On Super and Restricted Connectivity of Some Interconnection Networks*

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Abstract

The super (resp., edge-) connectivity of a connected graph is the minimum cardinality of a vertex-cut (resp., an edge-cut) whose removal does not isolate a vertex. In this paper, we consider the two parameters for a special class of graphs $G(G_0, G_1; M)$, proposed by Chen *et al* [*Applied Math. and Computation*, 140 (2003), 245-254], obtained from two k-regular k-connected graphs G_0 and G_1 with the same order by adding a perfect matching between their vertices. Our results improve ones of Chen *et al.* As applications, the super connectivity and the super edge-connectivity of the n-dimensional hypercube, twisted cube, cross cube, Möbius cube and locally twisted cube are all $2n-2$.

Keywords: Connectivity; super connectivity; restricted connectivity; hypercubes; twisted cubes; cross cubes; Möbius cubes; locally twisted cubes

AMS Subject Classification: 05C40 90B10

1 Introduction

We follow [19] for graph-theoretical terminology and notation not defined here. Throughout this paper, a graph $G = (V, E)$ always means a simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. The symbols $K_{1,n-1}$ and K_n denote a star graph and a complete graph with order n, respectively. For a subset $X \subset V(G)$, the symbol $\partial_G(X)$ the set of edges incident with some vertex

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in X. $\xi(G) = \min\{d_G(x) + d_G(y) : e = xy \in E(G)\} - 2$ is the minimum edge-degree of G.

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph G, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important measurements for fault-tolerance of the network [18]. The two parameters, however, have a obvious deficiency, that is to tacitly assume that all elements in any subset of G can potentially fail at the same time. To compensate for this shortcoming, Bauer *et al* [4] suggested the concept of the super connectedness. A connected graph is said to be super vertex-connected (resp., super edge-connected), if every minimum vertex-cut (resp., edge-cut) isolates a vertex. Many super connected graphs have been found in the literature (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 15, 16]). A quite natural problem is that if a connected graph G is super vertex-connected or super edge-connected then how many vertices or edges must be removed to disconnect G such that every component of the resulting graph contains no isolated vertices. This problem results in the concept of the super connectivity, introduced in [13] (see also [3, 15]).

A subset $F \subset V(G)$ is said to be nontrivial if it contains no $N_G(x)$ as its subset for some vertex $x \in V(G) \setminus F$, and a subset $B \subset E(G)$ is said to be nontrivial if it contains no $\partial_G(x)$ as its subset for some vertex $x \in V(G)$. A nontrivial vertex-set (reps., edge-set) S is called a nontrivial vertex-cut (resp., edge-cut) if $G - S$ disconnected. The super vertex-connectivity $\kappa_s(G)$ (resp., edge-connectivity $\lambda_s(G)$) of a connected graph G is defined as the minimum cardinality of a nontrivial vertex-cut (resp. edge-cut) if G has a nontrivial vertex-cut (resp., a nontrivial edge-cut), and does not exist otherwise, denoted by ∞.

Esfahanian and Hakimi [11, 12] generalized the notion of connectivity by introducing the concept of the restricted connectivity in point of view of network applications. A set $S \subset V(G)$ (resp., $S \subset E(G)$) is called a restricted vertex-set (resp., edge-set) if it contains no $N_G(x)$ (resp., $\partial_G(x)$) as its subset for any vertex $x \in V(G)$. A restricted vertex-set (resp., edge-set) S is called a restricted vertex-cut (resp., edge-cut) if $G - S$ is disconnected. The restricted vertex-connectivity (resp., edge-connectivity) of a connected graph G, denoted by $\kappa_r(G)$ (resp., $\lambda_r(G)$), is defined as the minimum cardinality of a restricted vertex-cut (resp., edge-cut) if G has a restricted vertex-cut (resp., edge-cut), and does not exist otherwise.

The four parameters $\kappa_s, \kappa_r, \lambda_s$ and λ_r in conjunction with κ and λ can provide more accurate measurements for fault tolerance of a large-scale interconnection network. What relationships exist between κ_s and κ_r, λ_s and λ_r?

From definitions, there is no difference between two concepts of nontrivial edge-cuts and restricted edge-cuts, and so $\lambda_s(G) = \lambda_r(G)$ for any graph G provided they exist. There are a number of journal papers on $\lambda_s(G)$ or $\lambda_r(G)$ (see, for example, [2, 3, 11, 12, 14, 17, 20]), due to the fact Esfahanian and Hakimi [12] solved the existence of $\lambda_r(G)$ for a graph G by proving the following proposition.

Proposition 1 If G is neither $K_{1,n}$ nor K_3, then $\lambda(G) \leq \lambda_r(G) \leq \xi(G)$.

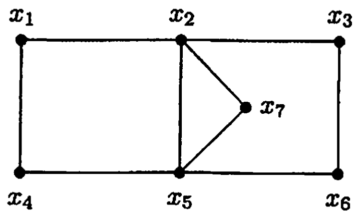

Figure 1: $\kappa_r(G)$ does not exist, while $\kappa_s(G) = 3$

However, there is a slightly difference between two concepts of nontrivial vertex-cuts and restricted vertex-cuts. For example, consider the graph G shown in Figure 1. The graph G has a unique nontrivial vertex-cut $S = \{x_2, x_5, x_7\}$, and no restricted vertex-cut, and so $\kappa_s(G) = 3$. A unique possible restricted vertex-cut is also S, however, it contains $N_G(x_7)$ as a subset. Thus, $\kappa_r(G)$ does not exist. Up to now, a few results on $\kappa_s(G)$ and $\kappa_r(G)$ for a graph G have been known. Indeed, the existence of $\kappa_s(G)$ and $\kappa_r(G)$ has not been yet solved for a general graph G. However, for a graph G if $\kappa_r(G)$ exists then $\kappa_s(G)$ exists and $\kappa_s(G) \leq \kappa_r(G)$ since any restricted vertex-cut is certainly a nontrivial vertex-cut. Conversely, if $\kappa_s(G)$ does not exist then $\kappa_r(G)$ does not exist. The following proposition holds obviously, which shows relationships between the super connectivity and the restricted connectivity.

Proposition 2 Let G be a connected graph, neither $K_{1,n}$ nor K_3. Then

(1) $\kappa_r(G) \geq \kappa_s(G) \geq \kappa(G)$, and if $\kappa_s(G) > \kappa(G) = \delta(G)$ then G is super-connected.

(2) $\lambda_r(G) = \lambda_s(G) \geq \lambda(G)$, and if $\lambda_s(G) > \lambda(G) = \delta(G)$ then G is super edge-connected.

In this paper, we consider a special class of graphs, proposed by Chen *et al* [10]. Let G_0 and G_1 be two k-regular k-connected graphs with n vertices, and M be an arbitrary perfect matching between the vertices of G_0 and G_1. The graph $G(G_0, G_1; M)$ is defined as a graph G with the vertex-set

$V(G) = V(G_0) \cup V(G_1)$, and the edge-set $E(G) = E(G_0) \cup E(G_1) \cup M$. We will call the edges in M cross-edges. The well-known n-dimensional hypercube Q_n, the twisted cube TQ_n, the cross cube CQ_n, the Möbius cube MQ_n and the locally twisted cube LTQ_n, each of them can be viewed as a special $G(G_0, G_1; M)$ for some two graphs G_0, G_1 and some perfect matching M. Chen *et al* [10] have shown that $G(G_0, G_1; M)$ is super vertex-connected if and only if either $n > k+1$ or $n = k+1$ with $k = 2$, and is super edge-connected if and only if $n > k+1$. Applying these results, they proved Q_n, TQ_n, CQ_n and MQ_n all are super vertex-connected and super edge-connected.

We, in the present paper, study the super connectivity and the restricted connectivity of the graph $G(G_0, G_1; M)$. As applications, we determine these parameters for Q_n, TQ_n, CQ_n, MQ_n and LTQ_n all being $2n-2$. The above-mentioned Chen *et al*'s results will be referred to as direct consequences of our results.

2 Main Results

The following theorem holds obviously, the proof is omitted here.

Theorem 1 $\kappa(G(G_0, G_1; M)) = \lambda(G(G_0, G_1; M)) = k+1$ if and only if $n \geq k+1$ for any $k \geq 1$. ∎

Theorem 2 Let $G = G(G_0, G_1; M)$ and $k \geq 2$. Then
(1) $\kappa_s(G) = \kappa_r(G) = k+1$ if and only if $n = k+1$ with $k \geq 3$;
(2) $k+1 < \kappa_s(G) \leq \kappa_r(G) \leq 2k$ if and only if either $n \geq k+2$; and
(3) $\kappa_s(G) = \kappa_r(G) = 2k$ if each of G_0 and G_1 contains no triangles.

Proof (1) Assume $\kappa_s(G) = \kappa_r(G) = k+1$. Then $n \geq k+1$ with $k \geq 2$ and there is a vertex-cut S with $|S| = k+1$ such that every component of $G - S$ contains no isolated vertices. Clearly, no component of $G - S$ is included in both G_0 and G_1 since $\kappa(G_i) = k$ for $i = 1, 2$ and $2k > k+1$ for $k \geq 2$. Moreover, $G - S$ contains exactly two components, the one in G_0 and the other in G_1. Let H be the only component of $G - S$ that is included in G_0. Then $N_G(H) = S$. Because every vertex in H is matched by M with a vertex in G_1, we have $|V(H)| = |N_G(H) \cap V(G_1)|$. It follows that

$$\begin{aligned} k+1 \leq \;& n = |V(G_0)| = |V(H)| + |S \cap V(G_0)| \\ =\;& |N_G(H) \cap V(G_1)| + |S \cap V(G_0)| \\ \leq\;& |S| = k+1, \end{aligned}$$

which gives $n = k+1$. Thus, both G_0 and G_1 are isomorphic to K_{k+1} since they are k-regular. Since $G - S$ has at least two components and every component has at least two vertices, thus, $2k+2 = 2n \geq (k+1)+4 = k+5$, which yields $k \geq 3$.

Conversely, we clearly have $k+1 = \kappa(G) \le \kappa_s(G) \le \kappa_r(G)$ by Theorem 1 and Proposition 2. We want to show $\kappa_r(G) \le k+1$ if $n = k+1$ with $k \ge 3$. Let u and v be any two adjacent vertices in G_0 and $S = N_G(u,v)$. Then $|S| = k+1$ since both G_0 and G_1 both are isomorphic to a complete graph K_{k+1}. Moreover, $G - S$ is disconnected since $2n - (k+1) - 2 \ge 2$ for $k \ge 3$. It is easy to verify that S is a restricted vertex-cut of G. Thus, $\kappa_r(G) \le |S| = k+1$.

(2) We note that $k \ge 2$ is required only by $n \ge k+2$ and to ensure $2k > k$. By the conclusion in (1), we only need to show that $\kappa_r(G) \le |S| \le 2k$ for $n \ge k+2$. To this aim, let us arbitrarily choose two adjacent vertices u and v in G_0 and $S = N_G(u,v)$. Then, $|S| \le 2k$. Clearly, $G - S$ is disconnected since $2n - (2k) - 2 \ge 2$ for $n \ge k+2$. Since for every vertex in G, at least one of its neighbors is not in S, S is a restricted vertex-cut of G. This shows that $\kappa_r(G)$ exists and $\kappa_r(G) \le |S| \le 2k$.

(3) Clearly, the hypothesis that G_i contains no triangles and $k \ge 2$ implies $n \ge k+2$. By the conclusion in (2), which also implies the existence of $\kappa_s(G)$, we only need to prove $\kappa_s(G) \ge 2k$ if each of G_0 and G_1 contains no triangles. To the end, we only need to show that for any nontrivial vertex-set F in G, if $|F| \le 2k-1$ then then $G - F$ is connected.

Let $F_0 = F \cap V(G_0)$, and $F_1 = F \cap V(G_1)$. Obviously, $F_0 \cap F_1 = \emptyset$. Thus, either $|F_0| \le k-1$ or $|F_1| \le k-1$. We can, without loss of generality, suppose that $|F_1| \le k-1$. Then $G_1 - F_1$ is connected since $\kappa(G_1) = k$. We show that any vertex u_0 in $G_0 - F_0$ can be connected to the connected graph $G_1 - F_1$. Let u_0u_1 be a cross-edge, where $u_1 \in V(G_1)$. If $u_1 \notin F_1$, then we are done. So we assume that $u_1 \in F_1$. Since F is a nontrivial vertex-set, there exists a vertex v_0 adjacent to u_0 in $G_0 - F_0$. Consider $N = N_G(u_0, v_0)$. Then $|N| = 2k > 2k-1$ since G contains no triangles. Thus, there is a vertex $x_0 \in N \cap V(G_0)$ such that the cross-edge x_0x_1, where $x_1 \in V(G_1)$, is not incident with any vertex in F. This implies that u_0 in $G_0 - F_0$ can be connected to $G_1 - F_1$ via the cross-edge x_0x_1.

Thus, we show that $|S| \ge 2k$ for any nontrivial vertex-cut S in G, that is, $\kappa_s(G) \ge 2k$ if each of G_0 and G_1 contains no triangles. The theorem follows. ∎

By Theorem 2 and Proposition 2, we obtain the following corollary immediately.

Corollary (Chen *et al* [10]) $G(G_0, G_1; M)$ is $(k+1)$-regular super connected if and only if either $n > k+1$ or $n = k+1$ with $k = 1, 2$. ∎

Theorem 3 Let $G = G(G_0, G_1; M)$ and $k \ge 1$. Then
(1) $\lambda_s(G) = \lambda_r(G) = k+1$ if and only if $n = k+1$;
(2) $k+1 < \lambda_s(G) = \lambda_r(G) \le 2k$ if and only if $n \ge k+2$; and

(3) $\lambda_s(G) = \lambda_r(G) = 2k$ if each of G_0 and G_1 contains no triangles.

Proof (1) Clearly, $n \geq k+1$ is necessary for $k \geq 1$. Assume $\lambda_s(G)) = k+1$, then there is a nontrivial edge-cut S with $|S| = k+1$ such that $G - S$ contains no isolated vertices. Let H be a component of $G - S$ and let $h = V(H)$. Then $\partial_G(H) = S$. We can easily verify that H is certainly included in one of G_0 and G_1. Without loss of generality, assume that H is included in G_0. Consider the degree-sum of vertices in H. Since G is $(k+1)$-regular, we have

$$\begin{aligned} h(h-1) \geq & \sum_{x \in V(H)} d_H(x) = \sum_{x \in V(H)} d_G(x) - |S| \\ = & \; h(k+1) - (k+1) = (h-1)(k+1), \end{aligned}$$

from which we have $h \geq k+1$. On the other hand, noting that $\partial_G(H) = S$ and every vertex in H is matched by M with a vertex in G_1, we have $|S \cap E(G_0)| = |S| - h \leq 0$, which implies $H = G_0$, $\partial_G(H) = S = M$, and so $h \leq |M| = |S| = k+1$. Thus, we have $n = h = k+1$.

Conversely, if $n = k+1$ with $k \geq 1$, then G_0 and G_1 are isomorphic to K_n. Clearly, the perfect matching M is a nontrivial edge-cut. Thus, $k+1 = \lambda(G) \leq \lambda_s(G) \leq |M| = n = k+1$, which yields $\lambda_r(G) = k+1$.

(2) By the conclusion in (1) and Proposition 1, the assertion holds clearly.

(3) By Proposition 2 and the conclusion in (2), we need to prove that $\lambda_s(G) \geq 2k$ if each of G_0 and G_1 contains no triangles.

Assume that F is a nontrivial edge-set of G. We need to prove that if $|F| \leq 2k-1$ then $G - F$ is connected. Since $\lambda(G_0) = \lambda(G_1) = k$, at least one of $G_0 - F$ and $G_1 - F$ is connected. We can, without loss of generality, suppose that $G_0 - F$ is connected. In order to prove that $G - F$ is connected, we only need to show that any vertex x_1 in G_1 can be connected to some vertex in $G_0 - F$.

If the cross edge x_0x_1 is not in F, then there is noting to do. Suppose that $x_0x_1 \in F$. Since F is a nontrivial edge-set, which does not isolate a vertex, there exists an edge x_1y_1 in G_1 such that $x_1y_1 \notin F$. Since G contains no triangles, then $|N_G(x_1, y_1)| = 2k > 2k-1$. Thus, there exists at least one $u_1 \in N_G(x_1, y_1)$ such that the cross-edge u_0u_1 is not F. Thus, x_1 can be connected to $G_0 - F$ via the cross-edge u_0u_1.

Thus, we show that $|S| \geq 2k$ for any nontrivial edge-cut S in G, that is, $\lambda_s(G) \geq 2k$.

The theorem follows. ∎

By Theorem 3 and Proposition 1, we obtain the following corollary immediately.

Corollary (Chen *et al* [10]) $G(G_0, G_1; M)$ is $(k+1)$-regular super edge-connected if and only if $n > k+1$ for any $k \geq 1$. ∎

3 Applications

Topologies of many interconnection networks can be viewed as $G(G_0, G_1; M)$ for some k-regular graphs G_0 and G_1, such as the hypercube Q_n, the twisted cube TQ_n, the cross cube CQ_n, the Möbius cube MQ_n and the locally twisted cube LTQ_n. Chen *et al* [10] have proved that each of these networks is super connected and super edge-connected. Applying our results, we immediately obtain that their super connectivity, super edge-connectivity, restricted connectivity and restricted edge-connectivity all are $2n-2$ for $n \geq 3$ and, thus, are super connected and super edge-connected. The proofs are omitted here.

Acknowledgements

The authors would like to express their gratitude to the anonymous referee for his/her suggestions and useful comments on the original manuscript, which result in the present version of this paper.

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