An Extremal Problem On Potentially $K_{r+1} - H$-graphic Sequences *

Chunhui Lai, Lili Hu
Department of Mathematics, Zhangzhou Teachers College,
Zhangzhou, Fujian 363000, P. R. of CHINA.
e-mail: zjlaichu@public.zzptt.fj.cn (Chunhui Lai)

Abstract

Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). A sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - H, n)$ for $n \geq 4r+10, r \geq 3, r+1 \geq k \geq 4$ where H is a graph on k vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices. We also determine the values of $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r+8, r \geq 3$.

Key words: graph; degree sequence; potentially $K_{r+1} - H$-graphic sequence
AMS Subject Classifications: 05C07, 05C35

1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d(v_1), d(v_2), ..., d(v_n))$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and

*Project Supported by NNSF of China(10271105), NSF of Fujian(Z0511034), Science and Technology Project of Fujian, Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering" , Project of Fujian Education Department and Project of Zhangzhou Teachers College.

such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d(v_1)+d(v_2)+...+d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. We denote $G+H$ as the graph with $V(G+H) = V(G)\bigcup V(H)$ and $E(G+H) = E(G)\bigcup E(H)\bigcup\{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m).

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and generalized by Turán [15]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5],[10],[11],[12] and [13].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Luo [14] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Lai [7] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin,Li and Mao[17] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r+1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$, and Yin and Li [16] further determined $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$. Moreover, Yin and Li in [16] also gave two sufficient conditions for a sequence $\pi \epsilon GS_n$ to be potentially A_{r+1}-graphic and two sufficient conditions for a sequence $\pi \epsilon GS_n$ to be potentially $K_{r+1} - e$-graphic. Yin [18] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r+5, r \geq 3$. Lai [8, 9] determined $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for $n \geq 5$. In this paper, we prove the following three theorems.

Theorem 1.1. If $r \geq 3$ and $n \geq 4r+8$, then $\sigma(K_{r+1} - P_2, n) = (r-1)(2n-r) - 2(n-r) + 2$.

Theorem 1.2. If $r \geq 3$ and $n \geq 4r+10$, then $\sigma(K_{r+1} - T_3, n) = (r-1)(2n-r) - 2(n-r)$.

Theorem 1.3. If $r \geq 3, r+1 \geq k \geq 4$ and $n \geq 4r+10$, then $\sigma(K_{r+1} -$

$H, n) = (r-1)(2n-r) - 2(n-r)$, where H is a graph on k vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices.

There are a number of graphs on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices (for example, the cycle on k vertices, the tree on k vertices, and the complete 2-partite graph on k vertices, etc).

2 Preparations

In order to prove our main result,we need the following notations and results.

Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n, 1 \le k \le n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{if } d_k \ge k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi_k' = (d_1', d_2', \cdots, d_{n-1}')$,where $d_1' \ge d_2' \ge \cdots \ge d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π.

Theorem 2.1[16] Let $n \ge r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \ge r$. If $d_i \ge 2r - i$ for $i = 1, 2, \cdots, r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2[16] Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3[16] Let $n \ge r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \ge r-1$. If $d_i \ge 2r - i$ for $i = 1, 2, \cdots, r-1$, then π is potentially $K_{r+1} - e$-graphic.

Theorem 2.4[16] Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r-1} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially $K_{r+1} - e$ -graphic.

Theorem 2.5[6] Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n$ and $1 \le k \le n$. Then $\pi \epsilon GS_n$ if and only if $\pi_k' \epsilon GS_{n-1}$.

Theorem 2.6[3] Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n$ with even $\sigma(\pi)$. Then $\pi \epsilon GS_n$ if and only if for any t,$1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{j=t+1}^{n} min\{t, d_j\}.$$

Theorem 2.7[5] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization

G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Lemma 2.1 [18] If $\pi = (d_1, d_2, \cdots, d_n) \epsilon NS_n$ is potentially $K_{r+1} - e$-graphic, then there is a realization G of π containing $K_{r+1} - e$ with the $r+1$ vertices $v_1, \cdots, v_{r+1}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \cdots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 [18] If $r \geq 3$ and $n \geq r+1$, then $\sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

3 Proof of Main results.

Lamma 3.1 Let $n \geq r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_r \geq r-1$ and $d_{r+1} \geq r-2$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r-2$, then π is potentially $K_{r+1} - P_2$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r-1$.

Subcase 1.1: $d_{r-1} \geq r+1$. Then π is potentially $K_{r+1} - e$-graphic by Theorem 2.3. Hence,π is potentially $K_{r+1} - P_2$-graphic.

Subcase 1.2: $d_{r-1} = r-1$. Then $d_{r-1} = d_r = d_{r+1} = r-1$.

If $d_{r+2} = r-1$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$, (2) $d'_1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r, d'_{(r-1)+1} = d'_r = d_{r+2} = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r+2} \leq r-2$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$, (2) $d'_1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r, d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r-2$. By Theorem 2.3, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Subcase 1.3: $d_{r-1} = r$. Then $d_{r+1} = r$ or $r-1$.

If $d_{r+1} = r$, then $d_{r-1} = d_r = d_{r+1} = r$. The residual sequence π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r-1) - (r-2)$ and $d'_{(r-1)+1} = d'_r \geq d_r - 1 = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Thus, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r+1} = r-1$, then $d_r = r-1$ or r.

If $d_r = r-1$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-1$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1)-(r-2)$ and $d'_{(r-1)+1} = d'_r = d_r = r-1$. According to Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-1} - 1, d_r\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_r = r$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1) - (r-2)$ and $d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Case 2: $d_{r+1} \leq r-2$, that is, $d_{r+1} = r-2$.

If $d_{r-1} < d_{r-2}$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1) - [(r-1)-1]$ and $d'_{(r-1)+1} = d'_r = d_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r-1} = d_{r-2} \geq r+2$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r-1) - [(r-1)-1]$ and $d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_{r-1}, d_r, d_1 - 1, \cdots, d_{r-2} - 1\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.2. Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r-2} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - P_2$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-1} \geq r$. Then π is potentially $K_{r+1} - e$-graphic by Theorem 2.4. Hence, π is potentially $K_{r+1} - P_2$-graphic.

Case 2: $d_{r-1} \leq r-1$, that is, $d_{r-1} = r-1$, then $d_{r-1} = d_r = d_{r+1} = \cdots = d_{2r+2} = r-1$ and π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_{(r-1)+1} = d'_r \geq r-1$ and $d'_{2(r-1)+2} = d'_{2r} \geq (r-1) - 1$. By Theorem 2.2, π'_{r+1} is potentially A_r-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.3. If $r \geq 3$ and $n \geq r+1$, then $\sigma(K_{r+1} - P_2, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

Proof. By Lemma 2.2, for $r \geq 3$ and $n \geq r+1$, $\sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$. Obviously, for $r \geq 3$ and $n \geq r+1$, $\sigma(K_{r+1} - P_2, n) \geq \sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

Lemma 3.4. If $r \geq 3, r+1 \geq k \geq 4$ and $n \geq r+1$, then $\sigma(K_{r+1} - H, n) \geq (r-1)(2n-r) - 2(n-r)$, for H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices.

Proof. Let

$$G = K_{r-2} + \overline{K_{n-r+2}}$$

Then G is a unique realization of $((n-1)^{r-2}, (r-2)^{n-r+2})$ and G clearly does not contain $K_{r+1} - H$, where the symbol x^y means x repeats y times in the sequence. Thus

$$\sigma(K_{r+1}-H, n) \geq (r-2)(n-1)+(r-2)(n-r+2)+2 = (r-1)(2n-r)-2(n-r).$$

The Proof of Theorem 1.1 According to Lemma 3.3, it is enough to verify that for $r \geq 3$ and $n \geq 4r+8$,

$$\sigma(K_{r+1} - P_2, n) \leq (r-1)(2n-r) - 2(n-r) + 2.$$

We now prove that if $n \geq 4r+8$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with

$$\sigma(\pi) \geq (r-1)(2n-r) - 2(n-r) + 2,$$

then π is potentially $K_{r+1} - P_2$-graphic.

If $d_{r-2} \leq r-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-1)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) \\ &= (r-1)(2n-r) - 2(n-r) \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Thus $d_{r-2} \geq r$.

If $d_r \leq r-2$, then

$$\begin{aligned} \sigma(\pi) &= \textstyle\sum_{i=1}^{r-1} d_i + \sum_{i=r}^{n} d_i \\ &\leq (r-1)(r-2) + \textstyle\sum_{i=r}^{n} min\{r-1, d_i\} + \sum_{i=r}^{n} d_i \\ &= (r-1)(r-2) + 2\textstyle\sum_{i=r}^{n} d_i \\ &\leq (r-1)(r-2) + 2(n-r+1)(r-2) \\ &= (r-1)(2n-r) - 2(n-r) - 2 \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Hence $d_r \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{aligned} \sigma(\pi) &= \textstyle\sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^{n} d_i \\ &\leq (r-1)(r-2) + \textstyle\sum_{i=r}^{n} min\{r-1, d_i\} + d_r + \sum_{i=r+1}^{n} d_i \\ &= (r-1)(r-2) + min\{r-1, d_r\} + d_r + 2\textstyle\sum_{i=r+1}^{n} d_i \\ &\leq (r-1)(r-2) + 2d_r + 2\textstyle\sum_{i=r+1}^{n} d_i \\ &\leq (r-1)(r-2) + 2(n-1) + 2(n-r)(r-3) \\ &= (r-1)(2n-r) - 2(n-r) \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Thus $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i=1,2,\cdots,r-2$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-P_2$-graphic by Lemma 3.1 or Lemma 3.2. If $d_{2r+2} \leq r-2$ and there exists an integer i,$1 \leq i \leq r-2$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}\sigma(\pi) &\leq (i-1)(n-1)+(2r+1-i+1)(2r-i-1) \\ &\quad +(r-2)(n+1-2r-2) \\ &= i^2+i(n-4r-2)-(n-1)+(2r-1)(2r+2) \\ &\quad +(r-2)(n-2r-1).\end{aligned}$$

Since $n \geq 4r+8$, it is easy to see that $i^2+i(n-4r-2)$, consider as a function of i, attains its maximum value when $i=r-2$. Therefore,

$$\begin{aligned}\sigma(\pi) &\leq (r-2)^2+(n-4r-2)(r-2)-(n-1) \\ &\quad +(2r-1)(2r+2)+(r-2)(n-2r-1) \\ &= (r-1)(2n-r)-2(n-r)+2-n+4r+7 \\ &< \sigma(\pi),\end{aligned}$$

which is a contradiction.

Thus, $\sigma(K_{r+1}-P_2,n) \leq (r-1)(2n-r)-2(n-r)+2$ for $n \geq 4r+8$.

The Proof of Theorem 1.2 According to Lemma 3.4, it is enough to verify that for $r \geq 3$ and $n \geq 4r+10$,

$$\sigma(K_{r+1}-T_3,n) \leq (r-1)(2n-r)-2(n-r).$$

We now prove that if $n \geq 4r+10$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with

$$\sigma(\pi) \geq (r-1)(2n-r)-2(n-r),$$

then π is potentially $K_{r+1}-T_3$-graphic.

If $d_{r-2} \leq r-1$, we consider the following cases.

(1)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)=(r-3)(n-1)+(r-1)(n-r+3)$, then $\pi=((n-1)^{r-3},(r-1)^{n-r+3})$. Obviously π is potentially $K_{r+1}-T_3$ graphic.

(2)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)<(r-3)(n-1)+(r-1)(n-r+3)$, then

$$\begin{aligned}\sigma(\pi) &< (r-3)(n-1)+(r-1)(n-r+3) \\ &= (r-1)(n-1)-2(n-1)+(r-1)(n-r+3) \\ &= (r-1)(2n-r)-2(n-r),\end{aligned}$$

which is a contradiction.

(3)Suppose $d_{r-2}<r-1$, then

$$\begin{aligned}\sigma(\pi) &< (r-3)(n-1)+(r-1)(n-r+3) \\ &= (r-1)(n-1)-2(n-1)+(r-1)(n-r+3) \\ &= (r-1)(2n-r)-2(n-r),\end{aligned}$$

which is a contradiction.

Thus, $d_{r-2} \geq r$ or π is potentially $K_{r+1} - T_3$ graphic.

If $d_r \leq r-2$, then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{r-1} d_i + \sum_{i=r}^{n} d_i \\
&\leq (r-1)(r-2) + \sum_{i=r}^{n} min\{r-1, d_i\} + \sum_{i=r}^{n} d_i \\
&= (r-1)(r-2) + 2\sum_{i=r}^{n} d_i \\
&\leq (r-1)(r-2) + 2(n-r+1)(r-2) \\
&= (r-1)(2n-r) - 2(n-r) - 2 \\
&< (r-1)(2n-r) - 2(n-r),
\end{aligned}$$

which is a contradiction. Hence $d_r \geq r-1$.

If $d_{r+1} \leq r-3$, we consider the following cases.

(1)Suppose $d_r = n-1$, then $d_1 \geq d_2 \geq \cdots \geq d_{r-1} \geq d_r = n-1$, therefore $d_1 = d_2 = \cdots = d_r = n-1$. Therefore $d_{r+1} \geq r$, which is a contradiction.

(2)Suppose $d_r \leq n-2$, then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^{n} d_i \\
&\leq (r-1)(r-2) + \sum_{i=r}^{n} min\{r-1, d_i\} + d_r + \sum_{i=r+1}^{n} d_i \\
&= (r-1)(r-2) + min\{r-1, d_r\} + d_r + 2\sum_{i=r+1}^{n} d_i \\
&\leq (r-1)(r-2) + 2d_r + 2\sum_{i=r+1}^{n} d_i \\
&\leq (r-1)(r-2) + 2(n-2) + 2(n-r)(r-3) \\
&= (r-1)(2n-r) - 2(n-r) - 2 \\
&< (r-1)(2n-r) - 2(n-r),
\end{aligned}$$

which is a contradiction.

Thus $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-2$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - T_3$ graphic($\pi = ((n-1)^{r-3}, (r-1)^{n-r+3})$) or π is potentially $K_{r+1} - P_2$-graphic by Lemma 3.1 or Lemma 3.2 . Therefore, π is potentially $K_{r+1} - T_3$-graphic. If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-2$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}
\sigma(\pi) &\leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\
&\quad +(r-2)(n+1-2r-2) \\
&= i^2 + i(n-4r-2) - (n-1) \\
&\quad +(2r-1)(2r+2) + (r-2)(n-2r-1).
\end{aligned}$$

Since $n \geq 4r+10$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i, attains its maximum value when $i = r-2$. Therefore,

$$\begin{aligned}
\sigma(\pi) &\leq (r-2)^2 + (n-4r-2)(r-2) - (n-1) \\
&\quad +(2r-1)(2r+2) + (r-2)(n-2r-1) \\
&= (r-1)(2n-r) - 2(n-r) - n + 4r + 9 \\
&< \sigma(\pi),
\end{aligned}$$

which is a contradiction.

Thus, $\sigma(K_{r+1} - T_3, n) \leq (r-1)(2n-r) - 2(n-r)$ for $n \geq 4r + 10$.

The Proof of Theorem 1.3 By Lemma 3.4, for $r \geq 3, r+1 \geq k \geq 4$ and $n \geq r+1$, $\sigma(K_{r+1} - H, n) \geq (r-1)(2n-r) - 2(n-r)$. Obviously, for $r \geq 3, r+1 \geq k \geq 4$ and $n \geq 4r+10$, $\sigma(K_{r+1} - H, n) \leq \sigma(K_{r+1} - T_3, n)$. By theorem 1.2, for $r \geq 3, r+1 \geq k \geq 4$ and $n \geq 4r+10$, $\sigma(K_{r+1} - T_3, n) = (r-1)(2n-r) - 2(n-r)$. Then $\sigma(K_{r+1} - H, n) = (r-1)(2n-r) - 2(n-r)$, for $r \geq 3, r+1 \geq k \geq 4$ and $n \geq 4r + 10$.

Acknowledgment

The authors thanks the referees for many helpful comments.

References

[1] B. Bollabás, Extremal Graph Theory, Academic Press, London, 1978.

[2] P. Erdös, On sequences of integers no one of which divides the product of two others and some related problems, Izv. Naustno-Issl. Mat. i Meh. Tomsk 2(1938), 74-82.

[3] P. Erdös and T. Gallai , Graphs with given degrees of vertices, Math. Lapok,11(1960),264-274.

[4] P.Erdös, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in Graph Theory, Combinatorics and Application, Vol. 1(Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.

[5] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G-graphic degree sequences,in Combinatorics, Graph Theory and Algorithms,Vol. 2 (Y. Alavi et al.,eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.

[6] D.J. Kleitman and D.L. Wang , Algorithm for constructing graphs and digraphs with given valences and factors,Discrete Math., 6(1973),79-88.

[7] Chunhui Lai, A note on potentially $K_4 - e$ graphical sequences, Australasian J. of Combinatorics 24(2001), 123-127.

[8] Chunhui Lai, An extremal problem on potentially $K_m - C_4$-graphic sequences, Journal of Combinatorial Mathematics and Combinatorial Computing, to appear

[9] Chunhui Lai, An extremal problem on potentially $K_m - P_k$-graphic sequences, submitted

[10] Jiong-Sheng Li and Zi-Xia Song, An extremal problem on the potentially P_k-graphic sequences, Discrete Math., 212(2000), 223-231.

[11] Jiong-Sheng Li and Zi-Xia Song, The smallest degree sum that yields potentially P_k-graphical sequences, J. Graph Theory,29(1998), 63-72.

[12] Jiong-sheng Li and Zi-Xia Song, On the potentially P_k-graphic sequences, Discrete Math. 195(1999), 255-262.

[13] Jiong-sheng Li, Zi-Xia Song and Rong Luo, The Erdös-Jacobson-Lehel conjecture on potentially P_k-graphic sequence is true, Science in China(Series A), 41(5)(1998), 510-520.

[14] Rong Luo, On potentially C_k-graphic sequences, Ars Combinatoria 64(2002), 301-318.

[15] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48(1941), 436-452.

[16] Jianhua Yin and Jiongsheng Li,Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size,Discrete Math.,301(2005) 218-227.

[17] Jianhua Yin, Jiongsheng Li and Rui Mao,An extremal problem on the potentially $K_{r+1} - e$-graphic sequences,Ars Combinatoria,74(2005),151-159.

[18] Mengxiao Yin, The smallest degree sum that yields potentially $K_{r+1} - K_3$-graphic sequences, submitted