The Cayley digraph associated to the Kautz digraph

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Abstract

de Bruijn digraphs and shuffle-exchange graphs are useful models for interconnection networks. They can be represented as group action graphs of the wrapped butterfly graph and the cube-connected cycles, respectively. The Kautz digraph has the similar definitions and properties to de Bruijn digraphs. It is d-regular and strongly d-connected, thus it is a group action graph. In this paper, we use another representation of the Kautz digraph and settle the open problem posed by M.-C. Heydemann in [6].
Keywords: Cayley digraph, Kautz digraph, wreath product of groups

1 Introduction

In designing interconnection networks or large scale parallel processing architectures, designers often require the following properties: small diameter, small degree, high connectivity, simple routing algorithm, simple network analyzability, and so forth. The class of Cayley digraphs has good properties for those requirements, and many researchers have proposed Cayley digraphs based on various groups [1, 4, 7, 9, 10, 12, 13]. In [2], they have been interested in the relation between group action

Table 1: Some Cayley digraphs and their associated graphs.

Cayley digraph	Group action graph
Wrapped butterfly graph $BF(d,n)$	de Bruijn digraph $B(d,n)$[2]
Cube-connected cycles $CCC(n)$	Shuffle-exchange graph $SE(n)$[2]
-Open [6]-	Kautz digraph $K(d,n)$

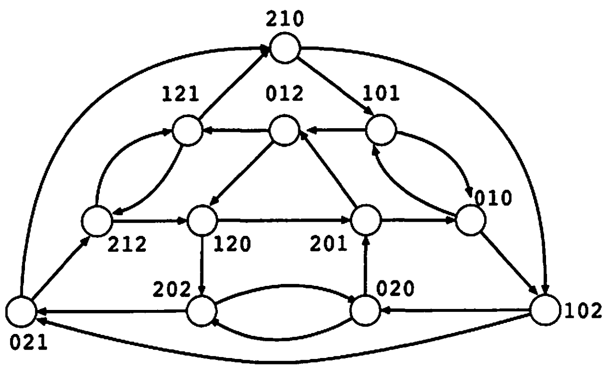

Figure 1: Kautz digraph $K(2,3)$.

graphs (GAG, for short) and Cayley digraphs, and have proved that every connected GAG is isomorphic to some Cayley coset graph (or Schreier coset graph). The advantage of Cayley coset graphs is the ability to simulate large associated Cayley digraphs with small slow-down. Some Cayley digraphs and their associated digraphs are shown in Table 1. In [11], the Cayley digraph associated to the regular digraph is called *Cayley cover.*

The d-ary n-dimensional Kautz digraph $K(d,n)$ is the digraph with the vertex set

$$V = \{x_0 x_1 \cdots x_{n-1} \in \mathbf{Z}_{d+1}^n : \ x_i \neq x_{i+1}, 0 \leq i \leq n-2\},$$

where $\mathbf{Z}_{d+1} = \{0, 1, \ldots, d\}$, and the arc set given by

$$\{(x_0 x_1 \cdots x_{n-1}, x_1 \cdots x_{n-1} x_n) : \ x_0 x_1 \cdots x_{n-1}, x_1 x_2 \cdots x_{n-1} x_n \in V\}.$$

The Kautz digraph $K(2,3)$ is illustrated in Figure 1. For any integers d, n, $K(d,n)$ is a regular digraph of out- and in-degree d and strongly

d-connected, thus it is a GAG and a coset digraph. M.-C. Heydemann proposed the following problem in [6]:

Problem 1 ([6]) *Give an explicit construction of the Cayley digraph* $\mathbf{Cay}(\mathbf{Gr}(\Pi), \Pi)$ *associated to* $K(d, n)$ *considered as* $\mathbf{GAG}(V, \Pi)$.

This problem has been studied in [3], [5] and [11]. In [3], they have treated the case when the Kautz digraph is a Cayley digraph in itself. In [5] and [11], they have treated the class of iterated line digraphs, including the de Bruijn digraph and the Kautz digraph. They have shown that the group generated by some regular digraph is a subgroup of the semidirect product of groups. In [11], when $d+1$ is a prime power, they have shown the explicit group and their construction using the finite field theory. That representation is valid in the above condition, however it seems that it is not easy to extend for any d. In [5], they have considered another representation based on permutation groups. Nevertheless, the target of their method is vast and it is difficult to say that it shows the group explicitly. For example, the cardinalities of the groups are not given. It seems difficult to know the structure of the groups from their representation.

In this paper, for $n \geq 2$ and any $d \geq 2$, we give a permutation group on the vertex set of the Kautz digraph $K(d, n)$ by using another representation of the vertex set. Second, we show the cycle decomposition of $K(d, n)$. In other words, we propose a view on the Kautz digraph as a GAG. Third, we show that the permutation group generated by our cycle decomposition is isomorphic to an explicit subgroup of the product of four cyclic groups. This construction is simple and clear. In addition, we construct a new class of Cayley cover of the Kautz digraph. These results settle the problem given by Heydemann.

This paper consists of two major parts. In Section 3, one major part, we discuss the Kautz digraph. We give the representation of the Kautz digraph and show a cycle decomposition of that digraph as a GAG. Section 4, the other major part, shows the group and their action on a specified set. Section 5.2 illustrates a relation between the Kautz digraph as a GAG in Section 3 and groups in Section 4. Section 5 gives a formal definition of the Cayley digraph associated to the Kautz digraph based on previous three sections.

2 Preliminaries

This paper treats both groups and graphs. In Section 2.1 and Section 2.2, we describe some group-theoretic definitions. In Section 2.3, we give definition of the Cayley digraph and their associated graphs.

Notations $\ominus_k$ and $\oplus_k$ are used to indicate modulo k subtraction and addition, respectively, when there exists a possibility of misreading, but we use notations $-$ and $+$ when those are correctly read.

2.1 Symmetric group, homomorphism and group action

Let S be any nonempty set. A bijection of S into itself is called a *permutation* of S. The set of all permutations of S forms a group under the composition, called the *symmetric group* on S and denoted $Sym(S)$. A *permutation group* is just a subgroup of some symmetric group.

Let Γ be a group and $\Delta = \{\delta_1, \delta_2, \ldots, \delta_k\}$ a subset of Γ. Δ is called a *generating set* of Γ if every element of Γ can be expressed as a finite multiplication of elements of Δ. Each element of the generating set is called a *generator*. For the converse, given a subset $\Delta = \{\delta_0, \delta_1, \ldots, \delta_k\}$ of a group Γ, a group $\mathbf{Gr}(\Delta)$ is a minimum subgroup of Γ which includes the subset Δ, that is, if $\gamma \in \mathbf{Gr}(\Delta)$, then $\gamma \cdot \delta_t \in \mathbf{Gr}(\Delta)$ where $t = 0, 1, \ldots, k$. Δ is said to *generate* Γ (or Γ is generated by Δ) if $\Gamma = \mathbf{Gr}(\Delta)$.

Let Γ be a group and S a nonempty set, and suppose that for each $x \in S$ and each $\gamma \in \Gamma$, we define an element of S denoted by $x \cdot \gamma$ (in other words, $(x, \gamma) \mapsto x \cdot \gamma$ is a mapping of $S \times \Gamma$ into S). Then we say that this defines an *action by right* of Γ on S (or Γ acts on S) if we have:

1. $x \cdot \epsilon = x$ for all $x \in S$ where ϵ is the identity element of Γ; and
2. $x \cdot (\alpha\beta) = (x \cdot \alpha) \cdot \beta$ for all $x \in S$ and all $\alpha, \beta \in \Gamma$.

Let Γ, Γ' be groups. A mapping $f : \Gamma \to \Gamma'$ is a *homomorphism* of Γ onto Γ' if for all $\alpha, \beta \in \Gamma$,

$$f(\alpha\beta) = f(\alpha)f(\beta).$$

Let $f : \Gamma \to \Gamma'$ be a homomorphism, and ϵ' be the identity element of Γ'. The *kernel* of f is the set $\ker(f) = \{\gamma \in \Gamma | f(\gamma) = \epsilon'\}$. A homomorphism of Γ onto $Sym(S)$ is called a (*permutation*) *representation* of Γ on S.

We see that each action of Γ on S gives rise to a representation of Γ on S, so we may think that permutation representations and group actions are different ways for describing the same situation.

Proposition 1 *Let Γ, Γ' be groups. A homomorphism $f : \Gamma \to \Gamma'$ is injective if and only if* $\ker(f) = \{\epsilon\}$ *where ϵ is the identity element of* Γ.

Let Γ be a group and S a set. A permutation representation of Γ on S is called *faithful* if it is an injective homomorphism from Γ to $Sym(S)$.

2.2 Wreath product of groups

Let $\mathbb{Z}_m$ be the cyclic group of order m, and Γ any group. *The wreath product* of the group Γ by $\mathbb{Z}_m$, denoted by $\Gamma \wr \mathbb{Z}_m$, is defined as follows: Each element of $\Gamma \wr \mathbb{Z}_m$ is denoted as

$$\pi = \langle \alpha; \beta(0) \rangle,$$

where $\alpha \in \mathbb{Z}_m$ and $\beta(0) = \beta_0\beta_1 \cdots \beta_{m-1} \in \Gamma^m$. Multiplication in the wreath product is executed in the following manner.

$$\langle \alpha; \beta(0) \rangle \cdot \langle \alpha'; \beta'(0) \rangle = \langle \alpha + \alpha'; \beta(0) \cdot \beta'(m - \alpha) \rangle,$$

where $\alpha + \alpha'$ is taken under modulo-m addition, and

$$\beta(0) \cdot \beta'(m - \alpha) = (\beta_0 \cdot \beta'_{m-\alpha}) \, (\beta_1 \cdot \beta'_{m-\alpha+1}) \cdots (\beta_{m-1} \cdot \beta'_{m-\alpha+(m-1)}).$$

For each i, $(\beta_i \cdot \beta'_{m-\alpha+i})$ is taken under multiplication of Γ.

2.3 Group action graphs and Cayley digraphs

Let V be a set and Π a set of permutations on V. A *group action graph* (GAG, for short) $GAG(V, \Pi)$ has a vertex set V, and there exists an arc (u, v) if and only if there exists a permutation $\pi \in \Pi$ such that $\pi(u) = v$.

Let Γ be a nontrivial finite group and $\Delta = \{\delta_1, \delta_2, \ldots, \delta_k\}$ a generating set of Γ. We associate a digraph called the *Cayley digraph of Γ with respect to Δ* and denote it by $\mathbf{Cay}(\Gamma, \Delta)$. The vertex set of $\mathbf{Cay}(\Gamma, \Delta)$ is the set of elements of Γ. For $\gamma_1, \gamma_2 \in \Gamma$, there exists an arc (γ_1, γ_2) generated by δ_i in $\mathbf{Cay}(\Gamma, \Delta)$ if and only if $\gamma_2 = \gamma_1\delta_i$.

3 A GAG representation of the Kautz digraph

In this section, we propose a GAG representation, that is, a vertex labeling and representation of the adjacency based on the labeling, of the Kautz digraph. From the definition of the Kautz digraph, the set of the difference of two consecutive letters in the vertex label is $\{1, 2, \ldots, d\}$. We may map $\{1, 2, \ldots, d\} = \mathbf{Z}_{d+1} \setminus \{0\}$ onto $\{0, 1, \ldots, d-1\} = \mathbf{Z}_d$ by subtracting one, that is, $i \in \mathbf{Z}_{d+1} \setminus \{0\}$ is mapped to $i - 1 \in \mathbf{Z}_d$. To use this property, we give another representation of the Kautz digraph.

Theorem 1 *For the Kautz digraph $K(d, n)$, there exists a vertex labeling from $\mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$ such that a vertex $v = (w; v_0 v_1 \cdots v_{n-2})$ in $V(K(d, n))$ is adjacent to the vertices $(w \oplus_{d+1} (v_0 + 1); v_1 v_2 \cdots v_{n-2}\, b)$ for some $b \in \mathbf{Z}_d$.*

Proof: For $a, b \in \mathbf{Z}_{d+1}$ where $a \neq b$, a function $\phi : \mathbf{Z}_{d+1}^2 \to \mathbf{Z}_d$ is defined by $\phi(a, b) = (b \ominus_{d+1} a) - 1$. A mapping $f : V(K(d, n)) \to \mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$ is defined as follows:

$$f(x_0 x_1 \cdots x_{n-1}) = (x_0; \phi(x_0, x_1)\phi(x_1, x_2) \cdots \phi(x_{n-2}, x_{n-1})).$$

First, we have to prove that f is bijective. Since $|V(K(d, n))| = |\mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}|$, it is sufficient to show that f is injective. Let $v = (w; v_0 v_1 \cdots v_{n-2}) \in \mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$. We assume that there exist vertices $x = x_0 x_1 \cdots x_{n-1}, y = y_0 y_1 \cdots y_{n-1} \in V(K(d, n))$ such that $f(x) = f(y) = v$. By the definition of f, $x_0 = y_0 = w$. Next, we notice the following property on v_0 with the fact of previous equation,

$$v_0 = (x_1 \ominus_{d+1} x_0) - 1 = (y_1 \ominus_{d+1} y_0) - 1.$$

To make this equation valid, we must have $x_1 = y_1$ since $0 \leq x_1 \leq d$ and $0 \leq y_1 \leq d$. We apply this relation sequentially to conclude that $x = y$. Hence, the mapping f is injective.

Next, we describe the adjacency of vertices with this labeling in the Kautz digraph. Let $u = u_0 u_1 \cdots u_{n-1}$, $v = u_1 u_2 \cdots u_{n-1} u_n \in V(K(d, n)), u_n \in \mathbf{Z}_{d+1} \setminus \{u_{n-1}\}$. By the definition of the Kautz digraph, $(u, v) \in A(K(d, n))$ and

$$\begin{aligned} f(u) &= (u_0; \phi(u_0, u_1)\, \phi(u_1, u_2) \cdots \phi(u_{n-2}, u_{n-1})), \\ f(u) &= (u_1; \phi(u_1, u_2)\, \phi(u_2, u_3) \cdots \phi(u_{n-1}, u_n)). \end{aligned}$$

$u_0 \oplus_{d+1} (\phi(u_0, u_1) + 1) = u_0 \oplus_{d+1} (((u_1 \ominus_{d+1} u_0) - 1) + 1) = u_0 \oplus_{d+1} u_1 \ominus_{d+1} u_0 = u_1$. Therefore, u_1 can be obtained from u_0, $\phi(u_0, u_1)$ and a constant 1. □

The vertex labeling given in Theorem 1 is called $(1, n-1)$*-labeling*. Now, we are ready to introduce a GAG representation of the Kautz digraph using $(1, n-1)$-labeling.

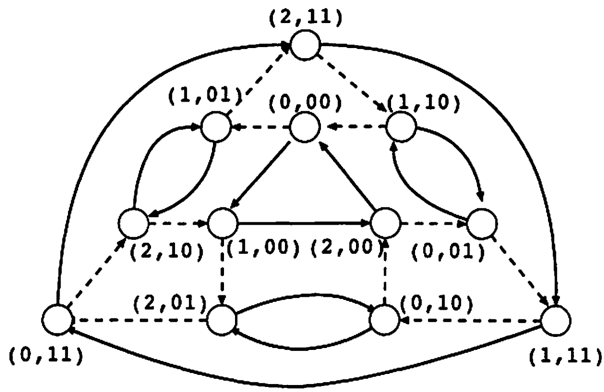

Figure 2: Kautz digraph $K(2,3)$ with $(1, n-1)$-labeling.

Theorem 2 *Let $n \geq 2$. The function set $\Psi = \{\psi_0, \psi_1, \ldots, \psi_{d-1}\}$ from $V(K(d,n))$ onto itself is defined as follows:*

$$(w; v_0 v_1 \cdots v_{n-2}) \cdot \psi_i = (w \oplus_{d+1} (v_0 + 1)\,;\, v_1 v_2 \cdots v_{n-2}\, (v_0 \oplus_d i)).$$

The function set Ψ is a permutation set on $V(K(d,n))$ and the Kautz digraph $K(d,n)$ can be arc-labeled so as to be a $GAG(V, \Psi)$.

Proof: By the definition, for each i, ψ_i is a function on $V(K(d,n))$. We now show that each ψ_i is injective. Suppose to the contrary, we assume that there exist vertices $u = (w_1; u_0 u_1 \cdots u_{n-2})$ and $v = (w_2; v_0 v_1 \cdots v_{n-2})$ such that $\psi_i(u) = \psi_i(v)$. By the definition of ψ_i, $u_i = v_i$ for $1 \leq i \leq n-2$. From $v_0 \oplus_d i = u_0 \oplus_d i$, $u_0 = v_0$. Moreover, from $w_1 \oplus_{d+1} (u_0 + 1) = w_2 \oplus_{d+1} (v_0 + 1)$, $w_1 = w_2$. We can conclude that $u = v$, and ψ_i is injective for any $i \in \{0, 1, \ldots, d-1\}$. Therefore, for each i, ψ_i is a permutation, and Ψ is a permutation set on $V(K(d,n))$. Moreover, we can conclude that $K(d,n)$ can be arc-labeled so as to be a $GAG(V, \Psi)$. □

In Figure 2, vertices in $K(2,3)$ are labeled by $(1, n-1)$-labeling, solid arcs and dashed arcs are labeled by ψ_0 and ψ_1, respectively.

4 Group based on the wreath product

In this section, we investigate the group and its action on the set. The set can be considered as the vertex set of the Kautz digraph.

Let $d \geq 2$ and $n \geq 2$ be integers. We consider the subset $\mathfrak{G}_{d,n}$ of the group $\Gamma = ((\mathbb{Z}_{d+1} \wr \mathbb{Z}_d) \wr \mathbb{Z}_{n-1}) \times \mathbb{Z}_2$ and the subset Π of $\mathfrak{G}_{d,n}$, defined as follows:

$$\mathfrak{G}_{d,n} = \Bigg\{ ((k; \langle x_0 ; \vec{y}_0(0)\rangle \ \langle x_1 ; \vec{y}_1(0)\rangle \ \cdots \langle x_{n-2} ; \vec{y}_{n-2}(0)\rangle), p) \\ \Bigg| k \in \mathbb{Z}_{n-1}, \ x_i \in \mathbb{Z}_d, \ y_{i,j} \in \mathbb{Z}_{d+1}, \\ \sum_{t=0}^{d-1} y_{0,t} \equiv \sum_{t=0}^{d-1} y_{1,t} \equiv \cdots \equiv \sum_{t=0}^{d-1} y_{n-2,t} \equiv 0 \pmod{d+1} \Bigg\},$$

where $p = 0$ if d is even, $p \equiv k \pmod 2$ if d and n are both odd, and $p \in \mathbb{Z}_2$ otherwise.

For each i and $t \in \mathbb{Z}_d$, $\vec{y}_i(t) = y_{i,t}\, y_{i,t+1} \cdots y_{i,d-1}\, y_{i,0}\, y_{i,1} \cdots y_{i,t-1}$.

$$\Pi = \left\{ \pi_m = \left(\left\langle 1; \langle m; 211\cdots 1\rangle \langle 0; \vec{0}\rangle \langle 0; \vec{0}\rangle \cdots \langle 0; \vec{0}\rangle \right\rangle, p \right) \middle| m \in \mathbb{Z}_d \right\},$$

where $p = 0$ if d is even and $p = 1$ if d is odd, and $\mathbf{0} = 00\cdots 0$.

In this paper, we use a letter $\gamma \in \mathfrak{G}_{d,n}$ to refer to the following element, since each element is represented in very long form;

$$\gamma = ((k; \langle x_0 ; \vec{y}_0(0)\rangle \langle x_1 ; \vec{y}_1(0)\rangle \langle x_2 ; \vec{y}_2(0)\rangle \cdots \langle x_{n-2} ; \vec{y}_{n-2}(0)\rangle), p).$$

It is clear that $\mathfrak{G}_{d,n}$ is closed under the multiplication, and we have the following lemma.

Lemma 1 *The subset $\mathfrak{G}_{d,n}$ forms a subgroup of $((\mathbb{Z}_{d+1} \wr \mathbb{Z}_d) \wr \mathbb{Z}_{n-1}) \times \mathbb{Z}_2$.*

4.1 Structure of the group $\mathfrak{G}_{d,n}$

We show that Π is a generating set of $\mathfrak{G}_{d,n}$. To treat Cayley digraphs and permutation groups, an important fact is to define their generating set clearly. The following theorem shows Π is a generating set of $\mathfrak{G}_{d,n}$. This is the main factor that determines the structure of the Cayley digraph.

Theorem 3 $\mathrm{Gr}(\Pi) = \mathfrak{G}_{d,n}$.

Proof: We prove $\mathbf{Gr}(\Pi) \subseteq \mathfrak{G}_{d,n}$ and $\mathfrak{G}_{d,n} \subseteq \mathbf{Gr}(\Pi)$ to show $\mathbf{Gr}(\Pi) = \mathfrak{G}_{d,n}$.

Since $\Pi \in \mathfrak{G}_{d,n}$ and $\mathfrak{G}_{d,n}$ forms a group, it is easy to verify that $\mathbf{Gr}(\Pi) \subseteq \mathfrak{G}_{d,n}$.

For the converse, we prove that any element in $\mathfrak{G}_{d,n}$ can be represented as some multiplication sequence of elements in Π. However, it is hard to represent any element in $\mathfrak{G}_{d,n}$ by using some elements in Π directly. Therefore, we should use more lucid set for easiness to verify that $\mathbf{Gr}(\Pi) \supseteq \mathfrak{G}_{d,n}$. We will consider a subset $\Pi' = \{\mu_0, \mu_1, \nu_0, \nu_1, \ldots, \nu_{d-2}\}$ which is a subset of $\mathfrak{G}_{d,n}$. We show that Π' can be represented by the multiplication of the elements of Π and every element in $\mathfrak{G}_{d,n}$ can be represented as a multiplication of elements of Π'.

$$
\begin{aligned}
\mu_0 &= (\langle 1; \langle 0;\vec{0}\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, p) \\
&= \pi_1^{(n-1)d} \cdot (\pi_0 \cdot \pi_1^{n-2})^d \cdot \pi_0,
\end{aligned}
$$

where $p = 0$ if d is even, $p = 1$ if d is odd.

$$
\begin{aligned}
\mu_1 &= (\langle 0; \langle 1;\vec{0}\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, 0) \\
&= \pi_1^{(n-1)d} \cdot \pi_1^{(n-1)(d-1)} \cdot \pi_2 \cdot \pi_1^{n-2}, \\
\nu_0 &= (\langle 0; \langle 0;100\cdots 0d\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, 0) \\
&= \pi_1^{(n-1)d} \cdot \pi_0 \cdot \pi_1^{n-2} \cdot \pi_1^{(n-1)(d-2)} \cdot \pi_2 \cdot \pi_1^{n-2}, \\
\nu_1 &= (\langle 0; \langle 0;010\cdots 0d\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, 0) \\
&= \pi_1^{(n-1)d} \cdot \pi_1^{n-1} \cdot \pi_0 \cdot \pi_1^{n-2} \cdot \pi_1^{(n-1)(d-3)} \cdot \pi_2 \cdot \pi_1^{n-2}, \\
&\ \vdots \\
\nu_t &= (\langle 0; \langle 0;\underbrace{00\cdots 0}_{t}10\cdots 0d\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, 0) \\
&= \pi_1^{(n-1)d} \cdot \pi_1^{(n-1)t} \cdot \pi_0 \cdot \pi_1^{n-2} \cdot \pi_1^{(n-1)(d-2-t)} \cdot \pi_2 \cdot \pi_1^{n-2}, \\
&\ \vdots \\
\nu_{d-2} &= (\langle 0; \langle 0;00\cdots 01d\rangle\langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle\rangle, 0) \\
&= \pi_1^{(n-1)d} \cdot \pi_1^{(n-1)(d-2)} \cdot \pi_0 \cdot \pi_1^{n-2} \cdot \pi_2 \cdot \pi_1^{n-2}.
\end{aligned}
$$

The element $\gamma \in \mathfrak{G}_{d,n}$ can be represented as

$$\begin{aligned}\gamma &= \nu_0^{y_{0,0}} \cdot \nu_1^{y_{0,1}} \cdot \nu_2^{y_{0,2}} \cdots \nu_{d-2}^{y_{0,d-2}} \cdot \mu_1^{x_0} \cdot \mu_0 \\ &\cdot \nu_0^{y_{1,0}} \cdot \nu_1^{y_{1,1}} \cdot \nu_2^{y_{1,2}} \cdots \nu_{d-2}^{y_{1,d-2}} \cdot \mu_1^{x_1} \cdot \mu_0 \\ &\cdot \nu_0^{y_{2,0}} \cdot \nu_1^{y_{2,1}} \cdot \nu_2^{y_{2,2}} \cdots \nu_{d-2}^{y_{2,d-2}} \cdot \mu_1^{x_2} \cdot \mu_0 \\ &\vdots \\ &\cdot \nu_0^{y_{i,0}} \cdot \nu_1^{y_{i,1}} \cdot \nu_2^{y_{i,2}} \cdots \nu_{d-2}^{y_{i,d-2}} \cdot \mu_1^{x_i} \cdot \mu_0 \\ &\vdots \\ &\cdot \nu_0^{y_{n-2,0}} \cdot \nu_1^{y_{n-2,1}} \cdot \nu_2^{y_{n-2,2}} \cdots \nu_{d-2}^{y_{n-2,d-2}} \cdot \mu_1^{x_{n-2}} \cdot \mu_0 \cdot \mu_0^{k},\end{aligned}$$

with elements in Π'.

When d is odd and n is even, some elements are unable to be represented by using only Π'. Although the element

$$\pi_1^{(n-1)d} = (\langle 0; \langle 0; \vec{0} \rangle \langle 0; \vec{0} \rangle \cdots \langle 0; \vec{0} \rangle \rangle, 1)$$

can be applied to invert p. We can represent any element in $\mathfrak{G}_{d,n}$ by Π' together with the above fact.

Therefore, $\mathfrak{G}_{d,n} \subseteq \mathbf{Gr}(\Pi)$ and $\mathfrak{G}_{d,n} = \mathbf{Gr}(\Pi)$. □

4.2 The action of $\mathfrak{G}_{d,n}$ on the set

In this section, we define an action of $\mathfrak{G}_{d,n}$ on some set related to the vertex set of the Kautz digraph.

Definition 1 *Let $\gamma \in \mathfrak{G}_{d,n}$, and $s = (w; \vec{v}(0)) \in \mathfrak{X} = \mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$. The mapping $\rho : \mathfrak{X} \times \mathfrak{G}_{d,n} \to \mathfrak{X}$ is defined as follows:*

$$s \cdot \gamma = \left(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot y_{j,t-v_j} + \frac{p(d+1)}{2}; \vec{v}(k) + \vec{x}(k) \right),$$

where $w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot y_{j,t-v_j} + \frac{p(d+1)}{2}$ is taken under modulo $d+1$ and each element of $\vec{v}(k) + \vec{x}(k) = (v_k + x_k)(v_{k+1} + x_{k+1}) \cdots (v_{k-1} + x_{k-1})$ is taken under modulo d.

We show that the mapping defined in Definition 1 is an action of $\mathfrak{G}_{d,n}$ on $\mathfrak{X}$.

Theorem 4 *The mapping ρ is an action of $\mathfrak{G}_{d,n}$ on $\mathfrak{X}$.*

Proof: From the definition of ρ, for any $s = (w; \vec{v}(0)) \in \mathfrak{X}$, we have

$$(w; \vec{v}(0)) \cdot (\langle 0; \langle 0; \vec{0}\rangle\langle 0; \vec{0}\rangle \cdots \langle 0; \vec{0}\rangle\rangle, 0) = (w; \vec{v}(0)).$$

Then, ρ satisfies the condition (1) of Section 2.1.

Let $\alpha, \beta \in \mathfrak{G}_{d,n}$ be denoted as

$$\begin{aligned}
\alpha &= (\langle a\,;\, \langle e_0\,; \vec{g}_0(0)\rangle\, \langle e_1\,; \vec{g}_1(0)\rangle \cdots \langle e_{n-2}\,; \vec{g}_{n-2}(0)\rangle, p_\alpha), \\
\beta &= (\langle b\,;\, \langle f_0\,; \vec{h}_0(0)\rangle\, \langle f_1\,; \vec{h}_1(0)\rangle \cdots \langle f_{n-2}\,; \vec{h}_{n-2}(0)\rangle, p_\beta).
\end{aligned}$$

Then,

$$\begin{aligned}
\alpha\beta = (\langle a+b\,; &\langle e_0 + f_{0-a}\,; \vec{g}_0(0) + \vec{h}_{0-a}(d-e_0)\rangle \\
&\langle e_1 + f_{1-a}\,; \vec{g}_1(0) + \vec{h}_{1-a}(d-e_1)\rangle \cdots \\
&\langle e_{n-2} + f_{n-2-a}\,; \vec{g}_{n-2}(0) + \vec{h}_{n-2-a}(d-e_{n-2})\rangle, p_\alpha + p_\beta).
\end{aligned}$$

To prove that ρ satisfies the condition (2) of Section 2.1, we show that $s \cdot (\alpha\beta) = (s \cdot \alpha) \cdot \beta$.

$$\begin{aligned}
s \cdot (\alpha\beta) &= \Bigg(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot (g_{j,t-v_j} + h_{j-a,(t-v_j)-e_j}) \\
&\quad + \frac{(p_\alpha + p_\beta)(d+1)}{2};\ \vec{v}(a+b) + \vec{e}(a+b) + \vec{f}(b) \Bigg) \\
&= \Bigg(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot g_{j,t-v_j} + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot h_{j-a,(t-v_j)-e_j} \\
&\quad + \frac{(p_\alpha + p_\beta)(d+1)}{2};\ \vec{v}(a+b) + \vec{e}(a+b) + \vec{f}(b) \Bigg),
\end{aligned}$$

$$\begin{aligned}
s \cdot \alpha &= \left(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot g_{j,t-v_j} + \frac{p_\alpha(d+1)}{2}; \vec{v}(a) + \vec{e}(a) \right), \\
(s \cdot \alpha) \cdot \beta &= \Bigg(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot g_{j,t-v_j} + \sum_{i=0}^{n-2} \sum_{s=0}^{d-1} s \cdot h_{i,s-(v_{a+i}+e_{a+i})} \\
&\quad + \frac{p_\alpha(d+1)}{2} + \frac{p_\beta(d+1)}{2}; \vec{v}(a+b) + \vec{e}(a+b) + \vec{f}(b) \Bigg).
\end{aligned}$$

In the last equation, the summation on the index i means the sum of all possible values of the first index of h and the indices of v and e because those indices are taken under modulo $n-1$. Therefore, if we set $i' \equiv i + a \pmod{n-1}$, the last equation can be rewritten as:

$$(s \cdot \alpha) \cdot \beta = \left(w + \sum_{j=0}^{n-2} \sum_{t=0}^{d-1} t \cdot g_{j,t-v_j} + \sum_{i'=0}^{n-2} \sum_{s=0}^{d-1} s \cdot h_{i'-a,s-(v_{i'}+e_{i'})} \right.$$
$$\left. + \frac{(p_\alpha + p_\beta)(d+1)}{2};\ \vec{v}(a+b) + \vec{e}(a+b) + \vec{f}(b) \right).$$

Thus,

$$s \cdot (\alpha\beta) = (s \cdot \alpha) \cdot \beta,$$

and ρ is an action of $\mathfrak{G}_{d,n}$ on $\mathfrak{X}$. □

Next we prove that ρ is a faithful action by showing that no non-identity element $\gamma \in \mathfrak{G}_{d,n}$ fixes every element $x \in \mathfrak{X}$.

Theorem 5 *The action ρ of $\mathfrak{G}_{d,n}$ on $\mathfrak{X}$ is faithful.*

Proof: It is easy to see that the element γ in $\mathfrak{G}_{d,n}$ which fixes $(0;0)$ satisfies: $x_0 = x_1 = \cdots = x_{n-2} = 0$. Elements that satisfies above condition and fixes the element $(0; 0\,0 \cdots 0\,1)$ have $k = 0$.

Before we go to the general proof, we should consider the special element

$$\tau = (\langle 0; \langle 0; \vec{0}\rangle \langle 0; \vec{0}\rangle \langle 0; \vec{0}\rangle \cdots \langle 0; \vec{0}\rangle\rangle, 1)$$

in $\mathfrak{G}_{d,n}$ where d is odd and n is even.

$$(0;\mathbf{0}) \cdot (\langle 0; \langle 0; \vec{0}\rangle \langle 0; \vec{0}\rangle \langle 0; \vec{0}\rangle \cdots \langle 0; \vec{0}\rangle\rangle, 1) = \left(0 + \frac{(d+1)}{2}; \vec{0}\right) \neq (0;\mathbf{0}).$$

Thus, $\tau \notin \ker(\rho)$. This fact makes it clear that if ρ is not a faithful action, then $\ker(\rho)$ consists of elements such that there exists at least one nonzero entries $y_{r,q}$ in γ where $r \in \mathbb{Z}_{n-1}, q \in \mathbb{Z}_d$. We assume that there exists such an element γ.

Let $s_{d-q} = (0; \underbrace{0 \cdots 0}_{r} (d-q)\, 0 \cdots 0)$ and $s_{d-q-1} = (0; \underbrace{0 \cdots 0}_{r} (d-q-1)\, 0 \cdots 0)$. By the assumption, actions of γ on s_{d-q} and s_{d-q-1} in $\mathfrak{X}$

are both fixed under ρ, that is,

$$\begin{aligned}
s_{d-q}\cdot\gamma &= (0;00\cdots0\,(d-q)\,0\cdots0)\cdot\gamma \\
&= \left(0+\sum_{j=0}^{r-1}\sum_{t=0}^{d-1} t\cdot y_{j,t}+\sum_{t=0}^{d-1} t\cdot y_{r,t-(d-q)}\right. \\
&\qquad \left.+\sum_{j=r+1}^{n-2}\sum_{t=0}^{d-1} t\cdot y_{j,t}+\frac{p(d+1)}{2};\ 00\cdots0\,(d-q)\,0\cdots0\right) \\
&= s_{d-q}, \\
s_{d-q-1}\cdot\gamma &= (0;00\cdots0\,(d-q-1)\,0\cdots0)\cdot\gamma \\
&= \left(0+\sum_{j=0}^{r-1}\sum_{t=0}^{d-1} t\cdot y_{j,t}+\sum_{t=0}^{d-1} t\cdot y_{r,t-(d-q-1)}\right. \\
&\qquad \left.+\sum_{j=r+1}^{n-2}\sum_{t=0}^{d-1} t\cdot y_{j,t}+\frac{p(d+1)}{2};\ 00\cdots0\,(d-q-1)\,0\cdots0\right) \\
&= s_{d-q-1},
\end{aligned}$$

We consider the first term in $s_{d-q}\cdot\gamma$ and $s_{d-q-1}\cdot\gamma$. In each case, if the element γ fixes both s_{d-q} and s_{d-q-1}, then

$$\sum_{t=0}^{d-1} t\cdot y_{r,t-(d-q)}-\sum_{t=0}^{d-1} t\cdot y_{r,t-(d-q-1)}\equiv 0 \pmod{d+1}.$$

Since the second index of y is taken under modulo d, we can rewrite $t-(d-q)$ and $t-(d-q-1)$ as $t+q$ and $t+q+1$, respectively. Unfolding and arranging the left hand side member in the above congruence equation, we have

$$\begin{aligned}
&\sum_{t=0}^{d-1} t\cdot y_{r,t+q}-\sum_{t=0}^{d-1} t\cdot y_{r,t+q+1} \\
&= 0\cdot y_{r,q}+1\cdot y_{r,q+1}+2\cdot y_{r,q+2}+\cdots+(d-1)\cdot y_{r,q+d-1} \\
&\qquad -(0\cdot y_{r,q+1}+1\cdot y_{r,q+2}+\cdots+(d-2)\cdot y_{r,q+d-1}+(d-1)\cdot y_{r,q}) \\
&= y_{r,q+1}+y_{r,q+2}+\cdots+y_{r,q+d-1}+y_{r,q}-d\cdot y_{r,q}.
\end{aligned}$$

From the definition of $\mathfrak{G}_{d,n}$, $\sum_{t=0}^{d-1} y_{r,t}\equiv 0 \pmod{d+1}$ and $-dy_{r,q}\equiv y_{r,q} \pmod{d+1}$, then

$$y_{r,q+1}+y_{r,q+2}+\cdots+y_{r,q+d-1}+y_{r,q}-d\cdot y_{r,q}\equiv y_{r,q}\equiv 0 \pmod{d+1}.$$

That is a contradiction to the hypothesis. Then no nonidentity element in $\mathfrak{G}_{d,n}$ fixes every elements in $\mathfrak{X}$ under the action ρ, and we conclude that ρ is a faithful action. □

5 The explicit construction of the Cayley digraphs based on the Kautz digraph

5.1 The action of elements in Π on the set

In this section, we investigate the action of the element $\pi_m \in \Pi$ on $\mathfrak{X} = \mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$ under the action ρ. This action is related to the permutation set in Theorem 2.

Lemma 2 *Under the action ρ, for each $m \in \mathbb{Z}_d$ the element $\pi_m \in \Pi$ acts on the set $\mathfrak{X}$ as follows:*

$$(w; \vec{v}(0)) \cdot \pi_m = (w + v_0 + 1; v_1\, v_2 \cdots v_{n-2}\, v_0 + m),$$

where $w + v_0 + 1$ and $v_0 + m$ are taken under modulo $d + 1$ and d, respectively.

Proof: Let $(w; \vec{v}(0)) \in \mathfrak{X}$.

Case 1: d is even

$$\begin{aligned}
&(w; \vec{v}(0)) \cdot (\left\langle 1; \langle m; 211 \cdots 1\rangle \langle 0; \vec{0}\rangle \cdots \langle 0; \vec{0}\rangle \right\rangle, 0) \\
=& \left(w + \sum_{t=0}^{d-1} t \cdot (1 + [t = v_0]);\ v_1\, v_2\, v_3 \cdots v_{n-2}\, v_0 + m \right) \\
=& (w + 0 + 1 + 2 + \cdots + (d-1) + v_0;\ v_1\, v_2\, v_3 \cdots v_{n-2}\, v_0 + m),
\end{aligned}$$

where $[t = v_0]$ is the Iverson's convention, that is, if $t = v_0$, then $[t = v_0] = 1$, otherwise, 0.

$$\begin{aligned}
&w + 0 + 1 + 2 + \cdots + (d-1) + v_0 = \frac{d(d-1)}{2} + w + v_0 \\
&= \frac{(d+1)(d-2)}{2} + w + v_0 + 1.
\end{aligned}$$

Since d is even, $d + 1 | ((d+1)(d-2)/2)$ and then

$$w + 0 + 1 + 2 + \cdots + (d-1) + v_0 \equiv w + v_0 + 1 \pmod{d+1}.$$

Then, $(w; \vec{v}(0)) \cdot \pi_m = (w + v_0 + 1;\ v_1\, v_2 \cdots v_{n-2}\, v_0 + m)$.

Case 2: d is odd

$$(w;\vec{v}(0)) \cdot (\left\langle 1; \langle m; 211\cdots 1\rangle \langle 0;\vec{0}\rangle \cdots \langle 0;\vec{0}\rangle \right\rangle, 1)$$
$$= \left(w + \sum_{t=0}^{d-1} t\cdot(1+[t=v_0]) + \frac{(d+1)}{2};\ v_1\, v_2\, v_3 \cdots v_{n-2}\, v_0 + m \right).$$

$$w+0+1+2+\cdots+(d-1)+\frac{(d+1)}{2}+v_0 = \frac{(d+1)(d-1)}{2}+w+v_0+1.$$

Since d is odd, $2|(d-1)$ therefore $d+1|((d+1)(d-1)/2)$. Then,

$$(w;\vec{v}(0)) \cdot \pi_m = (w+v_0+1;\ v_1\, v_2 \cdots v_{n-2}\, v_0 + m).$$

□

5.2 The group generated by $K(d,n)$ and the Cayley graph

In Theorem 2, we have described the GAG representation of the Kautz digraph. In this section, we investigate the structure of the group $\mathbf{Gr}(\Psi)$ generated by the Kautz digraph.

Theorem 6 *Let d, $n \geq 2$ be integers and Ψ a permutation set on $V(K(d,n))$ defined in Theorem 2. Then, $\mathfrak{G}_{d,n} = \mathbf{Gr}(\Psi)$.*

Proof: We can remark that the action ρ is a faithful action from $\mathfrak{G}_{d,n}$ onto $Sym(\mathfrak{X})$, there exists a subgroup Γ of $Sym(\mathfrak{X})$ which is isomorphic to $\mathfrak{G}_{d,n}$ and $\{\rho(\pi_0), \rho(\pi_1), \ldots, \rho(\pi_{d-1})\}$ is the generating set of Γ. On the other hand, by the definition $\mathbf{Gr}(\Psi)$ is also a subgroup of $Sym(\mathfrak{X})$. From Theorem 2 and Lemma 2, for each $m \in \mathbb{Z}_d$, there is a one-to-one correspondence between ψ_m and π_m, that is, $\rho(\pi_m) = \psi_m$. Since generating sets of Γ and $\mathbf{Gr}(\Psi)$ are equal, a group Γ is isomorphic to $\mathbf{Gr}(\Psi)$ and therefore $\mathfrak{G}_{d,n} = \mathbf{Gr}(\Psi)$. □

Based on the results in previous sections, we are ready to propose a new class of Cayley digraphs.

Definition 2 *Let d and $n \geq 2$ be integers.* The twofold butterfly digraph *$TBF(d,n)$ is a Cayley digraph* $\mathbf{Cay}(\mathfrak{G}_{d,n}, \Pi)$.

Proposition 2 *$TBF(d,n)$ is the Cayley cover of Kautz digraph $K(d,n)$.*

Proposition 3 *The order of $TBF(d,n)$ is $2(n-1)(d(d+1)^{(d-1)})^{(n-1)}$ if d is odd and n is even, and otherwise $(n-1)(d(d+1)^{(d-1)})^{(n-1)}$.*

The twofold butterfly digraph $TBF(2,3)$ is shown in Figure 3. Figure 5 and Figure 6 show the decomposition of $TBF(2,3)$ with respect to the generator π_0 and π_1, respectively. Figure 4 shows a vertex layout in each vertex block in Figure 3,5 and 6.

It follows from Theorem 6 and Definition 2 that the twofold butterfly digraph is the Cayley digraph associated to the Kautz digraph and we have the following theorem.

Theorem 7 *Let d and $n \geq 2$ be integers. The twofold butterfly digraph $TBF(d,n)$ is the Cayley digraph associated to the Kautz digraph $K(d,n)$ considered as the GAG of Theorem 2.*

6 Conclusion

In this paper, we have investigated the subgroup of some products of four cyclic groups and introduced the twofold butterfly digraph. The twofold butterfly digraph and the Kautz digraph are associated graphs. These results settle the problem posed by M. -C. Heydemann. This study is based on the $(1, n-1)$-labeling of the Kautz digraph.

A The difference from the previous results

In [5] and [11], the 1-factorization, that is, GAG representation of Kautz digraph $K(r, k+1)$ has been defined as follows:

$$(x, h_0, h_1, \ldots, h_{k-1})f_i^* = (x + h_0, h_1, h_2, \ldots h_{k-1}, h_0 i), \quad 1 \leq i \leq r$$

where $r+1$ is a prime and $(x, h_0, h_1, \ldots, h_{k-1}) \in \mathbb{Z}_{r+1} \times (\mathbb{Z}_{r+1} \setminus \{0\})^k$, by using the property that $K(r, k+1)$ is isomorphic to the k-line digraph of complete symmetric digraph (without loops) K_{r+1}. When $r = 2$, GAG representation in [5] and [11] are same as this paper. They also investigated the permutation group that is generated by $K(r, k+1)$. The group is isomorphic to the group $(\mathbb{Z}_{r+1} \rtimes \mathbb{Z}_r)^k \rtimes \mathbb{Z}_k$ where symbol $\rtimes$ means the semidirect product of two groups. Their cardinality is $k(r(r+1))^k$. Comparing with Proposition 3, it is clear that the groups are different.

If GAG representations are the same, permutation groups that are derived are also the same. Therefore, GAG representations are different when $r \neq 2$.

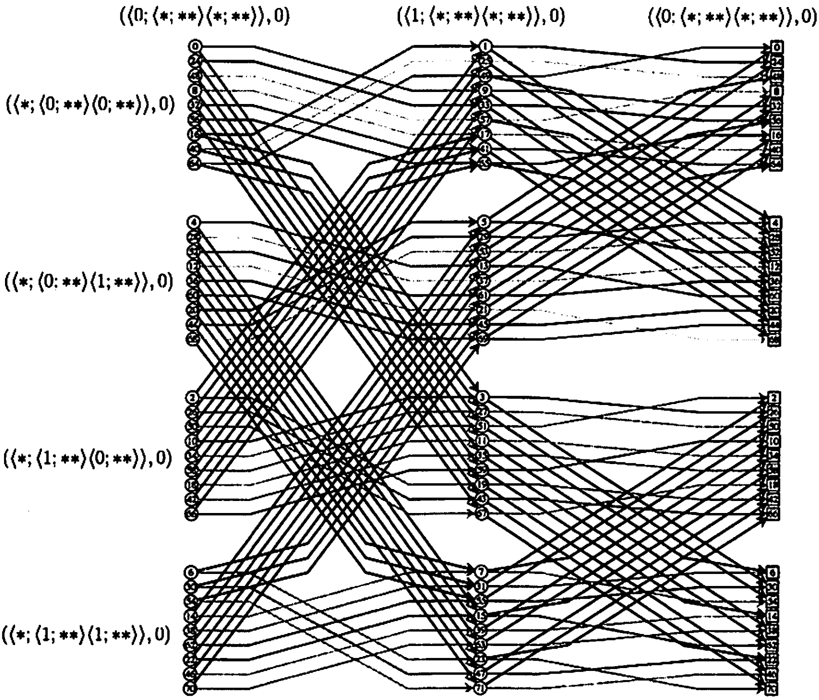

Figure 3: Twofold butterfly digraph $TBF(2,3)$.
Remark:Box-shaped vertices are replicated for visualization.

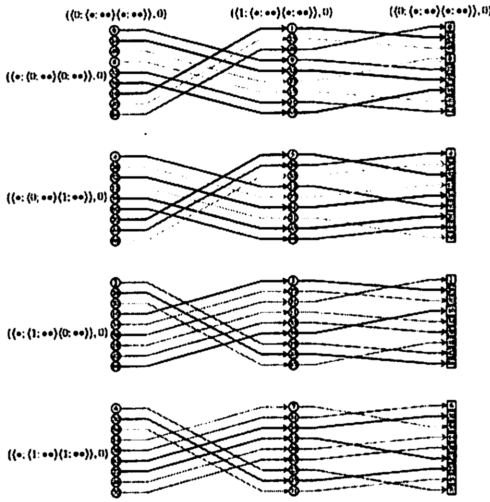

Figure 5: π_0 arcs in $TBF(2,3)$.

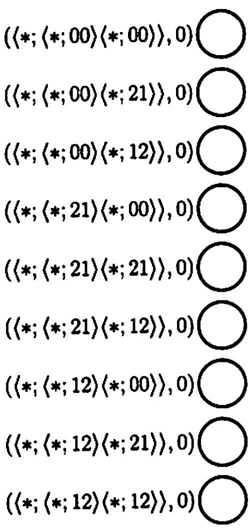

Figure 4: Detail of vertex blocks in Fig. 3, 5 and 6.

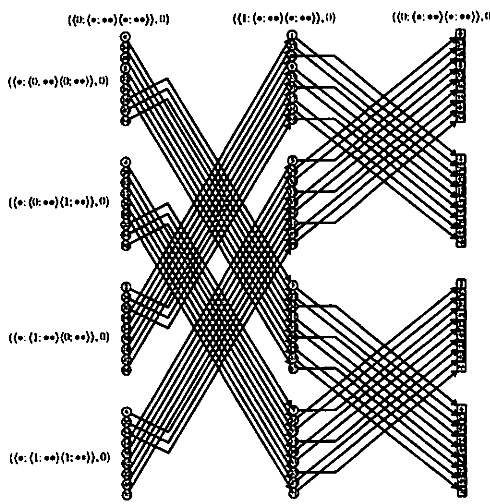

Figure 6: π_1 arcs in $TBF(2,3)$.

Remark:Box-shaped vertices are replicated for visualization.

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