A summation formula related to q-Eulerian polynomials

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Abstract

Brenti (J. Combin. Theory Ser. A 91 (2000)) considered a q-analogue of the Eulerian polynomials by enumerating permutations in the symmetric group S_n with respect to the numbers of excedances and cycles. Here we establish a connection between these q-Eulerian polynomials and some infinite generating functions.

Keywords: q-Eulerian polynomials; Summation formula

1 Introduction

Let S_n denote the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$. An *excedance* in a permutation $\pi := \pi(1)\pi(2)\cdots\pi(n)$ is an index i such that $\pi(i) > i$. As usual, we will denote by $\operatorname{exc}(\pi)$ the number of excedances of π, and by $\operatorname{cyc}(\pi)$ the number of cycles of π. For example, the permutation $\pi = 315426 \in S_6$ has the cycle decomposition $\pi = (1,3,5,2)(4)(6)$, so $\operatorname{exc}(\pi) = 2$ and $\operatorname{cyc}(\pi) = 3$. It is evident that $\operatorname{exc}(\pi) \leq n - \operatorname{cyc}(\pi)$ for $\pi \in S_n$. The ordinary Eulerian polynomials $A_n(x)$ are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in S_n} x^{\operatorname{exc}(\pi)+1} \quad \text{for } n \geq 1.$$

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Let $A(n,k)$ be the coefficient of x^k in $A_n(x)$. The number $A(n,k)$ is called an Eulerian number (see, e.g., [2, Chapter. VI]). They satisfy the symmetry relation

$$A(n,k) = A(n, n-k+1) \quad \text{for } 1 \le k \le n.$$

The historical origin of the ordinary Eulerian polynomials is the following summation formula:

$$\sum_{i\ge 0} i^n x^i = \frac{A_n(x)}{(1-x)^{n+1}}. \tag{1}$$

Recently, Brenti [1] considered a q-analogue of the ordinary Eulerian polynomials defined by

$$A_0(x;q) = 1, \quad A_n(x;q) = \sum_{\pi\in S_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)} \quad \text{for } n \ge 1.$$

Clearly, $A_n(x) = xA_n(x;1)$ for $n \ge 1$. Brenti obtained [1, Proposition 7.2] the recursion

$$A_{n+1}(x;q) = (nx+q)A_n(x;q) + x(1-x)\frac{\partial}{\partial x}A_n(x;q) \tag{2}$$

and showed [1, Theorem 7.5] that $A_n(x;q)$ has only simple real zeros when q is a positive rational number.

2 Results

The first few of the q-Eulerian polynomials $A_n(x;q)$ are listed below:

$$\begin{aligned}
A_0(x;q) &= 1,\\
A_1(x;q) &= q,\\
A_2(x;q) &= q(x+q),\\
A_3(x;q) &= q[x^2+(1+3q)x+q^2],\\
A_4(x;q) &= q[x^3+(4+7q)x^2+(1+4q+6q^2)x+q^3],\\
A_5(x;q) &= q[x^4+(11+15q)x^3+(11+30q+25q^2)x^2+\\
&\quad (1+5q+10q^2+10q^3)x+q^4].
\end{aligned}$$

Looking at the above list carefully, we observe the connection between these polynomials and some infinite generating functions, which is a generalization of (1).

Theorem 1. *For all nonnegative integers n, we have*

$$\sum_{i\geq 0} i^n \binom{q+i-1}{i} x^i = \frac{x^n}{(1-x)^{n+q}} A_n\left(\frac{1}{x};q\right) \tag{3}$$

in $\mathbf{Z}(q)[[x]]$.

Proof. Set $B_n(x;q) = x^n A_n(x^{-1};q)$. From (2), it is not difficult to verify that

$$B_0(x;q) = 1, \quad B_{n+1}(x;q) = (n+q)xB_n(x;q) + x(1-x)\frac{\partial}{\partial x}B_n(x;q) \quad \text{for } n \geq 1.$$

Let

$$C_n(x;q) = (1-x)^{n+q} \sum_{i\geq 0} i^n \binom{q+i-1}{i} x^i \quad \text{for } n \geq 0. \tag{4}$$

We have

$$x\frac{\partial}{\partial x}\left[\frac{C_n(x;q)}{(1-x)^{n+q}}\right] = \frac{C_{n+1}(x;q)}{(1-x)^{n+q+1}},$$

which is equivalent to

$$(n+q)xC_n(x;q) + x(1-x)\frac{\partial}{\partial x}C_n(x;q) = C_{n+1}(x;q).$$

It follows from (4) and the Binomial Theorem (see, e.g., [3, p. 16]) that $C_0(x;q) = 1$. Hence $C_n(x;q)$ satisfies the same recursion and initial conditions as $B_n(x;q)$, so they agree. This completes the proof of the theorem. □

Acknowledgements

The author thanks the anonymous referees for their careful reading and corrections.

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