Divisor orientations of powers of paths and powers of cycles

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Abstract.

In this paper, we prove that for any positive integers k, n with $k \geq 2$, the graph P_n^k is a divisor graph if and only if $n \leq 2k+2$, where P_n^k is the k^{th} power of the path P_n. For powers of cycles we show that C_n^k is a divisor graph when $n \leq 2k+2$, but is not a divisor graph when $n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3$, where C_n^k is the k^{th} power of the cycle C_n. Moreover, for odd n with $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$, we show that the graph C_n^k is not a divisor graph.

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1 Introduction

A graph G is a *divisor graph* if there is a bijection $f : V(G) \to S$, for some finite set S of positive integers such that $uv \in E(G)$ if and only if

$gcd(f(u), f(v)) = min\{f(u), f(v)\}$, that is either $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$. The function f is called a *divisor labeling* of G.

The length $g(n)$ of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, \ldots, n\}$ was studied in [6], [8], and [9]. The concept of divisor graph involving finite nonempty sets of integers rather than positive integers was introduced in [10]. It is shown in [10] that odd cycles of length greater than 3 are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs as shown in [5]. Divisor graphs do not contain induced odd cycles of length greater than 3, but they may contain triangles, for example the complete graphs are divisor graphs, see [5].

Quite different view points of visualizing divisor graphs were introduced in [5] through providing two characterizations of divisor graphs, one in terms of strongly convex digraphs and the other in terms of extreme vertices. The latter one is stated in the next section, for it will be frequently used in this paper. The present paper will investigate which powers of paths and cycles are divisor graphs. A complete characterization is obtained for powers of paths, while for powers of cycles, there are still for each integer $k \geq 8$ few cases not settled yet, namely C_n^k when n is even and $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$, where C_n^k is the k^{th} power of the cycle C_n, they are likely to be not divisor graphs.

For undefined notions and terminology, the reader is referred to [2] and [7].

2 Preliminaries

The following proposition was proved in [5].

Proposition 1. *Every induced subgraph of a divisor graph is a divisor graph.*

If a component of a disconnected graph G is not a divisor graph, then G is not a divisor graph. But if all components of G are divisor graphs, then G is a divisor graph, as explained in the next proposition.

Proposition 2. *A graph G is a divisor graph if and only if each component of G is a divisor graph.*

Proof. Let $G_1, G_2, ..., G_r$ be the components of G, and let f_i be a divisor labeling of G_i for $i = 1, 2, ..., r$. Let $p_1, p_2, ..., p_r$ be distinct primes none of which appears in any of the labelings f_i for $i = 1, 2, ..., r$. Now define f on $V(G)$ as follows: for $i = 1, 2, ..., r$, if $x \in V(G_i)$, then $f(x) = p_i f_i(x)$. Clearly f is a divisor labeling of G, and hence G is a divisor graph. The converse follows by Proposition 1. □

In a digraph D, a *transmitter* is a vertex having indegree 0, a *receiver* is a vertex having outdegree 0, while a vertex v is a *transitive vertex* if it has both positive outdegree and positive indegree such that $(u, w) \in E(D)$ whenever (u, v) and $(v, w) \in E(D)$. We will call an orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex, a *divisor orientation* of G. The previous two propositions can be directly deduced from the following characterization of divisor graphs given in [5].

Theorem 1. *A graph G is a divisor graph if and only if G has a divisor orientation.*

A vertex of a digraph D is an *extreme vertex* if it is a transmitter with positive outdegree, a receiver with positive indegree , or a transitive vertex, see [3]. Therefore, a nontrivial connected graph G is a divisor graph if and only if G has an orientation in which every vertex is an extreme vertex. Nontrivial connected divisor graphs where characterized as the graphs of order $n \geq 2$ whose upper orientable hull number equals n, see [4].

If D is an orientation of a graph G, then every transmitter (receiver) in D is a receiver (transmitter) in the converse of D, and every transitive vertex in D is also transitive in the converse of D. Thus we have the following result.

Theorem 2. *If D is a divisor orientation of a graph G, then the converse of D is also a divisor orientation of G.*

3 Powers of paths

In this section, we characterize which powers of paths are divisor graphs. We start by showing that P_7^2 is not a divisor graph.

Theorem 3. *Let P_7 be the path 12...7. Then each of the graphs P_7^2, $P_7^2 - 12$, $P_7^2 - \{12, 67\}$, $P_7^2 + 17$, $(P_7^2 - 12) + 17$, and $(P_7^2 - \{12, 67\}) + 17$ is not a divisor graph.*

Proof. Let G be one of the graphs P_7^2, $P_7^2 - 12$, $P_7^2 - \{12, 67\}$, $P_7^2 + 17$, $(P_7^2 - 12) + 17$, or $(P_7^2 - \{12, 67\}) + 17$. Assume to the contrary that G is a divisor graph. Then, by Theorem 1, G has a divisor orientation D. In view of Theorem 2, suppose that $(4, 5) \in E(D)$. Then, since $25, 47 \notin E(G)$, we must have $(4, 2), (7, 5) \in E(D)$. Then we get $(3, 5), (4, 6) \in E(D)$, because $73, 62 \notin E(G)$. But also $25, 15, 36 \notin E(G)$, which implies that $(3, 2), (3, 1), (4, 3) \in E(D)$. Now since $(4, 3), (3, 1) \in E(D)$ but $(4, 1) \notin E(D)$, this leads to a contradiction. □

The following theorem determines precisely when a power of a path is a divisor graph.

Theorem 4. *For any integer $k \geq 2$, the graph P_n^k is a divisor graph if and only if $n \leq 2k+2$.*

Proof. Let P_n be the path $12...n$.

Case 1. $n > 2k+2$.

Then $\{1, 2, 3, k+2, k+3, k+4, 2k+3\}$ induces in P_n^k a graph which is isomorphic to P_7^2. Thus, by Proposition 1 and Theorem 3, P_n^k is not a divisor graph.

Case 2. $n = 2k+2$.

Let D be the orientation of P_n^k with $E(D) = A \cup B \cup C$, where

$$A = \{(i, j) : 1 \leq i \leq k, i < j \leq k+1\},$$

$$B = \{(i, j) : k+2 \leq i \leq 2k+1, i < j \leq 2k+2\},$$

and

$$C = \{(j, i) : k+2 \leq j \leq 2k+1, j-k \leq i \leq k+1\}.$$

Now we will show that D is a divisor orientation of P_n^k. Clearly, $1, k+2$ are transmitters, while $k+1$, $2k+2$ are receivers. So, let $i \in V(D) - \{1, k+1, k+2, 2k+2\}$ and let aib be a directed path in D. Distinguish two subcases.

Case 2.1: $1 < i < k+1$.

Then $b \in \{i+1, ..., k+1\}$ and either $a \in \{1, ..., i-1\}$ or $a \in \{k+2, ..., k+i\}$. Thus either $ab \in A \subseteq E(D)$ or $ab \in C \subseteq E(D)$, respectively. Therefore i is a transitive vertex in D.

Case 2.2: $k+2 < i < 2k+2$.

Then $a \in \{k+2, ..., i-1\}$ and either $b \in \{i-k, ..., k+1\}$ or $b \in \{i+1, ..., 2k+2\}$. Thus either $ab \in C \subseteq E(D)$ or $ab \in B \subseteq E(D)$, respectively. Therefore i is a transitive vertex in D.

Thus D is a divisor orientation of P_n^k. Hence P_n^k is a divisor graph.

Case 3. $n < 2k+2$.

Then P_n^k is isomorphic to an induced subgraph of P_{2k+2}^k, and hence by Proposition 1 and Case 2 above, P_n^k is a divisor graph. □

Note that the divisor orientation of P_{2k+2}^k given in the proof of the previous theorem can be generated from the divisor labeling f of P_{2k+2}^k defined by:

$$f(i) = \begin{cases} 2^{i-1}3^i & , \quad 1 \leq i \leq k+1 \\ 2^{i-(k+1)} & , \quad k+2 \leq i \leq 2k+2 \end{cases},$$

where P_{2k+2} is the path $12...(2k+2)$.

A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, see [1]. For any integer $k \geq 2$, if G is a graph with

$diam(G) = d \geq 2k+2$, then there exist vertices $x_0, x_1, \cdots, x_{d-1}$ and x_d such that $d(x_0, x_d) = d$ and the path $P: x_0x_1 \cdots x_d$ is a shortest x_0-x_d path. Since P is a shortest path in G, it is isometric, and hence every subpath of P is also isometric in G. In particular, since $d \geq 2k+2$, the path $x_0x_1 \cdots x_{2k+2}$ is an isometric path of order $2k+3$ in G. Thus G contains an isometric subgraph which is isomorphic to P_{2k+3}. Therefore the graph G^k contains an induced subgraph that is isomorphic to P^k_{2k+3}, which is not a divisor graph. Thus we have the following result.

Corollary 1. *For any integer $k \geq 2$, if G is a graph of diameter $d \geq 2k+2$, then G^k is not a divisor graph.*

4 Powers of cycles

In this section, we investigate which powers of cycles are divisor graphs. In [10], it was shown that any odd cycle of length greater than 3 is not a divisor graph. This fact will be used in the proof of some of the following results.

We start by considering n modulo $2k$.

Lemma 1. *For any integer $k \geq 2$, if $n > 4k$ and $n \equiv i \pmod{2k}$, for some $i \in \{1, 2, ..., k\}$, then C_n^k is not a divisor graph.*

Proof. Let C_n be the cycle $12...n1$. Then the cycle C: $1(1+k)(1+2k)...(1+tk)1$, where $t = \frac{n-i}{k}$, is an induced cycle of length $t+1$ in C_n^k. Clearly, t is even, so C is an odd cycle. Since $n > 4k$, we have $n - i > 4k - i \geq 4k - k$, thus $t = \frac{n-i}{k} > 3$. Hence, C has length greater than 3. Therefore C_n^k is not a divisor graph. □

Lemma 2. *For any integer $k \geq 2$, if $n > 2k$ and $n \equiv i \pmod{2k}$, for some $i \in \{k+1, k+2, ..., 2k\}$, then C_n^k is not a divisor graph.*

Proof. Let C_n be the cycle $12...n1$, and let $t = \frac{n-i}{k}$. Then the vertex $1 + (t+1)k$ is adjacent to 1 in C_n^k, because

$$\begin{aligned} d_{C_n}(1+(t+1)k, 1) &= d_{C_n}(1+(t+1)k, n) + 1 \\ &= (tk+i) - (1+tk+k) + 1 \\ &= i - k \\ &\leq 2k - k \\ &= k. \end{aligned}$$

Therefore the cycle C: $1(1+k)(2+k)(2+2k)(1+3k)(1+4k)...(1+(t+1)k)1$ is an induced cycle of length $t+3$ in C_n^k, note that $1+3k \leq n$ and if $n \leq 4k$, then $t+1 = 3$ and hence C has length 5. But t is even, so C is an odd cycle. Finally, since $n > 2k$, we have $n - i > 2k - i \geq 2k - 2k = 0$, thus $t = \frac{n-i}{k} > 0$. Therefore, C_n^k contains the induced odd cycle C of length greater than 3, hence C_n^k is not a divisor graph. □

Combining the previous two lemmas, we obtain the next corollary.

Corollary 2. *For any integer $k \geq 2$, if $n \geq 3k+1$, then C_n^k is not a divisor graph.*

Proof. By Lemma 1 and Lemma 2, C_n^k is not a divisor graph for each $n > 4k$. Again by Lemma 2, C_n^k is not a divisor graph for each $n \in \{2k+(k+1), 2k+(k+2), ..., 2k+(2k)\} = \{3k+1, ..., 4k\}$, hence the result follows immediately. □

To improve the result of the previous corollary, we turn to consider n modulo $k+1$.

Lemma 3. *For any integer $k \geq 2$, if $n > 2(k+1)$ and $n \equiv i \pmod{k+1}$, for some $i \in \{\lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, ..., k\}$, then C_n^k is not a divisor graph.*

Proof. Let C_n be the cycle $12...n1$, and let $t = \frac{n-i}{k+1}$. Then the vertex $1 + t(k+1)$ is not adjacent to $(k+1) - (k-i)$, because

$$\begin{aligned} d_{C_n}(1 + t(k+1), (k+1) - (k-i)) &= d_{C_n}(1 + t(k+1), n) + 1 \\ &\quad + d_{C_n}(1, (k+1) - (k-i)) \\ &= (i-1) + 1 + i \\ &= 2i \\ &> k. \end{aligned}$$

Then the cycle C: $1(1(k+1)-(k-i))(1+1(k+1))(2(k+1)-(k-i))(1+2(k+1))...(t(k+1)-(k-i))(1+t(k+1))1$ is an induced cycle of length $2t+1$ in C_n^k. Since $n > 2(k+1)$, we have $n - i > 2k+2-i \geq 2k+2-k = k+2$, thus $t = \frac{n-i}{k+1} > \frac{k+2}{k+1} > 1$. Hence, C has length greater than 3. Therefore C_n^k is not a divisor graph. □

Using the previous lemma, we obtain the following improvement of Corollary 2.

Theorem 5. *For any integer $k \geq 2$, if $n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3$, then C_n^k is not a divisor graph.*

Proof. By Corollary 2, the statement is true for $n \geq 3k+1$. Now by Lemma 3, the statement is true for $n \in \{2(k+1)+\lfloor \frac{k}{2} \rfloor, 2(k+1)+\lfloor \frac{k}{2} \rfloor + 2, ..., 2(k+1)+k\} = \{2k+\lfloor \frac{k}{2} \rfloor + 3, 2k+\lfloor \frac{k}{2} \rfloor + 4, ..., 3k+1\}$, hence the result follows immediately. □

It is worth mentioning that the result of Theorem 5 can be shown, in view of Theorem 3, by producing an induced subgraph H of C_n^k which is isomorphic to one of the two graphs $P_7^2, P_7^2 + 17$. Take H namely to be the subgraph of C_n^k induced by $\{1, \frac{k}{2}+1, \frac{k}{2}+2, k+2, n-k, n-\frac{k}{2}, n-\frac{k}{2}+1\}$ if n is even and by $\{1, \frac{k-1}{2}+2, \frac{k-1}{2}+3, k+2, n-k, n-\frac{k-1}{2}, n-\frac{k-1}{2}+1\}$ if n is odd, where in both cases C_n is the cycle $12...n1$. This proof is shorter, but it does not include the fact that C_n^k, where $n \geq 2k+\lfloor \frac{k}{2} \rfloor + 3$, contains an induced odd cycle of length greater than 3, which is shown in the proofs of Lemmas 1,2, and 3 above. The following theorem assures that $2k+\lfloor \frac{k}{2} \rfloor + 3$ is the minimum value of n for which C_n^k contains an induced odd cycle of length greater than 3.

Theorem 6. *For any integer $k \geq 2$, the graph C_n^k contains an induced odd cycle of length greater than 3 if and only if $n \geq 2k+\lfloor \frac{k}{2} \rfloor + 3$.*

Proof. If $n \geq 2k+\lfloor \frac{k}{2} \rfloor + 3$, then, by the proofs of Lemmas 1, 2, and 3, the graph C_n^k contains an induced odd cycle of length greater than 3. Suppose $n < 2k+\lfloor \frac{k}{2} \rfloor + 3$, and let C_n be the cycle $12...n1$. Assume to the contrary that C_n^k contains an induced odd cycle C: $1a_1b_1a_2b_2...a_tb_t1$, for some $t \geq 2$. Then $b_1 \geq k+2$, and hence $b_2 \geq 2k+3$. Then, since $3k+4 > 2k+\lfloor \frac{k}{2} \rfloor + 3 > n$, we must have $t = 2$. But $d_{C_n}(b_2, n) < (2k+\lfloor \frac{k}{2} \rfloor + 3) - (2k+3) = \lfloor \frac{k}{2} \rfloor$, thus $d_{C_n}(b_2, 1) = d_{C_n}(b_2, n) + 1 \leq \lfloor \frac{k}{2} \rfloor$. Then, since $a_1b_2 \notin E(C_n^k)$, we have $a_1 \geq \lfloor \frac{k}{2} \rfloor + 2$. But also $a_1a_2 \notin E(C_n^k)$, which implies that $a_2 \geq \lfloor \frac{k}{2} \rfloor + k + 3$. Then $d_{C_n}(a_2, 1) = d_{C_n}(a_2, n) + 1 < (2k+\lfloor \frac{k}{2} \rfloor + 3) - (\lfloor \frac{k}{2} \rfloor + k + 3) + 1 = k+1$, which means that $1a_2 \in E(C_n^k)$, a contradiction. □

Although for $n < 2k+\lfloor \frac{k}{2} \rfloor + 3$, the graph C_n^k does not contain an induced odd cycle of length greater than 3, C_n^k need not be a divisor graph as we will see in the sequel.

Now we consider cases of n for which C_n^k is a divisor graph.

Theorem 7. *For any integer $k \geq 2$, if $n \leq 2k+2$, then C_n^k is a divisor graph.*

Proof. For $n < 2k+2$, the graph C_n^k is a complete graph and hence is a divisor graph. Suppose $n = 2k+2$. Then C_n^k is isomorphic to the graph $\overline{(k+1)K_2}$, so let $V(C_n^k) = \{1, \bar{1}, 2, \bar{2}, ..., k+1, \overline{k+1}\}$ where $i\bar{i} \notin E(C_n^k)$ for $i \in \{1, 2, ..., k+1\}$. Define the orientation D of C_n^k as follows: $E(D) =$

$\{ij : i < j\} \cup \{i\bar{j} : i < j\} \cup \{\bar{i}\bar{j} : i < j\} \cup \{\bar{i}j : i \leq j\}$. Clearly, the vertices $1, \bar{1}$ are transmitters, while the vertices $k+1, \overline{k+1}$ are receivers. Let $v \in V(C_n^k) - \{1, \bar{1}, k+1, \overline{k+1}\}$. Say $v = i$, for some $1 < i < k+1$ (the case $v = \bar{i}$ is similar). Let aib be a directed path in D. Then $a = i_1$ or $\bar{i_1}$, for some $1 \leq i_1 < i$, and $b = i_2$ or $\bar{i_2}$, for some $i < i_2 < k+1$. Thus $ab \in E(D)$, hence v is a transitive vertex. Therefore D is a divisor orientation of C_n^k. □

Next, we will investigate when C_n^k is a divisor graph for some cases of n between $2k+2$ and $2k + \lfloor \frac{k}{2} \rfloor + 3$. We start by the following result.

Theorem 8. *Let C be an odd cycle in a graph G such that for each vertex $v \in V(C)$, the two neighbors of v in C are not adjacent in G. Then G is not a divisor graph.*

Proof. Let C be the cycle $12...(2k+1)1$. Assume to the contrary that G is a divisor graph. Let D be a divisor orientation of G. In view of Theorem 2, we can suppose that $(1,2) \in E(D)$. Since $13 \notin E(G)$, we must have $(3,2) \in E(D)$, and hence 2 is a receiver in the corresponding orientation D_C of C. By the same argument, we have $4, 6, ..., 2k$ are receivers and $3, 5, ..., 2k+1$ are transmitters in D_C, a contradiction since $(2k+1, 1)$ and $(1,2) \in E(D)$ but $(2k+1, 2) \notin E(D)$. □

We use the previous theorem to show that for any odd integer n with $n > 2k+2$, the graph C_n^k is not a divisor graph.

Theorem 9. *For any integer $k \geq 2$, if n is an odd integer with $n > 2k+2$, then C_n^k is not a divisor graph.*

Proof. Since n is odd, the element $a = \lfloor \frac{k}{2} \rfloor + 1$ has an odd order in the cyclic group $\mathbb{Z}_n$. But $3a = 3\lfloor \frac{k}{2} \rfloor + 3 < 2k+3 \leq n$, which implies that $|a| > 3$, where $|a|$ denotes the order of a in $\mathbb{Z}_n$. But $n \geq 2k+3$ and $a > \frac{k}{2}$, which implies that for each $1 \leq i \leq |a|-1$, the two vertices ia and $(i+2)a$ are not adjacent (note that for $j = 1, 2, ..., |a|+1$, the numbers ja are taken modulo n). Thus the cycle C: $(a)(2a)...(|a|\,a)(a)$ is an odd cycle in C_n^k such that the two neighbors of any vertex of C are not adjacent in C_n^k. Therefore, by the previous theorem, C_n^k is not a divisor graph. □

By Theorems 7, 5 and 9, we obtain the following characterizations for the cases when $k = 2, 3$.

Corollary 3. *C_n^2 is a divisor graph if and only if $n \leq 6$.*

Corollary 4. *C_n^3 is a divisor graph if and only if $n \leq 8$.*

For each value of $k \in \{4,5,6,7\}$ there is exactly one missing case for n to obtain a complete characterization of all divisor graphs C_n^k. We investigate these cases in the following lemma. Note that for each $k \geq 8$, there are more than one missing case.

Lemma 4. *Let $k \in \{4,5,6,7\}$. Then C_{2k+4}^k is not a divisor graph.*

Proof. Let C_{2k+4} be the cycle $12...(2k+4)1$. Assume to the contrary that C_{2k+4}^k is a divisor graph. By Theorems 1 and 2, let $(1,2) \in E(D)$, for some divisor orientation D of C_{2k+4}^k. Since $1(k+2) \notin E(C_{2k+4}^k)$, we must have $(k+2,2) \in E(D)$. But also $(k+2)(2k+4) \notin E(C_{2k+4}^k)$, which implies that $(2k+4,2) \in E(D)$. Since $(2k+4)(k+1), (k+5)2 \notin E(C_{2k+4}^k)$, we get $(k+1,2), (2k+4,k+5) \in E(D)$. Now since $(k+1,2) \in E(D)$ but $(k+1)(2k+2) \notin E(C_{2k+4}^k)$, we must have $(2k+2,2) \in E(D)$. Hence $(k-1,2) \in E(D)$, because $(2k+2)(k-1) \notin E(C_{2k+4}^k)$. Thus, since $(k+3)2 \notin E(C_{2k+4}^k)$, we have $(k-1,k+3) \in E(D)$. Now we distinguish two cases:

Case 1. $k \in \{4,5\}$.

Since $(k-1,k+3) \in E(D)$ but $(k-1)(k+5) \notin E(C_{2k+4}^k)$, we get $(k+5,k+3) \in E(D)$, a contradiction since we now have $(2k+4,k+5), (k+5,k+3) \in E(D)$ but $(2k+4,k+3) \notin E(D)$.

Case 2. $k \in \{6,7\}$.

Since $(k-1,k+3) \in E(D)$ but $(k-1)(2k) \notin E(C_{2k+4}^k)$, we get $(2k,k+3) \in E(D)$. Then, since $2(k+3) \notin E(C_{2k+4}^k)$, we have $(2k,2) \in E(D)$. Thus, since $(2k)(k-3) \notin E(C_{2k+4}^k)$, we must have $(k-3,2) \in E(D)$. But $(k+3)2 \notin E(C_{2k+4}^k)$, which implies that $(k-3,k+3) \in E(D)$. Then, since $(k-3)(k+5) \notin E(C_{2k+4}^k)$, we have $(k+5,k+3) \in E(D)$. Now we have $(2k+4,k+5), (k+5,k+3) \in E(D)$ but $(2k+4,k+3) \notin E(D)$, a contradiction. □

For $k \in \{4,5,6,7\}$, the graph C_n^k is (by Theorem 7) a divisor graph when $n \leq 2k+2$, and is (by Theorem 5) not a divisor graph when $n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3$. By Theorem 9 and Lemma 4, the graph C_n^k is not a divisor graph when $2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$. So we have the following result.

Corollary 5. *Let $k \in \{4,5,6,7\}$. Then C_n^k is a divisor graph if and only if $n \leq 2k+2$.*

For an integer $k \geq 8$, the case when n is even and $2k+3 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$ is not yet settled. It is shown in Lemma 4 that C_{12}^4,C_{14}^5,C_{16}^6, and C_{18}^7 are not divisor graphs, we conjecture this would be the case also when $k \geq 8$.

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