, \\ 2n+1, & \text{for odd } m \leq 2n-1. \end{cases}$

Proof of Theorem 2. By Lemma 1, $R(C_m, B_n) \geq (\chi(B_n)-1)(|V(C_m)|-1) + s(B_n) = 2m-1$. We only need to show that $R(C_m, B_n) \leq 2m-1$.

Firstly, we prove that $R(C_m, B_n) = 2m-1$ for $m \geq 2n \geq 4$. Suppose to the contrary that there exists a graph G on $2m-1$ vertices such that $G \in \mathcal{R}(C_m, B_n)$ and $m \geq 2n \geq 4$. Then by Lemma 5, we have $\delta(G) \geq 2m-1-1-(R(C_m, K_{1,n})-1) = m-1$. It is easy to see that G is connected. Since $\overline{G}$ contains no B_n, then G is a nonbipartite graph. Otherwise, there is a partite set with order more than $n+2$. By Lemma 3, G is weakly pancyclic with $g(G) = 3$ or 4. By Lemma 4, $c(G) \geq m$. Thus, G contains a C_m, a contradiction.

Secondly, we will show that $R(C_m, B_n) = 2m-1$ for $m = 2n-1 \geq 7$. Suppose to the contrary that there exists a graph G on $2m-1 = 4n-3$ vertices such that $G \in \mathcal{R}(C_m, B_n)$. Since m is odd and by Lemma 5, we have $\delta(G) \geq 2m-1-1-(R(C_m, K_{1,n})-1) = 2n-4$. Now, we have the following Claims.

Claim 1 *G is connected.*

Proof. Otherwise, there are at least more than two components. Let H_1 be a component of G and $H_2 = G - H_1$. Since $\delta(G) \geq 2n-4$, we have $|V(H_i)| \geq 2n-3$ for $i = 1, 2$. Since $\overline{G}$ contains no B_n, we have H_i are complete graphs for $i = 1, 2$. Since $\delta(G) \geq 2n-4$, we have $\max(|V(H_i)|) \geq 2n-1, i = 1, 2$. Then we have that G contains a cycle of order $m = 2n-1$, a contradiction. □

Claim 2 *G is 2-connected.*

Proof. Otherwise, let $v \in V(G)$ be a vertex cut, H_1 be a component of $G - \{v\}$ and $H_2 = G - \{v\} - H_1$. We have $|V(H_i)| \geq 2n-4$ for $i = 1, 2$. Since $\overline{G}$ contains no B_n, H_i is complete graph for $i = 1, 2$. Since G contains no C_m, we have $\max(|V(H_i)|) \leq 2n-2, i = 1, 2$. Hence $H_i \cong K_{2n-2}$ for $i = 1, 2$. Since $d(v) \geq 2n-4$, we have that either $G[V(H_1) \cup \{v\}]$ or $G[V(H_2) \cup \{v\}]$ contains a C_m, a contradiction. □

Now, we return to the proof of the theorem. By Lemma 4, $c(G) \geq 2(2n-4) = 4n-8 > 2n-1$. Since $\overline{G}$ contains no B_n, then G is a nonbipartite graph. By Lemma 3, G contains C_m, a contradiction. □

3 Some small Ramsey numbers $R(C_m, B_n)$

We use the powerful tool *nauty* to generate all nonisomorphic graphs on 10 vertices and *shortg*[8] to reject graph isomorphism. By the definition of $R(C_m, B_n)$, we need to extend $\mathcal{R}(C_m, B_n; t)$ to $\mathcal{R}(C_m, B_n; t+1)$ by increasing t until $\mathcal{R}(C_m, B_n; t+1) = \emptyset$. In order to reach this goal, we apply one-vertex extension method similar to [11]. For some m and n, the values of $R(C_m, B_n)$ are shown in Table 2 and data for them in Table 1, and the statistics for $|\mathcal{R}(C_m, B_n)|$ gives Theorem 3.

Table 1: The number of nonisomorphic $(C_m, B_n; k)$-graphs

k	$\|\mathcal{R}(C_6,B_6)\|$	$\|\mathcal{R}(C_6,B_7)\|$	$\|\mathcal{R}(C_6,B_8)\|$	$\|\mathcal{R}(C_6,B_9)\|$	$\|\mathcal{R}(C_6,B_{10})\|$
10	3333	18180	28706	30247	30247
11	695	14075	87449	139691	146332
12	243	2195	66539	451327	719784
13	109	334	10845	371597	2511619
14	23	40	1491	55836	2217275
15	5	5	100	7469	298812
16	0	0	2	1339	21008
17	0	0	0	69	5776
18	0	0	0	0	189
19	0	0	0	0	0

k	$\|\mathcal{R}(C_8,B_5)\|$	$\|\mathcal{R}(C_8,B_6)\|$	$\|\mathcal{R}(C_8,B_7)\|$	$\|\mathcal{R}(C_7,B_6)\|$	$\|\mathcal{R}(C_7,B_7)\|$
10	4944	36585	143905	13436	52040
11	1860	17240	207310	7831	75562
12	113	3915	172063	2153	40245
13	34	519	83913	403	9199
14	4	281	1718	206	13778
15	0	135	1044	81	407
16	0	21	601	12	167
17	0	2	306	0	27
18	0	0	115	0	5
19	0	0	14	0	0
20	0	0	0	0	0

k	$\|\mathcal{R}(C_7,B_8)\|$	$\|\mathcal{R}(C_8,B_8)\|$	$\|\mathcal{R}(C_9,B_6)\|$	$\|\mathcal{R}(C_9,B_7)\|$
10	76020	20018	196262	535878
11	288978	813895	122880	884018
12	434614	1320322	47997	496324
13	324618	1265049	10826	198506
14	479028	180598	326	35879
15	80038	3377	158	1368

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Table 1 – continued from previous page

k	$\lvert\mathcal{R}(C_7,B_8)\rvert$	$\lvert\mathcal{R}(C_8,B_8)\rvert$	$\lvert\mathcal{R}(C_9,B_6)\rvert$	$\lvert\mathcal{R}(C_9,B_7)\rvert$
16	158149	1995	23	816
17	101	1129	2	437
18	5	550	0	199
19	0	228	0	25
20	0	32	0	2
21	0	5	0	0
22	0	0	0	0

Theorem 3 $R(C_6,B_6) = 16$, $R(C_6,B_7) = 16$, $R(C_6,B_8) = 17$, $R(C_6,B_9) = 18$, $R(C_6,B_{10}) = 19$, $R(C_8,B_5) = 15$, $R(C_8,B_6) = 18$, $R(C_8,B_7) = 20$, $R(C_7,B_6) = 17$, $R(C_7,B_7) = 19$, $R(C_7,B_8) = 19$, $R(C_8,B_8) = 22$, $R(C_9,B_6) = 18$, $R(C_9,B_7) = 21$.

Table 2: Known and new values of $R(C_m,B_n)$ for $m \geq 3, 2 \leq n \leq 12$.(New values and lower bounds are in bold font.)

m / n	3	4	5	6	7	8	9	10	11	12
2	7	7	9	11	13	15	17	19	21	23
3	9	9	10	11	13	15	17	19	21	23
4	11	11	11	12	13	15	**17**	19	21	23
5	13	12	13	14	15	**15**	**17**	**19**	**21**	23
6	15	13	15	**16**	**17**	**18**	**18**		**21**	**23**
7	17	16	17	**16**	**19**	**20**	**21**			
8	19	17	19	**17**	**19**	**22**	**≥ 23**			
9	21	18	21	**18**			**≥ 25**	**≥ 26**		
10	23	19	23	**19**				**≥ 28**		

4 Lower bounds for $R(C_n, B_n)$ and $R(C_n, B_{n-1})$

Theorem 4 *If $n \geq 3$, then $R(C_n,B_n) \geq 3n-2$, $R(C_n,B_{n-1}) \geq 3n-4$.*

Proof. The graph $3K_{n-1}$ is a $(C_n,B_n;3n-3)$-graph, hence $R(C_n,B_n) \geq 3n-2$. The graph obtained by adding one vertex v to the graph $3K_{n-2}$ such that v is adjacent to each vertex of $3K_{n-2}$ is a $(C_n,B_{n-1};3n-5)$-graph, hence $R(C_n,B_{n-1}) \geq 3n-4$. □

5 Summary of $R(C_m, B_n)$

Table 2 shows the known and new values of $R(C_m, B_n)$ for $m \geq 3, 2 \leq n \leq 12$ (see [11, 10]). By Theorem 4 we can give lower bounds for some cycle-book Ramsey numbers. We list four such bounds in Table 2. Although we think these four lower bounds are interesting and may be the exact values, but it seems difficult to decide them by computing.

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