Fundamental relation on (m, n)-ary hypermodules over (m, n)-ary hyperrings

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Abstract

In this paper, the class of (m, n)-ary hypermodules is introduced and several properties and examples are found. (m, n)-ary hypermodules are a generalization of hypermodules. On the other hand, we can consider (m, n)-ary hypermodules as a good generalization of (m, n)-ary modules. We define the fundamental relation ϵ^* on the (m, n)-ary hypermodules M as the smallest equivalence relation such that M/ϵ^* is an (m, n)-ary modules, and then some related properties are investigated.

1 Introduction

The notion of an n-ary group was introduced by Dörnte [6], which is a natural generalization of group. The notion of n-ary hypergroup was first introduced by Davvaz and Vougiouklis as a generalization of n-ary group [3], and studied mainly by many authors [4, 5, 7, 8, 9].

Let H be a non-empty set and h be a mapping $h : H \times H \longrightarrow \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then h is called a

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binary hyperoperation on H [1]. We denoted by H^n the cartesian product $H \times \ldots \times H$, where H appears n times and an element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $h : H^n \longrightarrow \wp^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity of hyperoperation.*

Let h be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ be non-empty subsets of H. We define

$$h(A_1, \ldots, A_n) = \cup\{h(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}.$$

We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. In this convention

$$h(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, x_{j+1}, \ldots, x_n)$$

will be written $h(x_1^i, y_{i+1}^j, x_{j+1}^n)$.

If h is an n-ary groupoid and $t = l(n-1)+1$, then the $t-$ary hyperoperation $h_{(l)}$ given by

$$h_{(l)}(x_1^{l(n-1)+1}) = h(h(\ldots, h(h(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(l-1)(n-1)+2}^{l(n-1)+1}),$$

will be denoted by $h_{(l)}$.

A non-empty set H with an n-ary hyperoperation $h : H^n \longrightarrow P^*(H)$ will be called an *n-ary hypergroupoid* and will be denoted by (H, h). An n-ary hypergroupoid (H, h) will be an *n-ary semihypergroup* if and only if the following associative axiom holds:

$$h(x_1^{i-1}, h(x_i^{n+i-1}), x_{n+i}^{2n-1})) = h(x_1^{j-1}, h(x_j^{n+j-1}), x_{n+j}^{2n-1}))$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$. An n-ary semihypergroup (H, h), in which the equation $b \in h(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leq i \leq n$, is called an *$n-$ ary hypergroup.*

A recent book [2] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Now, we consider the notion of (m, n)-ary hyperrings.

Definition 1.1. An *(m, n)-ary hyperring* is an algebraic hyperstructure $< R, f, g >$, which satisfies the following axioms:

(1) (R, f) is an m-ary hypergroup,

(2) (R, g) is an n-ary hypersemigroup,

(3) the n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e.,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)),$$

for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R,\ 1 \le i \le n$.

$< R, f, g >$ is called an m-ary hyperring if $m = n$. An m-ary hyperring R is a hyperring if $m = 2$.

Definition 1.2. Let M be a non-empty set. Then, $M = (M, h, k)$ is an *(m,n)-ary hypermodule* over an (m,n)-ary hyperring R, if (M,h) is an m-ary hypergroup and the map

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$

satiessfies in the following conditions:

(1) $k(r_1^{n-1}, h(x_1^m)) = h(k(r_1^{n-1}, x_1), \dots, k(r_1^{n-1}, x_m))$,

(2) $k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) = h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \dots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$,

(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = k(r_1^{n-1}, k(r_m^{n+m-2}, x))$.

If k is a scalar n-ary hyperoperation, $S_1, \dots, S_{n-1}$ are non-empty subsets of R and $M_1 \subseteq M$, we set

$$k(S_1, \dots, S_{n-1}, M_1) = \cup\{k(r_1, \dots, r_{n-1}, x) \mid r_i \in S_i, i = 1, \dots, n-1, x \in M_1\}.$$

An (m,n)-ary hypermodule M is an *$R-$hypermodule*, if $m = n = 2$.

EXAMPLE 1. Let R be a hyperring and M be an $R-$hypermodule. Then, R with m-ary hyperoperation $f(r_1^m) = \sum_{i=1}^{m} r_i$, and n-ary hyperoperation $g(r_1^n) = \prod_{i=1}^{n} r_i$, is an (m,n)-ary hyperring. Also, M with hyperoperation h with $h(x_1^m) = \sum_{i=1}^{m} x_i$, where $x_i \in M$, is an (m,n) hypergroup. Now, we define the scalar n-ary hyperoperation k with

$$k(r_1, \dots, r_{n-1}, x) := (\prod_1^n r_i) \cdot x.$$

Then M is an (m,n)-ary hypermodule over (m,n)-ary hyperring R.

EXAMPLE 2. Let $(R, +, \cdot)$ be a hyperring and $(M, +)$ be an R–hypermodule. If N is a subhypermodule of M then set:

$$\begin{aligned}
h(x_1^m) &= \sum_{i=1}^{m} x_i + N, && \forall\, x_1^m \in M,\\
f(r_1^m) &= \sum_{i=1}^{m} r_i, && \forall\, r_1^m \in R,\\
g(x_1^n) &= \prod_{i=1}^{n} r_i, && \forall\, r_1^n \in R,\\
k(r_1^{n-1}, x) &= (\sum_{i=1}^{n-1} r_i) \cdot x + N, && \forall\, r_1^{n-1} \in R,\ \forall x \in M.
\end{aligned}$$

Then (M, h, k) is an (m, n)-ary hypermodule over (m, n)-ary hyperring (R, f, g).

Definition 1.3. Let (M, h, k) be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. A non-empty subset $N \subseteq M$ is called an *(m, n)-ary subhypermodule* of M, if (N, h, k) is an (m, n)-ary hypermodule over the (m, n)-ary hyperring R.

Let (M_1, h_1, k) and (M_2, h_2, k) be two (m, n)-ary hypermodule over an (m, n)-ary hyperring R. A *homomorphism* from M_1 to M_2 is a mapping $\phi : M_1 \longrightarrow M_2$ such that

(1) $\phi(h_1(a_1, \ldots, a_m)) = h_2(\phi(a_1), \ldots, \phi(a_m))$,

(2) $\phi(k(r_1, \ldots, r_{n-1}, a)) = k(r_1, \ldots, r_{n-1}, \phi(a))$.

Lemma 1.4. *Let (M, h, k) be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. Then N is an (m, n)-ary subhypermodule M over the (m, n)-ary hyperring R if and only if the following conditions hold:*

(1) If the equation $b \in h(a_1^{i-1}, x_i, a_{i+1}^m)$ is solvable at the place $i = 1$ and $i = m$ or at least one place $1 < i < m$, for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m, b \in N$.

(2) For any $r_1, r_2, \ldots, r_{n-1} \in R$ and $y \in N$ imply that

$$k(r_1, r_2, \ldots, r_{n-1}, y) \subseteq N.$$

Proof. N is an m-ary hypergroup by Theorem 2.3 of [3] , since k is a closed scalar n-ary hyperoperation on N, then N is an (m, n)-ary subhypermodule over (m, n)-ary hyperring R. □

Lemma 1.5. *Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-ary hypermodules over an (m, n)-ary hyperring R and $\phi : M_1 \longrightarrow M_2$ a homomorphism. Then*

(1) If S is an (m, n)-ary subhypermodule of M_1 over an (m, n)-ary hyperring R, then $\phi(S)$ is an (m, n)-ary subhypermodule of M_2.

(2) If K is an (m, n)-ary subhypermodule of M_2 over an (m, n)-ary hyperring R, such that $\phi^{-1}(K) \neq \emptyset$, then $\phi^{-1}(K)$ is an (m, n)-ary subhypermodule of M_1.

Proof. (1) We know that $\phi(S)$ is an m-ary subhypergroup of M_2. Now, let $r_1, r_2, \ldots, r_{n-1} \in R$ and $y \in \phi(S)$, then there exists $x \in S$ such that $\phi(x) = y$. Hence $k(r_1, \ldots r_{n-1}, y) = k(r_1, \ldots, r_{n-1}, \phi(x)) = \phi(r_1, \ldots, r_{n-1}, x) \in \phi(S)$.

(2) The proof of this part is similar to (1). □

Definition 1.6. Let (M, h, k) be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. An equivalence relation ρ on M is called *compatible* if $a_1 \ \rho \ b_1, \ldots, \ a_m \ \rho \ b_m$, then for all $a \in h(a_1, \ldots, a_m)$ there exists $b \in h(b_1, \ldots, b_m)$ such that $a\rho b$, and if $r_1, \ldots, r_{n-1} \in R$, and $x\rho y$, then for all $a \in k(r_1, \ldots, r_{n-1}, x)$ there exists $b \in k(r_1, \ldots, r_{n-1}, y)$ such that $a\rho b$.

Let (M, h, k) be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R and ρ be an equivalence relation on M. Then ρ is a *strongly compatible relation* if

$$a_i\rho b_i \quad \text{for all} \quad 1 \leq \mathrm{i} \leq \mathrm{m} \quad \text{then,} \quad h(a_1, \ldots a_m) \ \overline{\overline{\rho}} \ h(b_1, \ldots, b_m),$$

and for every $r_1, \ldots, r_{n-1} \in R$ and $x\rho y$, then

$$k(r_1, \ldots, r_{n-1}, x) \ \overline{\overline{\rho}} \ k(r_1, \ldots, r_{n-1}, y).$$

We recall the following theorem from [9].

Theorem 1.7. *Let (H, f) be an m-ary hypergroup and let ρ be an equivalence relation on H. Then the relation ρ is strongly compatible if and only if the quotient $(H/\rho, f/\rho)$ is an m-ary group.*

Now, we introduce the strong compatible relation Γ on an (m, n)-ary hyperring R.

Definition 1.8. Let (R, f, g) be an (m, n)-ary hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, when $s = k(m-1) + 1$, we define the relation $\Gamma_{k;l_1^s}$, as follows:

$x \ \Gamma_{k;l_1^s} \ y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1) + 1$, $i = 1, \ldots, s$ such that

$$\{x, y\} \subseteq f_{(k)}(u_1, \ldots, u_s),$$

where for every $i = 1, \dots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$.

Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}^*} \Gamma_k$. Then the relation Γ is reflexive and symmetric. Let Γ^* be the transitive closure of relation Γ.

Theorem 1.9. *[10]. The relation Γ^* is a strongly compatible relation on both m-ary hypergroup (R, f) and n-ary semihypergroup (R, g) and the quotient $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m, n)-ary ring.*

Theorem 1.10. *Let (M, h, k) be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R and ρ be an equivalence relation on M. Then the following conditions are equivalent.*

(1) The relation ρ is strongly compatible.

(2) If $r_1, \dots, r_{n-1} \in R$, $x_1^m, a, b \in M$ and $a\rho b$, then for every $(i = 1, \dots, m)$, we have

$$h(x_1^{i-1}, a, x_{i+1}^m)\ \overline{\overline{\rho}}\ h(x_1^{i-1}, b, x_{i+1}^m)$$

and

$$k(r_1, \dots, r_{n-1}, a)\ \overline{\overline{\rho}}\ k(r_1, \dots, r_{n-1}, b).$$

(3) The quotient $(M/\rho, h/\rho, k/\rho)$ is an (m, n)-ary module over an (m, n)-ary hyperring R. In the other word, M is an m-ary group and the scalar n-ary hyperoperation k is singleton.

Proof. We show that (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (1) Let $a_i\ \rho\ b_i$, where $i = 1, \dots, m$. By (2) we have

$$\begin{aligned} h(a_1, \dots, a_m) &\ \overline{\overline{\rho}}\ h(a_1, \dots, a_{m-1}, b_m) \\ &\ \overline{\overline{\rho}}\ h(a_1, \dots, a_{m-2}, b_{m-1}, b_m) \\ &\ \vdots \\ &\ \overline{\overline{\rho}}\ h(a_1, b_2, \dots, b_m) \\ &\ \overline{\overline{\rho}}\ h(b_1, \dots, b_m). \end{aligned}$$

Since $\overline{\overline{\rho}}$ is transitive, thus $\overline{\overline{\rho}}$ is strongly compatible for h.

Now, let $r_1, \dots, r_{n-1} \in R$ and $a\rho b$, hence

$$k(r_1, \dots, r_{n-1}, a)\ \overline{\overline{\rho}}\ k(r_1, \dots, r_{n-1}, b).$$

Since $\overline{\overline{\rho}}$ is transitive, then ρ is strongly compatible.

(1) $\Rightarrow$ (3) Define

$$h/\rho(\rho(a_1), \dots, \rho(a_m)) := \{\rho(a) | a \in h(a_1, \dots, a_m)\} = \rho(h(a_1^m))$$

and

$$k/\rho(r_1,\ldots,r_{n-1},\rho(a)) := \{\rho(x)|x \in k(r_1,\ldots,r_{n-1},a)\} = \rho(k((r_1^{n-1},a)).$$

Since ρ is a compatible relation, then we conclude that h/ρ and k/ρ are well-defined. Also ρ is strongly compatible, so $(M/\rho, h/\rho)$ is an m-ary group by Theorem 1.7.

(1) Now, we have

$$\begin{aligned}
&h/\rho(k/\rho(r_1^{n-1},\rho(x_1)),\ldots,k/\rho(r_1^{n-1},\rho(x_m)))\\
&= h/\rho(\rho(k(r_1^{n-1},x_1)),\ldots,\rho(k(r_1^{n-1},x_m)))\\
&= \rho(k(r_1^{n-1},h(x_1^m)))\\
&= \bigcup_{x\in h(x_1^m)} \rho(k(r_1^{n-1},x)).
\end{aligned}$$

In the other hand

$$\begin{aligned}
k/\rho(r_1^{n-1},h/\rho(\rho(x_1),\ldots,\rho(x_m))) &= k/\rho(r_1^{n-1},\rho(h(x_1^m)))\\
&= \rho(k(r_1^{n-1},h(\rho(x_1),\ldots,\rho(x_m)))\\
&= \bigcup_{x\in h(x_1^m)} \rho(k(r_1^{n-1},x)).
\end{aligned}$$

(2) We have

$$\begin{aligned}
&k/\rho(r_1^{i-1},f(s_1^m),r_{i+1}^{n-1},\rho(x))\\
&= \rho(k(r_1^{i-1},f(s_1^m),r_{i+1}^{n-1},x))\\
&= \rho(h(k(r_1^{i-1},s_1,r_{i+1}^{n-1},x),\ldots,k(r_1^{i-1},s_m,r_{i+1}^{n-1},x))).
\end{aligned}$$

In the other hand

$$\begin{aligned}
&h/\rho(k/\rho(r_1^{i-1},s_1,r_{i+1}^{n-1},\rho(x)),\ldots,k/\rho(r_1^{i-1},s_m,r_{i+1}^{n-1},\rho(x)))\\
&= h/\rho(\rho(k(r_1^{i-1},s_1,r_{i+1}^{n-1},x),\ldots,k(r_1^{i-1},s_m,r_{i+1}^{n-1},x)))\\
&= \rho(h(k(r_1^{i-1},s_1,r_{i+1}^{n-1},x),\ldots,k(r_1^{i-1},s_m,r_{i+1}^{n-1},x))).
\end{aligned}$$

(3) We have

$$k/\rho(r_1^{i-1},g(r_i^{i+n-1}),r_{i+m}^{n+m-2},\rho(x)) = \rho(k(r_1^{i-1},g(r_i^{i+n-1}),r_{i+m}^{n+m-2},x)).$$

In the other hand

$$\begin{aligned}
k/\rho(r_1^{n-1},k/\rho(r_m^{n+m-2}),\rho(x))) &= k/\rho(r_1^{n-1},\rho(k(r_m^{n+m-2},x)))\\
&= \rho(k(r_1^{n-1},(k(r_m^{n+m-2},x))).
\end{aligned}$$

(3) $\Rightarrow$ (1) Now, let $(M/\rho, h/\rho, k/\rho)$ be an (m,n)-ary module. Let $a_i\ \rho\ b_i$, where $i = 1, \ldots, m$, since $(M/\rho, h/\rho)$ is an m-ary group, so

$$h/\rho(\rho(a_1), \ldots, \rho(a_m)) = \{\rho(x) \mid x \in h(a_1, \ldots, a_m)\}$$

and

$$h/\rho(\rho(b_1), \ldots, \rho(b_m)) = \{\rho(x) \mid x \in h(b_1, \ldots, a_m)\}$$

are singleton. Thus, for every $y \in h(a_1, \ldots, a_m)$ and $z \in h(b_1, \ldots, b_m)$ we have $h/\rho(\rho(a_1), \ldots, \rho(a_m)) = \rho(y)$ and $h/\rho(\rho(b_1), \ldots, \rho(b_m)) = \rho(z)$. But $\rho(a_i) = \rho(b_i)$ and so we obtain $\rho(y) = \rho(z)$ for every $y \in h(a_1, \ldots, a_m)$ and $z \in h(b_1, \ldots, b_m)$. Therefore $h(a_1, \ldots, a_m)\ \overline{\overline{\rho}}\ h(b_1, \ldots, b_m)$.

Now, let $r_1, \ldots, r_{n-1} \in R$ and $a\rho b$, since $(M/\rho, h/\rho, k/\rho)$ is an (m,n)-ary module over $(m,n)-ary$ ring R, so $k/\rho(r_1, \ldots, r_{n-1}, \rho(a)) = \{\rho(x) | x \in k(r_1, \ldots, r_{n-1}, a\}$ and $k/\rho(r_1, \ldots, r_{n-1}, \rho(b)) = \{\rho(y) | y \in k(r_1, \ldots, r_{n-1}, b\}$ are singleton. Thus for every $x \in k(r_1, \ldots, r_{n-1}, a)$ and $y \in k(r_1, \ldots, r_{n-1}, b)$ we have $(k/\rho(r_1, \ldots, r_{n-1}, \rho(a)) = \rho(x)$ and $(k/\rho(r_1, \ldots, r_{n-1}, \rho(b)) = \rho(y)$. But $\rho(a) = \rho(b)$ and so $\rho(x) = \rho(y)$ for every $x \in k(r_1, \ldots, r_{n-1}, a)$ and $y \in k(r_1, \ldots, r_{n-1}, b)$. Therefore $k(r_1, \ldots, r_{n-1}, a)\ \overline{\overline{\rho}}\ k(r_1, \ldots, r_{n-1}, b)$. □

Theorem 1.11. *Let (M, h, k) be an (m,n)-ary hypermodule over (m,n)-ary hyperring (R, f, g) and δ be a strongly compatible relation on f and g. Let ρ be a strongly compatible relation on h such that $\rho(k(r_1^{n-1}, x_i)) = k(\delta(r_1), \ldots, \delta(r_{n-1}), \rho(x_i))$. Then $(M/\rho, h/\rho, k/\rho)$ is an (m,n)-ary module on (m,n)-ary ring $(R/\delta, f/\delta, g/\delta)$.*

Proof. By Theorem 1.9., we have the quotient $(R/\delta, f/\delta, g/\delta)$ is an (m,n)-ary ring. Also, we know $(M/\rho, h/\rho)$ is an m-ary group, by Theorem 1.4. Define the scalar n-ary hyperoperation

$$k/\rho(\delta(r_1), \ldots, \delta(r_{n-1}), \rho(x)) = \rho(k(r_1, \ldots, r_{n-1}, x)).$$

Since $\rho(k(r_1^{n-1}, x)) = k(\delta(r_1), \ldots, \delta(r_{n-1}), \rho(x))$, by Theorem 1.10, k/ρ has all scalar n-ary hyperoperation properties. □

Let R be a hyperring and M be a hypermodule over R. We recall the definition of relation ϵ on M as follows [12]:

$$x\epsilon y \;\Leftrightarrow\; x, y \in \sum_{i=1}^{n} m_i'; \quad m_i' = m_i \quad \text{or} \quad m_i' = \sum_{j=1}^{n_i} \Big(\prod_{k=1}^{k_{ij}} x_{ijk}\Big) z_i,$$
$$m_i \in M, \quad x_{ijk} \in R, \quad z_i \in M.$$

The equivalence relation ϵ^* (transitive closure of ϵ) was first introduced by Vougiouklis, and studied mainly by many authors concerning hypermodules. The fundamental relation ϵ^* on M, defined as the smallest equivalence relation such that the quotient M/ϵ^* be a module over the corresponding fundamental ring such that M/ϵ^* as a group is not abelian, see [11, 12] .

Now, let M be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. We define the relation ϵ on M.

Definition 1.12. Let M be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. We define

$$x \,\epsilon\, y \Longleftrightarrow \begin{cases} \{x,y\} \subset h_{(a)}(u_1^r), & r = a(m-1)+1 \\ u_i = m_i \quad \text{or} \quad k(v_{i1}^{in-1}, x_i), & x_i \in M \\ v_{ij} = f_{(b_{ij})}(w_{ij1}^{ijs_{ij}}), & s_{ij} = b_{ij}(m-1)+1 \\ w_{ijk} = g_{(c_{ijk})}(x_{ijk1}^{ijkt_{ijk}}), & t_{ijk} = c_{ijk}(n-1)+1, \quad x_{ijkt} \in R. \end{cases}$$

The following example shows that the relation ϵ on an (m,n)-ary hypermodule is not transitive, in general.

EXAMPLE 3. Let $H = \{a,b,c,d\}$ and $h(a,\ldots,a) = \{b,c\}$ and for every $x_1^m \in M$, $h(x_1^m) = \{c,d\}$, where $x_i \neq a$, and $1 \leq i \leq m$. Then (M,h) is an m-ary semihypergroup. If R be an arbitrary (m,n)-ary hyperring then for every $r_1^{n-1} \in R$ and $x \in M$, we define $k(r_1^{n-1},x) = \{c,d\}$. Then (M,h,k) is an (m,n)-ary hypermodule. We have $b\epsilon c$ and $c\epsilon d$ so $b\epsilon^* d$ but $b \not\epsilon d$. Hence ϵ is not transitive.

Theorem 1.13. *The relation ϵ^* is a strongly compatible relation on M, as (m,n)-ary hypermodule, on both m-ary hyperoperation h and scalar n-ary hyperoperation k.*

Proof. If $a_1\ \epsilon^*\ b_1,\ldots,a_m\ \epsilon^*\ b_m$, then $\epsilon^*(a_1) = \epsilon^*(b_1),\ldots,\epsilon^*(a_m) = \epsilon^*(b_m)$. For every $a \in h(a_1,\ldots,a_m)$ and $b \in h(b_1,\ldots,b_m)$ we have

$$\begin{aligned} \epsilon^*(a) &= \epsilon^*(h(a_1,\ldots,a_m)) \\ &= h/\epsilon^*(\epsilon^*(a_1),\ldots,\epsilon^*(a_m)) \\ &= h/\epsilon^*(\epsilon^*(b_1),\ldots,\epsilon^*(b_m)) \\ &= \epsilon^*(h(b_1,\ldots,b_m)) \\ &= \epsilon^*(b). \end{aligned}$$

Now, let $r_1,\ldots,r_{n-1} \in R$, $a_1,b_1 \in M$ and $a_1\epsilon^* b_1$, then for every $a \in k(r_1,\ldots,r_{n-1},a_1)$ and $b \in k(r_1,\ldots,r_{n-1},b_1)$, we have

$$\begin{aligned}\epsilon^*(a) &= \epsilon^*(k(r_1,\dots,r_{n-1},a_1)) \\ &= k/\epsilon^*((r_1,\dots,r_{n-1},\epsilon^*(a_1)) \\ &= k/\epsilon^* k(r_1,\dots,r_{n-1},\epsilon^*(b_1)) \\ &= \epsilon^*(b).\end{aligned}$$

□

Theorem 1.14. *Let (M,h,k) be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. Then the quotient $(M/\epsilon^*, h/\epsilon^*)$ is an (m,n)-ary hypermodule over an (m,n)-ary hyperring R, where*

$$h/\epsilon^*(\epsilon^*(a_1),\dots,\epsilon^*(a_m)) := \{\epsilon^*(a) | a \in h(a_1,\dots,a_m)\} = \epsilon^*(h(a_1^m))$$

and

$$k/\epsilon^*(r_1,\dots,r_{n-1},\epsilon^*(a)) := \{\epsilon^*(x) | x \in k(r_1,\dots,r_{n-1},a)\} = \epsilon^*(k((r_1^{n-1},a)).$$

Proof. We shall use the following abbreviated notation:
the sequence $\epsilon^*(a_i),\epsilon^*(a_{i+1}),\dots,\epsilon^*(a_j)$ will be denoted by ${\epsilon^*}_{a_i}^{a_j}$. Since ϵ^* is a compatible relation, then we conclude that h/ϵ^* and k/ϵ^* are well-defined. Also

(1) We have

$$\begin{aligned}& h/\epsilon^*(k/\epsilon^*(r_1^{n-1},\epsilon^*(x_1)),\dots,k/\epsilon^*(r_1^{n-1},\epsilon^*(x_m)) \\ &= h/\epsilon^*(\epsilon^*(k(r_1^{n-1},x_1)),\dots,\epsilon^*(k(r_1^{n-1},x_m)) = \epsilon^*(k(r_1^{n-1},h(x_1^m)) \\ &= \bigcup_{x\in h(x_1^m)} \epsilon^*(k(r_1^{n-1},x)).\end{aligned}$$

In the other hand

$$\begin{aligned}k/\epsilon^*(r_1^{n-1},h(\epsilon^*(x_1),\dots,\epsilon^*(x_m))) &= k/\epsilon^*(r_1^{n-1},\epsilon^*(h(x_1^m))) \\ &= \epsilon^*(k(r_1^{n-1},h(\epsilon^*(x_1),\dots,\epsilon^*(x_m)) \\ &= \bigcup_{x\in h(x_1^m)} \epsilon^*(k(r_1^{n-1},x)).\end{aligned}$$

(2) We have

$$\begin{aligned}& k/\epsilon^*(r_1^{i-1},f(s_1^m),r_{i+1}^m,\epsilon^*(x)) \\ &= \epsilon^*(k(r_1^{i-1},f(s_1^m),r_{i+1}^m,x)) \\ &= \epsilon^*(h(k(r_1^{i-1},s_1,r_{i+1}^m,x),\dots,k(r_1^{i-1},s_m,r_{i+1}^m,x))).\end{aligned}$$

In the other hand

$$\begin{aligned}
&h/\epsilon^*(k/\epsilon^*(r_1^{i-1}, s_1, r_{i+1}^{n-1}, \epsilon^*(x)), \ldots, k/\epsilon^*(r_1^{i-1}, s_m, r_{i+1}^{n-1}, \epsilon^*(x)))\\
&= h/\epsilon^*(\epsilon^*(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, \epsilon^*(k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)))\\
&= \epsilon^*(h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))).
\end{aligned}$$

(3) We have

$$k/\epsilon^*(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, \epsilon^*(x)) = \epsilon^*(k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x)).$$

In the other hand

$$\begin{aligned}
k/\epsilon^*(r_1^{n-1}, k/\epsilon^*(r_m^{n+m-2}, \epsilon^*(x))) &= k/\epsilon^*(r_1^{n-1}, \epsilon^*(k(r_m^{n+m-2}, x)))\\
&= \epsilon^*(k(r_1^{n-1}, (k(r_m^{n+m-2}, x))).
\end{aligned}$$

□

The natural map $\pi : M \longrightarrow M/\epsilon^*$, where $\pi(x) = \epsilon^*(x)$ is an onto homomorphism.

Definition 1.15. Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-ary hypermodules over an (m, n)-ary hyperring R and let $\phi : M_1 \longrightarrow M_2$ be a homomorphism. Then, the kernel ϕ is defined by

$$ker\phi = \{(a, b) \in M_1 |\ \phi(a) = \phi(b)\}.$$

It is easy to see that $ker\phi$ is a compatible relation.

Theorem 1.16. *Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-ary hypermodules over an (m, n)-ary hyperring R, and let $\phi : M_1 \longrightarrow M_2$ be a homomorphism. Then there exists a compatible relation θ on M_1 and a homomorphism $\psi : M_1/\theta \longrightarrow M_2$ such that $\psi \circ \pi = \phi$.*

Proof. We consider $\theta = ker\phi$. Now, let $\theta(a) \in M_1/\theta$ and define $\psi(\theta(a)) = \phi(a)$. □

Theorem 1.17. *Let ρ and θ be compatible relation on (m, n)-ary hypermodules (M, h, k) over an (m, n)-ary hyperring R, such that $\rho \subseteq \theta$. Then, there exists a compatible relation μ on $(M/\rho, h/\rho, k/\rho)$ such that $(M/\rho)/\mu$ is isomorphic to M/θ, as (m, n)-ary hypermodules.*

Proof. We consider the map $\phi : M/\rho \longrightarrow M/\theta$ by $\phi(\rho(x)) = \theta(x)$. Since $\rho \subseteq \theta$, ϕ is well-defined. Clearly ϕ is a homomorphism. Now, by Theorem 1.16, there exists a compatible relation μ and a monomorphism ψ :

$(M/\rho)/\mu \longrightarrow M/\theta$ such that $\psi \circ \pi = \phi$, and so ψ is an isomorphism as (m,n)-ary hypermodule. □

Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m,n)-ary hypermodules over an (m,n)-ary hyperring R. Define the direct hyperproduct $(M_1 \times M_2, h_1 \times h_2, k_1 \times k_2)$ to the (m,n)-ary hypermodule whose universe is the set $M_1 \times M_2$ and such that for $a_i \in M_i$, $a'_i \in M_2$, $1 \le i \le m$,

$$\begin{aligned}&(h_1 \times h_2)((a_1, a'_1), \ldots, (a_m, a'_m))\\ &= \{(a, a') \mid a \in h_1(a_1, \ldots, a_m), a' \in h_2(a'_1, \ldots, a'_m)\},\end{aligned}$$

and

$$\begin{aligned}&(k_1 \times k_2)(r_1, \ldots, r_{n-1}, (x, x'))\\ &= \{(a, a') \mid a \in k_1(r_1, \ldots, r_{n-1}, x),\ a' \in k_2(r_1, \ldots, r_{n-1}, x')\}.\end{aligned}$$

The mapping $\pi_i : M_1 \times M_2 \longrightarrow M_i$, $i = 1, 2$, defined by $\pi_i((a_1, a_2)) = a_i$, is called the *projection map* on the ith coordinate of $M_1 \times M_2$, also the mapping $\pi_i : M_1 \times M_2 \longrightarrow M_i$ is an onto homomorphism.

2 Fundamental (m,n)-ary hypermodules

If (M, h, k) is an (m,n)-ary hypermodule, then $\hat{\epsilon}$ denoted the *transitive closure* of the relation $\epsilon = \bigcup_{a \ge 0} \epsilon_a$, where ϵ_0 is the diagonal, i.e., $\epsilon_0 = \{(x,x) | x \in M\}$ and for every integer $a \ge 1$, ϵ_a is the relation defined as follows:

$$x \epsilon_a y \quad \text{if and only if} \quad \{x, y\} \subseteq h_{(a)},$$

for some $a \in \mathbb{N}$. If $x\epsilon_0 y$ (i.e., $x = y$) then we write $\{x, y\} \subseteq u_{(0)}$. We define ϵ^* as the smallest equivalence relation such that the quotient $(M/\epsilon^*, h/\epsilon^*, k/\epsilon^*)$ is an (m,n)-ary module over an (m,n)-ary hyperring R, where M/ϵ^* is the set of all equivalence classes. The ϵ^* is called *fundamental equivalence relation.*

Lemma 2.1. *Let (M, h, k) be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R, then for every $a \in \mathbb{N}^*$, we have $\epsilon_a \subseteq \epsilon_{a+1}$.*

Proof. Let $x\epsilon_a y$, then there exists $a \in \mathbb{N}$, and $u_1, \ldots, u_r$, where $r = a(m-1)+1$, such that $\{x, y\} \subseteq h_{(a)}(u_1^r)$. By producibility of h, there exist $u'_1, \ldots, u'_m$, such that $u_1 \subseteq h(u'_1, \ldots, u'_m)$. So

$$\begin{aligned}\{x, y\} &\subseteq h_{(a)}(u_1^r) = h_{(a)}(u_1 \ldots, u_r)\\ &\subseteq h_{(a)}(h(u'_1, \ldots, u'_m), u_2, \ldots u_r) = h_{(a+1)}(u'^p_1, u_2^r).\end{aligned}$$

This means $x\epsilon_{a+1}y$. □

Corollary 2.2. Let (M,h,k) be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R, then for every $a \in \mathbb{N}$, we have $\epsilon_a^* \subseteq \epsilon_{a+1}^*$.

Theorem 2.3. *The fundamental relation ϵ^* is the transitive closure of the relation ϵ, i.e., ($\epsilon^* = \widehat{\epsilon}$).*

Proof. By Theorem 4.1 of [3], we know that the quotient $M/\widehat{\epsilon}$ is an m-ary hypergroup, where $h/\widehat{\epsilon}$ is defined in the usual manner

$$h/\widehat{\epsilon}(\widehat{\epsilon}(x_1),\ldots,\widehat{\epsilon}(x_m)) = \{\widehat{\epsilon}(y) | y \in h(\widehat{\epsilon}(x_1),\ldots,\widehat{\epsilon}(x_m))\}$$

for all $x_1,\ldots,x_m \in M$.

Now, we prove that $M/\widehat{\epsilon}$ is an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. The scalar n-ary hyperoperation $k/\widehat{\epsilon}$ in $M/\widehat{\epsilon}$ is defined in the usual manner:

$$k/\widehat{\epsilon}(r_1,\ldots,r_{n-1},\widehat{\epsilon}(x)) = \{\widehat{\epsilon}(y) | y \in k(r_1,\ldots,r_{n-1},x)\},$$

for all $r_1,\ldots,r_{n-1} \in H$ and $x \in M$. Suppose $a \in \widehat{\epsilon}(x)$, then we have $a\widehat{\epsilon}x$, if there exist $x_1,\ldots,x_m$ such that $x_1 = a,\ldots,x_m = x$ such that $\{x_i,x_{i+1}\} \subseteq h_{(i)}$. So every element $z \in k(r_1,\ldots,r_{n-1},x_i)$ is equivalent to every element to $k(r_1,\ldots,r_{n-1},x_{i+1})$. Therefore $k/\epsilon^*(r_1,\ldots,r_{n-1},\epsilon^*(x))$ is singleton. So we can write $k/\epsilon^*(r_1,\ldots,r_{n-1},\epsilon^*(x)) = \epsilon^*(y)$ for all $y \in k(r_1,\ldots,r_{n-1},\epsilon^*(x))$.

Moreover, since k has n-ary hypermodule scalar properties, consequently, $k/\widehat{\epsilon}$ has (m,n)-ary hypermodule scalar properties.

Now, let θ be an equivalence relation on M such that M/θ is an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. Then for all $x_1,\ldots,x_m \in M$, we have $h/\theta(\theta(x_1),\ldots,\theta(x_m)) = \theta(y)$ for all $y \in h(\theta(x_1),\ldots,\theta(x_m))$. Also $k/\theta(r_1,\ldots,r_{n-1},\theta(x)) = \theta(z)$, for all $z \in k(r_1,\ldots,r_{n-1},\theta(x))$. But also, for every $x_1,\ldots,x_m,x \in M$, $r_1,\ldots,r_{n-1} \in R$, $A_i \subseteq \theta(x_i),(i = 1,\ldots,m)$ and $A \subseteq \theta(x)$, we have

$$h/\theta(\theta(x_1),\ldots,\theta(x_m)) = \theta(h(x_1,\ldots,x_m)) = \theta(h(A_1,\ldots,A_m))$$

and

$$k/\theta((r_1,\ldots,r_{n-1},\theta(x)) = \theta(k(r_1,\ldots,r_{n-1},x)) = \theta(k(r_1,\ldots,r_{n-1},A)).$$

Therefore, $\theta(a) = \theta(u_{(i)})$ for all $i \geq 0$ and for all $a \in h_u$ or k. So for every $a \in M, x \in \epsilon(a)$ implies $x \in \theta(a)$. But θ is transitively closed, so we obtain $x \in \epsilon^*(a)$ implies $x \in \theta(a)$. Hence, the relation ϵ^* is the smallest equivalence relation on M such that M/ϵ^* is an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. □

Theorem 2.4. *ϵ^* is a strongly compatible relation.*

Proof. Since ϵ^* is an equivalence relation such that $(M/\epsilon^*, h/_{\epsilon^*}, k/\epsilon^*)$ is an (m,n)-ary module over an (m,n)-ary hyperring R, by Theorem 1.10, ϵ^* is a strongly compatible relation. □

Theorem 2.5. *Let (M,h,k) be an (m,n)-ary hypermodule over (m,n)-ary hyperring (R,f,g). Then, $(M/\epsilon^*, h/\epsilon^*)$ is an (m,n)-ary module over (m,n)-ary ring $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$.*

Proof. By Theorem 2.4, ϵ^* is a strongly compatible relation on M, and by Theorem 1.7, $(M/\epsilon^*, h/\epsilon^*)$ is an m-ary group. Also, by Theorem 1.9, $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m,n)-ary ring. Now, let $r_1,\ldots,r_{n-1} \in R$, $x \in M$ and define

$$k_{\epsilon^*}(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\epsilon^*(x)) := k(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\epsilon^*(x)).$$

If $x \in h_a(u_1,\ldots,u_r)$ and $r_i \in f_{k_i}(u'_1,\ldots,u'_s)$, then

$$\begin{aligned}&k(\Gamma^*(r_1)\ldots,\Gamma^*(r_{n-1}),\epsilon^*(x))\\ &\subseteq k(f_{k_1},\ldots,f_{k_{n-1}},h_a(u_1,\ldots,u_r))\\ &= h_a(k(f_{k_1},\ldots,f_{k_{n-1}},u_1),\ldots,k(f_{k_1},\ldots,f_{k_{n-1}},u_r)).\end{aligned}$$

So, for every $r'_1\Gamma^*r_1,\ldots.r'_{n-1}\Gamma^*r_{n-1}$ and $y\epsilon^*x$, we have

$$\begin{aligned}&k(\Gamma^*(r'_1)\ldots,\Gamma^*(r'_{n-1}),\epsilon^*(y))\\ &\subseteq h_a(k(f_{k_1},\ldots,f_{k_{n-1}},u_1),\ldots,k(f_{k_1},\ldots,f_{k_{n-1}},u_r)).\end{aligned}$$

Since M is an (m,n)-ary hypermodule over (m,n)-ary hyperring R, the properties of M as an (m,n)-ary hypermodule, guarantee that the m-ary group M/ϵ^* is an $(m,n) - ary\ R/\Gamma^*$-module. □

Theorem 2.6. *Let $A = (A, h_1, k_1)$ and $B = (B, h_2, k_2)$ be two (m,n)-ary hypermodules over an (m,n)-ary hyperring R and let ϵ^*_A, ϵ^*_B and $\epsilon^*_{A\times B}$ be fundamental equivalence relations on A, B and $A\times B$ respectively. Then*

$$\phi : A\times B/\epsilon^*_{A\times B} \cong A/\epsilon^*_A \times B/\epsilon^*_B,$$

as (m,n)-ary modules over an (m,n)-ary hyperring R.

Proof. First we define the relation $\hat{\epsilon}$ on $A\times B$ as follows:

$$(a_1,b_1)\ \hat{\epsilon}\ (a_2,b_2) \Leftrightarrow a_1\ \epsilon^*_A\ a_2 \text{ and } b_1\ \epsilon^*_B\ b_2,$$

$\hat{\epsilon}$ is an equivalence relation. We define h on $(A\times B)/\hat{\epsilon}$ as follows:

$$h(\hat{\epsilon}(a_1,b_1),\ldots,\hat{\epsilon}(a_m,b_m)) = \hat{\epsilon}(a,b),$$

for all $a \in h_1(\epsilon^*_A(a_1), \ldots, \epsilon^*_A(a_m))$, $b \in h_2(\epsilon^*_B(b_1), \ldots, \epsilon^*_B(b_m))$, and

$$k(r_1, \ldots, r_{n-1}, \widehat{\epsilon}(a_1, b_1)) = \widehat{\epsilon}(a, b)$$

for all $a \in k(r_1, \ldots, r_{n-1}, \epsilon^*_A(a_1))$, $b \in k(r_1, \ldots, r_{n-1}, \epsilon^*_B(b_1))$. Since $A = (A, h_1, k_1)$ and $B = (B, h_2, k_2)$ are (m, n)-ary hypermodule, consequently, $(A \times B)/\widehat{\epsilon}$ is an (m, n)-ary hypermodule. Now, let θ be an equivalence relation on $A \times B$ such that $(A \times B)/\theta$ is an (m, n)-ary hypermodule. Similar to the proof of Theorem 2.3, we get

$$(a_1, b_1)\ \widehat{\epsilon}\ (a_2, b_2) \ \Rightarrow\ (a_1, b_1)\ \theta\ (a_2, b_2).$$

Therefore the relation $\widehat{\epsilon}$ is the smallest equivalence relation on $A \times B$ such that $(A \times B)/\widehat{\epsilon}$ is an (m, n)-ary hypermodule, i.e., $\widehat{\epsilon} = \epsilon^*_{A \times B}$. Now, we consider the map $\phi : A/\epsilon^*_A \times B/\epsilon^*_B \to (A \times B)/\epsilon^*_{A \times B}$, by $\phi(\epsilon^*_A(a), \epsilon^*_B(b)) = \epsilon^*_{A \times B}(a, b)$. □

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