Several Identities Involving Second-order Recurrent Sequences

Hong Hu

Department of Mathematics, Huaiyin Normal University,

Huaian 223300, Jiangsu Province, P.R.China

E-mail:hysyhh@163.com

Abstract: Let $\{w_n\}$ be a second-order recurrent sequence. Several identities about the sums of products of second-order recurrent sequences were obtained and the relationship between the second-order recurrent sequences and the recurrence coefficient revealed. Some identities about Lucas sequences, Lucas numbers and Fibonacci numbers were also obtained.

Key words: Second-order recurrent sequences; Lucas sequences; Lucas numbers; Fibonacci numbers.

2000 Mathematics Subject Classification: 11B39, 11B37

1 Introduction

Let $\mathbf{Z}$ and $\mathbf{R}$ denote the ring of the integers and the field of real numbers, respectively. For a field $\mathbf{F}$, we put $\mathbf{F}^* = \mathbf{F} \setminus \{0\}$. Fix $A \in \mathbf{R}$ and $B \in \mathbf{R}^*$, and let $\mathcal{L}(A, B)$ consist of all those second-order recurrent sequences $w_n = w_n(a, b; A, B)$ of complex numbers satisfying the recursion:

$$w_0 = a, \quad w_1 = b, \quad w_{n+2} = Aw_{n+1} - Bw_n \quad \text{for } n = 0, \pm 1, \pm 2, \cdots. \tag{1}$$

For sequences in $\mathcal{L}(A, B)$, the corresponding characteristic equation is $x^2 - Ax + B = 0$, whose roots $(A \pm \sqrt{A^2 - 4B})/2$ are denoted by α and β. If

$A \in \mathbf{R}$ and $\Delta = A^2 - 4B \geq 0$, then we have

$$\alpha = \frac{A - sg(A)\sqrt{\Delta}}{2} \qquad \text{and} \qquad \beta = \frac{A + sg(A)\sqrt{\Delta}}{2},$$

where $sg(A) = 1$ if $A > 0$, and $sg(A) = -1$ if $A < 0$. It is well known that(see [1] [3])

$$w_n = \frac{(b - a\beta)\alpha^n + (a\alpha - b)\beta^n}{\alpha - \beta}, \qquad \text{for } n \in Z.$$

We shall denote by $u_n = w_n(0, 1; A, B)$ and $v_n = w_n(2, A; A, B)$ the sequences defined for the Lucas sequences, where $A \in R$ and $B \in R^*$. If $A = 1$ and $B = -1$, then those $F_n = u_n$ and $L_n = v_n$ are called Fibonacci numbers and Lucas numbers, respectively.

In [2] H. Feng and Z. Zhang defined the sequences $W_n = w_n(0, b; A, B)$, $U_m = w_m(0, 1; A, B)$ and $V_m = w_m(2, A; A, B)$, denoted by $\sigma_i(n, k)$ the summation of all products of choosing i elements from $n + k - i + 1, n + k - i + 2, \ldots, n + 2k - 1$ but not containing any two consecutive elements, i.e.,

$$\sigma_i(n, k) = \sum \prod_{t=1}^{i} (n + k - i + j_t),$$

where the summation is taken over all i-tuples with positive integer coordinates $j_1, j_2, \ldots, j_i$ such that $1 \leq j_1 < j_2 < \cdots < j_i \leq k + i - 1$ and $|j_r - j_s| \geq 2$ for $1 \leq r \neq s \leq i$. They obtained the following summation

$$\sum_{a_1 + a_2 + \cdots + a_k = n} W_{ma_1} W_{ma_2} \cdots W_{ma_k} = \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4B^m)^{k-1}}$$

$$\cdot \sum_{i=1}^{k-1} (-2B^m)^i V_m^{k-1-i} \langle n - k + 1 \rangle_{k-1-i} \sigma_i(n - k + 1, k - 1) \cdot W_{m(n-i)}, \quad (2)$$

where $\langle n \rangle_k = n(n+1)(n+2)\cdots(n+k-1)$.

In this paper we obtain the following theorems.

Theorem 1.1 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k be positive integer,

$$\sum_{a_0 + a_1 + \cdots + a_k = n} w_{a_0} w_{a_1} \cdots w_{a_k} = \sum_{i=0}^{k+1} \sum_{j=0}^{[\frac{n-i}{2}]} \binom{k+1}{i}$$

$$\cdot \binom{n+k-i-2j}{k} \binom{n+k-i-j}{j} A^{n-i-2j}(-B)^j a^{k+1-i}(b-aA)^i. \tag{3}$$

Theorem 1.2 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k, m be positive integers,

$$\sum_{a_0+a_1+\cdots+a_k=n} w_{ma_0} w_{ma_1} \cdots w_{ma_k} = \sum_{i=0}^{k+1} \sum_{j=0}^{[\frac{n-i}{2}]} \binom{k+1}{i}$$

$$\cdot \binom{n+k-i-2j}{k} \binom{n+k-i-j}{j} v_m^{n-i-2j}(-B^m)^j a^{k+1-i}(w_m - av_m)^i. \tag{4}$$

2 Proofs of Theorems 1.1 and 1.2

Lemma 2.1 Let A and B be two real numbers with $B \neq 0$, $u_n(A,B) = w_n(0,1;A,B)$ and $w_n(A,B) = w_n(a,b;A,B)$, we have

$$\sum_{n=0}^{\infty} u_n(A,B)t^n = \frac{t}{1-At+Bt^2}, \tag{5}$$

and

$$\sum_{n=0}^{\infty} w_n(A,B)t^n = \frac{a+(b-aA)t}{1-At+Bt^2}. \tag{6}$$

Proof

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(A,B)t^n &= \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} t^n \\
&= \frac{1}{\alpha-\beta}\left(\sum_{n=0}^{\infty} \alpha^n t^n - \sum_{n=0}^{\infty} \beta^n t^n\right) \\
&= \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha t} - \frac{1}{1-\beta t}\right) \\
&= \frac{t}{(1-\alpha t)(1-\beta t)} \\
&= \frac{t}{1-At+Bt^2},
\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(A,B)t^n &= \sum_{n=0}^{\infty} \frac{(b-a\beta)\alpha^n + (a\alpha - b)\beta^n}{\alpha-\beta} t^n \\ &= \frac{b-a\beta}{\alpha-\beta}\cdot\frac{1}{1-\alpha t} + \frac{a\alpha-b}{\alpha-\beta}\cdot\frac{1}{1-\beta t}) \\ &= \frac{a+(b-aA)t}{1-At+Bt^2}.\end{aligned}$$

Lemma 2.2 Let A and B be two real numbers with $B \neq 0$, $u_n(A,B) = w_n(0,1;A,B)$,

$$u_n(A,B) = \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-1-i}{i} A^{n-1-2i}(-B)^i. \tag{7}$$

Proof

$$\begin{aligned}u_n(A,B) &= [t^n]\frac{t}{1-At+Bt^2} \\ &= [t^n]\sum_{n=0}^{\infty} t(At-Bt^2)^n \\ &= \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-1-i}{i} A^{n-1-2i}(-B)^i.\end{aligned}$$

where $[t^n]f(t)$ denotes the coefficient of t^n in the expansion of function $f(t)$.

Proof of Theorem 1.1 We define

$$w_n(Ax,B) = w_n(a,b;Ax,B), \qquad u_n(Ax,B) = w_n(0,1;Ax,B),$$

and

$$G(x,t) = \sum_{n=0}^{\infty} w_n(Ax,B)t^n, \qquad F(x,t) = \sum_{n=0}^{\infty} u_n(Ax,B)t^{n-1}.$$

By Lemma 2.1 we obtain

$$G(x,t) = \frac{a+(b-aAx)t}{1-Axt+Bt^2}, \qquad F(x,t) = \frac{1}{1-Axt+Bt^2}.$$

Let $\frac{\partial^k F(x,t)}{\partial x^k}$ denote the $k-$th partial derivation of $F(x,t)$ for x, and

$u_n^{(k)}(Ax, B)$ denote the $k-$th partial derivation of $u_n(Ax, B)$. Then we get

$$\begin{aligned}
\frac{\partial F(x,t)}{\partial x} &= \frac{At}{(1-Axt+Bt^2)^2} &= \sum_{n=0}^{\infty} u_{n+1}^{(1)}(Ax,B)t^n, \\
\frac{\partial^2 F(x,t)}{\partial x^2} &= \frac{2!(At)^2}{(1-Axt+Bt^2)^3} &= \sum_{n=0}^{\infty} u_{n+2}^{(2)}(Ax,B)t^{n+1}, \\
& \cdots & \cdots \\
\frac{\partial^k F(x,t)}{\partial x^k} &= \frac{k!(At)^k}{(1-Axt+Bt^2)^{k+1}} &= \sum_{n=0}^{\infty} u_{n+k}^{(k)}(Ax,B)t^{n+k-1}.
\end{aligned}$$

by Lemma 2.2, we have

$$\begin{aligned}
u_n^{(k)}(Ax,B) &= \left\{ \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-1-i}{i} (Ax)^{n-1-2i}(-B)^i \right\}^{(k)} \\
&= k! \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-1-i}{i} \binom{n-1-2i-k}{k} \\
&\cdot A^{n-1-2i}(-B)^i x^{n-1-2i-k},
\end{aligned}$$

Thus,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{a_0+a_1+\cdots+a_k=n} w_{a_0}(Ax,B)w_{a_1}(Ax,B)\cdots w_{a_k}(Ax,B)t^n \\
=& (\sum_{n=0}^{\infty} w_n(Ax,B)t^n)^{k+1} \\
=& \frac{[a+(b-aAx)t]^{k+1}}{(1-Axt+Bt^2)^{k+1}} \\
=& \frac{[a+(b-aAx)t]^{k+1}}{k!A^k t^k} \sum_{n=0}^{\infty} u_{n+k+1}^{(k)}(Ax,B)t^{n+k} \\
=& \frac{1}{k!A^k} \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i}(b-aAx)^i t^i \sum_{n=0}^{\infty} u_{n+k+1}^{(k)}(Ax,B)t^n,
\end{aligned}$$

Compare the coefficients of t^n on both sides, we have

$$\sum_{a_0+a_1+\cdots+a_k=n} w_{a_0}(Ax,B)w_{a_1}(Ax,B)\cdots w_{a_k}(Ax,B)$$

$$= \frac{1}{k!A^k}\left[\sum_{i=0}^{k+1}\binom{k+1}{i}a^{k+1-i}(b-aAx)^i u^{(k)}_{n+k+1-i}(Ax,B)\right]$$
$$= \frac{1}{k!A^k}\sum_{i=0}^{k+1}\binom{k+1}{i}a^{k+1-i}(b-aAx)^i k!$$
$$\cdot\sum_{j=0}^{[\frac{n-i}{2}]}\binom{n+k-i-j}{i}\binom{n+k-i-2j}{k}A^{n+k-i-2j}(-B)^j x^{n-i-2j}. \tag{8}$$

Theorem 1.1 follows by taking $x = 1$ in (8). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 The following identity is well known (see[3]) that

$$w_{mn} = v_m w_{m(n-1)} - B^m w_{m(n-2)}.$$

Let $w_n^* = w_{mn}$, we have

$$w_n^* = v_m w_{n-1}^* - B^m w_{n-2}^*.$$

Now, we defined

$$R(x,t) = \sum_{n=0}^{\infty} w_n^*(v_m x, B^m)t^n,$$

so Theorem 1.2 follows from Theorem 1.1.

3 Corollaries of Theorems 1.1 and 1.2

In the case $u_n = w_n(0,1;A,B)$, $v_n = w_n(2,A;A,B)$, $F_n = w_n(0,1;1,-1)$ and $L_n = w_n(2,1;1,-1)$, (3) turns out to be

Corollary 3.1 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k be positive integer,

$$\sum_{a_0+a_1+\cdots+a_k=n} u_{a_0}u_{a_1}\cdots u_{a_k} =$$

$$\sum_{i=0}^{[\frac{n-k-1}{2}]}\binom{n-1-i}{i}\binom{n-1-2i}{k}A^{n-2i-k-1}(-B)^i, \tag{9}$$

and

$$\sum_{a_0+a_1+\cdots+a_k=n} v_{a_0}v_{a_1}\cdots v_{a_k} = \sum_{i=0}^{k+1}\sum_{j=0}^{[\frac{n-i}{2}]}\binom{k+1}{i}\binom{n+k-i-2j}{k}$$

$$\cdot\binom{n+k-i-j}{j}(-1)^{i+j}2^{k+i-1}A^{n-2j}B^{j}. \quad (10)$$

Remark 3.2 The result of Y.Yan [4] is essentially our (9) in the special case $A=1$ and $B=1$.

Corollary 3.3 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k be positive integer,

$$\sum_{a_0+a_1+\cdots+a_k=n} L_{a_0}L_{a_1}\cdots L_{a_k} =$$

$$\sum_{i=0}^{k+1}\sum_{j=0}^{[\frac{n-i}{2}]}\binom{k+1}{i}\binom{n+k-i-2j}{k}\binom{n+k-i-j}{j}(-1)^{i}2^{k+i-1}. \quad (11)$$

In the case $\{w_{mn}\}=\{u_{mn}\}$, $\{w_{mn}\}=\{v_{mn}\}$, $\{w_{mn}\}=\{F_{mn}\}$ and $\{w_{mn}\}=\{L_{mn}\}$, (4) becomes

Corollary 3.4 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k, m be positive integers,

$$\sum_{a_0+a_1+\cdots+a_k=n} u_{ma_0}u_{ma_1}\cdots u_{ma_k} =$$

$$\sum_{i=0}^{[\frac{n-k-1}{2}]}\binom{n-2i-1}{k}\binom{n-i-1}{i}L_m^{n-2i-k-1}(-B^m)^i u_m^{k+1}, \quad (12)$$

and

$$\sum_{a_0+a_1+\cdots+a_k=n} v_{ma_0}v_{ma_1}\cdots v_{ma_k} = \sum_{i=0}^{k+1}\sum_{j=0}^{[\frac{n-i}{2}]}\binom{k+1}{i}$$

$$\cdot\binom{n+k-i-2j}{k}\binom{n+k-i-j}{j}(-1)^{i+j}2^{k-i+1}v_m^{n-2j}B^{mj}. \quad (13)$$

Corollary 3.5 Let $a_0, a_1, \cdots, a_k$ be non-negative integers, and k, m be positive integers,

$$\sum_{a_0+a_1+\cdots+a_k=n} F_{ma_0}F_{ma_1}\cdots F_{ma_k} =$$

$$\sum_{i=0}^{[\frac{n-k-1}{2}]} \binom{n-2i-1}{k} \binom{n-i-1}{i} (-1)^{(m+1)i} L_m^{n-2i-k-1} F_m^{k+1}, \quad (14)$$

and

$$\sum_{a_0+a_1+\cdots+a_k=n} L_{ma_0} L_{ma_1} \cdots L_{ma_k} = \sum_{i=0}^{k+1} \sum_{j=0}^{[\frac{n-i}{2}]} \binom{k+1}{i}$$

$$\cdot \binom{n+k-i-2j}{k} \binom{n+k-i-j}{j} (-1)^{i+j+mj} 2^{k-i+1} L_m^{n-2j}. \quad (15)$$

Example 3.6 Let $m = 2, 3$, (14) becomes

$$\sum_{a_0+a_1+\cdots+a_k=n} F_{2a_0} F_{2a_1} \cdots F_{2a_k} =$$

$$\sum_{i=0}^{[\frac{n-k-1}{2}]} \binom{n-2i-1}{k} \binom{n-i-1}{i} (-1)^i 3^{n-2i-k-1}, \quad (16)$$

and

$$\sum_{a_0+a_1+\cdots+a_k=n} F_{3a_0} F_{3a_1} \cdots F_{3a_k} =$$

$$\sum_{i=0}^{[\frac{n-k-1}{2}]} \binom{n-2i-1}{k} \binom{n-i-1}{i} 2^{2n-4i-k-1}. \quad (17)$$

Example 3.7 Let $m = 2, 3$, (15) becomes

$$\sum_{a_0+a_1+\cdots+a_k=n} L_{2a_0} L_{2a_1} \cdots L_{2a_k} = \sum_{i=0}^{k+1} \sum_{j=0}^{[\frac{n-i}{2}]} \binom{k+1}{i}$$

$$\cdot \binom{n+k-i-2j}{k} \binom{n+k-i-j}{j} (-1)^{i+j} 2^{k+1-i} 3^{n-2j}, \quad (18)$$

and

$$\sum_{a_0+a_1+\cdots+a_k=n} L_{3a_0} L_{3a_1} \cdots L_{3a_k} = \sum_{i=0}^{k+1} \sum_{j=0}^{[\frac{n-i}{2}]} \binom{k+1}{i}$$

$$\cdot \binom{n+k-i-2j}{k} \binom{n+k-i-j}{j} (-1)^i 2^{2n+k+1-i-4j}. \quad (19)$$

References

[1] A.F.Horadam. *Basic Properties of a Certain Generalized Sequence of Numbers.* The Fibonacci Quarterly 4.3(1966): 274-282.

[2] H.Feng and Z.Zhang. *Computational Formulas for Convoluted Generalized Fibonacci and Lucas numbers.* The Fibonacci Quarterly 41.2(2003): 144-151.

[3] Stanley Ranbinowitz. *Algorithimic Manipulation of Second-order Linear Recurrences.* The Fibonacci Quarterly. 37.2(1999): 162-177.

[4] Y.Yi and W.Zhang. *Some Identities Involving the Fibonacci Polynomials.* The Fibonacci Quarterly 40.4(2002):314-348.

Chromaticity of Turán Graphs with Certain Matching or Star Deleted

G.C. Lau[a,c,1], Y.H. Peng[b,c]

[a]Faculty of Computer Science & Mathematics
Universiti Teknologi MARA (Segamat Campus)
Johor, Malaysia

[b]Department of Mathematics, and
[c]Institute for Mathematical Research
Universiti Putra Malaysia
43400 UPM Serdang, Malaysia

ABSTRACT

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H, $P(H, \lambda) = P(G, \lambda)$ implies H is isomorphic to G. In this paper, we study the chromaticity of Turán graphs with deleted edges that induce a matching or a star. As a by-product, we obtain new families of chromatically unique graphs.

AMS Mathematical Subject Classification (2000): 05C15;05C60

Keywords: Chromatic Polynomial; Chromatic Uniqueness; Turán Graph

1 Introduction

All graphs considered in this paper are finite and simple. For a graph G, we denote by $P(G; \lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (simply *χ-equivalent*), denoted $G \sim H$ if $P(G) = P(H)$. A graph G is said to be *chromatically unique* (simply *χ-unique*), if $H \sim G$ implies that $H \cong G$. A family $\mathcal{G}$ of graphs is said to be chromatically-closed (simply *χ-closed*) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. Many families of χ-unique graphs are known (see [4, 5]).

For a graph G, let $e(G)$, $v(G)$, $t(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles and chromatic number of G. By $\overline{G}$, we denote

[1]Corresponding author. E-mail: geeclau@yahoo.com

the complement of G. Let O_n be an edgeless graph with n vertices. Also let $Q(G)$ and $K(G)$ respectively be the number of induced subgraphs C_4 and complete subgraphs K_4 in G. Suppose S is a set of s edges of G. Denote by $G - S$ the graph obtained from G by deleting all edges in S, and by $\langle S \rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, let $K(p_1, p_2, \ldots, p_t)$ be a complete t-partite graph with partition sets V_i such that $|V_i| = p_i$ for $i = 1, 2, \ldots, t$. The Turán graph, denoted $T = K(t_1 \times p, t_2 \times (p+1))$ is the unique complete t-partite graph having $t_1 \geq 1$ partite sets of size p and t_2 partite sets of size $p+1$ (where $t_1 + t_2 = t$). It is shown in [2] that Turán graphs are χ-unique. In this paper, we study the chromaticity of Turán graph with deleted edges that induced a matching or a star. As a by-product, we obtain new families of χ-unique graphs.

2 Preliminary results and notations

Let $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ be the family $\{K(p_1, p_2, \ldots, p_t) - S \mid S \subset E(G)$ and $|S| = s\}$. For $p_1 \geq s+1$, we denote by $K_{i,j}^{-K(1,s)}(p_1, p_2, \ldots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a $K(1, s)$ with center in V_i and all the end-vertices in V_j, and by $K_{i,j}^{-sK_2}(p_1, p_2, \ldots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a matching with end-vertices in V_i and V_j. For convenience, we let $\mathcal{T}^{-s} = \mathcal{K}^{-s}(t_1 \times p, t_2 \times (p+1))$, $T_{i,j}^{-K(1,s)} = K_{i,j}^{-K(1,s)}(t_1 \times p, t_2 \times (p+1))$ and $T_{i,j}^{-sK_2} = K_{i,j}^{-sK_2}(t_1 \times p, t_2 \times (p+1))$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. If G is of order n, then $P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$ (see [8]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots,$ if $G \sim H$.

For a graph G with n vertices, the polynomial $\sigma(G, x) = \sum_{k=1}^{n} \alpha(G, k)x^k$ is called the *σ-polynomial* of G (see [1]), and the polynomial $h(G, x) = \sum_{k=1}^{n} \alpha(\overline{G}, k)x^k$ is called the adjoint polynomial of G (see [6]). Clearly, the conditions $P(G, \lambda) = P(H, \lambda)$, $\sigma(G, x) = \sigma(H, x)$ and $h(\overline{G}, x) = h(\overline{H}, x)$ are equivalent for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy | x \in V(G)$ and $y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t-partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [9].

Lemma 2.1. (Koh and Teo [4]) *Let G and H be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1, 2, \ldots,$ and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [1]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

The above lemma is equivalent to the following:

Remark. *Let G and H be two disjoint graphs. Then*

$$h(\overline{G} + \overline{H}, x) = h(\overline{G}, x)h(\overline{H}, x).$$

In particular,

$$h(\overline{K(n_1, n_2, \ldots, n_t)}, x) = \prod_{i=1}^{t} h(K_{n_i}, x).$$

Lemma 2.3. (Liu [7]) *Let G be a graph with $e \in E(G)$. If $e = uv$ does not belong to any triangle of G, then*

$$h(G,x) = h(G - e, x) + xh(G - \{u,v\}, x),$$

where $G - e$ (respectively $G - \{u,v\}$) denote the graph obtained from G by deleting the edge e (respectively the vertices u and v).

Denote by $\beta(G)$ the minimum real root of $h(G,x)$.

Lemma 2.4. (Zhao [10]) *Let G be a connected graph such that G contains H as a proper subgraph. Then*

$$\beta(G) < \beta(H).$$

Suppose $G = K(p_1, p_2, \ldots, p_t)$ and $H = G - S$ for a set S of s edges of G. Define $\alpha_k(H) = \alpha(H,k) - \alpha(G,k)$ for $k \geq t+1$.

Lemma 2.5. (Zhao [10]) *Let $G = K(p_1, p_2, \ldots, p_t)$ and $H = G - S$. If $p_1 \geq s + 1$, then*

$$s \leq \alpha_{t+1}(H) = \alpha(H, t+1) - \alpha(G, t+1) \leq 2^s - 1,$$

$\alpha_{t+1}(H) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H) = 2^s - 1$ if and only if $\langle S \rangle = K(1, s)$.

Lemma 2.6. (Dong et al. [3]) *Let p_1, p_2 and s be positive integers with $3 \leq p_1 \leq p_2$, then*

(i) *$K_{1,2}^{-K(1,s)}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_2 - 2$,*

(ii) *$K_{2,1}^{-K(1,s)}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_1 - 2$, and*

(iii) *$K^{-sK_2}(p_1, p_2)$ is χ-unique for $1 \leq s \leq p_1 - 1$*

In [10], Zhao obtained the following results on Turán graph with s edges deleted for $t_1 + t_2 \geq 5$.

Lemma 2.7. (Zhao [10]) *Let $s \geq 1$ and $t_1 \geq 1$. If $p \geq s + 2$, then $\mathcal{T}^{-s}$ is χ-closed.*

Lemma 2.8. (Zhao [10]) *Suppose $s \geq 1$ and $p \geq s + 2$, then*

(i) *every $T_{i,j}^{-K(1,s)}$ is χ-unique for any (i, j) where $1 \leq i \neq j \leq t$ and $|V_i| = |V_j| = p$, or $|V_i| = p$, $|V_j| = p + 1$, or $|V_i| = p + 1$, $|V_j| = p$, or $|V_i| = |V_j| = p + 1$.*

(ii) *$T_{1,2}^{-sK_2}$ is χ-unique if $t_1 = 2$.*

Note that Lemmas 2.7 and 2.8 hold for $t = 2, 3$ and 4 (see the proofs of Theorem 6.6.2 to Theorem 6.6.4 in [10]). Observe that if $G \in \mathcal{T}^{-s}$ such that $\alpha_{t+1}(G) = s$, then Lemma 2.7 also holds for $p > 2 + \log_2 s$ (see the proof of Theorem 6.6.2 in [10]).

Lemma 2.9. *If $p > 2 + \log_2 s$ and $G \in \mathcal{T}^{-s}$ such that $\alpha_{t+1}(G) = s$, then $\mathcal{T}^{-s}$ is χ-closed.*

It follows that Lemma 2.8(ii) also holds for $p > 2 + \log_2 s$.

By Lemmas 2.5 and 2.8, every graph $G \in \mathcal{T}^{-s}$ with $\alpha_{t+1}(G) = 2^s - 1$ is χ-unique if $p \geq s + 2$. However, only one family of graphs G is known to be χ-unique when $\alpha_{t+1}(G) = s$. We now give a necessary condition for two graphs G and H in $\mathcal{T}^{-s}$ (with $\alpha_{t+1}(G) = s$) to be χ-equivalent.

Suppose $F = K(p_1, p_2, \ldots, p_t)$. For $G = F - S$, denote by $t_i(G)$ the number of triangles in F that contain i deleted edges in S for $i = 1, 2, 3$. Suppose $G \in \mathcal{T}^{-s}$. An edge $e = uv$ in S is of Type A (respectively, Type B and Type C) if $u \in V_i$, $v \in V_j$ for $1 \le i < j \le t_1$ (respectively, for $1 \le i \le t_1$, $t_1 + 1 \le j \le t$, and for $t_1 + 1 \le i < j \le t$). Denote by $s_1(G)$ (respectively, $s_2(G)$ and $s_3(G)$) the number of Type A (respectively, Type B and Type C) edges in S.

Lemma 2.10. *Suppose G and H are two graphs in $\mathcal{T}^{-s}$ with $\alpha_{t+1}(G) = s$. If $G \sim H$, then $s_2(G) + 2s_3(G) + t_2(G) = s_2(H) + 2s_3(H) + t_2(H)$.*

Proof. Let $G = F - S$ and $H = F - S'$ be two graphs in $\mathcal{T}^{-s}$ with $\alpha_{t+1}(G) = s$. If $H \sim G$, then Lemma 2.1 implies that $\alpha_{t+1}(H) = s$ and $t(G) = t(H)$. Observe that both $\langle S \rangle$ and $\langle S' \rangle$ contain no K_3 subgraphs. So, $t_3(G) = t_3(H) = 0$. Note that $|S| = |S'| = s = \sum_{i=1}^{3} s_i(G) = \sum_{i=1}^{3} s_i(H)$. We now consider $t(G)$ and $t(H)$. Note that both G and H has t_1 and t_2 partite sets of size p and $p + 1$ respectively. Clearly,

$$\begin{aligned} t(G) &= t(F) - t_1(G) + t_2(G) \\ &= t(F) - s_1(G)\left((t_1 - 2)p + t_2(p+1)\right) - \\ &\quad s_2(G)\left((t_1 - 1)p + (t_2 - 1)(p+1)\right) - \\ &\quad s_3(G)\left(t_1 p + (t_2 - 2)(p+1)\right) + t_2(G) \\ &= t(F) - s\left(t_1 p + t_2(p+1)\right) + 2sp + \\ &\quad s_2(G) + 2s_3(G) + t_2(G). \end{aligned}$$

Similarly,

$$t(H) = t(F) - s[t_1 p + t_2(p+1)] + 2sp + s_2(H) + 2s_3(H) + t_2(H).$$

It follows immediately that $s_2(G) + 2s_3(G) + t_2(G) = s_2(H) + 2s_3(H) + t_2(H)$. □

In what follows, we let F be the Turán graph with $\chi(F) = t \ge 2$.

3 Turán graph with a matching deleted

For a graph $G \in \mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively, Type 2, and Type 3) if the vertices of the induced C_4 are in exactly two (respectively, three, and four) partite sets of $V(G)$. An example of induced C_4 of Type 1, 2 and 3 is shown in Figure 1.

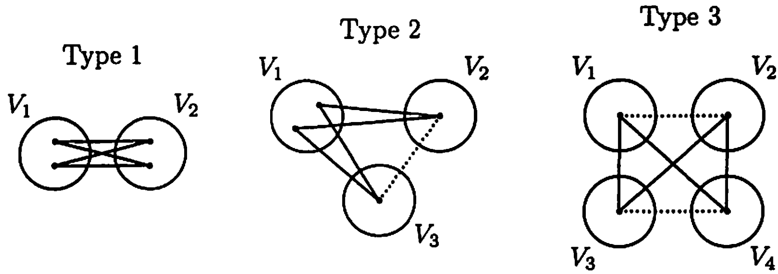

Figure 1. Three types of induced C_4 in a t-partite graph. Vertices joined by dotted lines are not adjacent.

Let S_{ij} $(1 \le i \le t, 1 \le j \le t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \ge 0$. We also say S is *ideal* if and only if $S = S_{ij}$ for some i and j, $1 \le i \le t$, $1 \le j \le t$.

Lemma 3.1. *For integer $t \ge 3$, let $F = K(p_1, p_2, \ldots, p_t)$ and $G = F - S$ for a set S of s edges in F. Suppose S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1\le i<j\le t} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - \sum_{1\le i<j<l\le t} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1 \le i < j \le t \\ 1 \le k < l \le t \\ i < k}} s_{ij}s_{kl} + \sum_{1\le i<j\le t}\left[s_{ij} \sum_{k\notin\{i,j\}} \binom{p_k}{2}\right] + \\ &\quad \sum_{\substack{1 \le i < j \le t \\ 1 \le i < k < l \le t \\ j \notin \{k,l\}}} s_{ij}s_{kl}, \end{aligned}$$

and

$$\begin{aligned} K(G) &= K(F) - \sum_{1\le i<j\le t}\left[s_{ij} \sum_{\substack{1 \le k < l \le t \\ \{i,j\}\cap\{k,l\} = \emptyset}} p_k p_l\right] + \\ &\quad \sum_{\substack{1 \le i < j \le t \\ 1 \le i < k < l \le t \\ j \notin \{k,l\}}} s_{ij}s_{kl}. \end{aligned}$$

Proof. Let $Q_1(G)$ (respectively, $Q_2(G)$ and $Q_3(G)$) be the number of Type 1 (respectively, Type 2 and Type 3) induced C_4 in G. Observe that $S = \bigcup_{1 \le i < j \le t} S_{ij}$. Hence,

$$\begin{aligned} Q_1(G) &= \sum_{1 \le i < j \le t} \binom{p_i}{2}\binom{p_j}{2} - \sum_{1 \le i < j \le t} (p_i - 1)(p_j - 1)s_{ij} + \sum_{1 \le i < j \le t} \binom{s_{ij}}{2} \\ &= Q(F) - \sum_{1 \le i < j \le t} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \le i < j < l \le t} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1 \le i < j \le t \\ 1 \le k < l \le t \\ i < k}} s_{ij}s_{kl}. \end{aligned}$$

Note that $\sum_{1 \le i < j < l \le t} s_{ij}s_{il} \ge 0$ and $\sum_{\substack{1 \le i < j \le t \\ 1 \le k < l \le t \\ i < k}} s_{ij}s_{kl} \ge 0$, and the equality holds if S is ideal.

We now find $Q_2(G)$. For three distinct indices i, j, k; $1 \le i, j, k \le t$, let v_iv_j be an edge in S such that $v_i \in V_i$ and $v_j \in V_j$, and let v_k, v'_k be two distinct vertices in V_k. It is clear that $v_iv_kv_jv'_kv_i$ is a Type 2 induced C_4 in G. Since the number of 2-element subsets of V_k is $\binom{p_k}{2}$, we have

$$Q_2(G) = \sum_{1 \le i < j \le t} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{p_k}{2} \right].$$

We now find $Q_3(G)$. For four distinct indices i, j, k, l; $1 \le i, j, k, l \le t$, let v_iv_j and v_kv_l be two edges in S such that $v_a \in V_a$ for $a \in \{i, j, k, l\}$. It is clear that $v_iv_kv_jv_lv_i$ is a Type 3 induced C_4 in G. Hence,

$$Q_3(G) = \sum_{\substack{1 \le i < j \le t \\ 1 \le i < k < l \le t \\ j \notin \{k,l\}}} s_{ij}s_{kl}.$$

Note that $Q_3(G) = 0$ if for any two distinct edges in S, say v_iv_j and v_kv_l, $\{i, j\} \cap \{k, l\} \ne \emptyset$. We now find $K(G)$. Observe that each K_4 subgraph in F has at most two edges in S. Let $K_m(G)$ be the number of K_4 subgraphs in F that contains m edges in S for $m = 1, 2$. Hence, $K(G) = K(F) - K_1(G) + K_2(G)$. Clearly,

$$K(F) = \sum_{1 \le i < j < k < l \le t} p_ip_jp_kp_l.$$

Let v_iv_j be an edge in S such that $v_i \in V_i$ and $v_j \in V_j$. Then, the number of K_4 subgraphs in F that contains v_iv_j is $\sum_{1\leq k<l\leq t} p_kp_l$ where $\{i,j\} \cap \{k,l\} = \emptyset$. Hence,

$$K_1(G) = \sum_{1\leq i<j\leq t} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq t \\ \{i,j\} \cap \{k,l\} = \emptyset}} p_kp_l \right].$$

Observe that there is a one-to-one correspondence between the set of Type 3 induced C_4 in G and the set of K_4 subgraph in F that contains two edges in S. Hence, $K_2(G) = Q_3(G)$.

This completes the proof. □

We now present two main results that extend Lemma 2.8(ii) above.

Theorem 3.1. *Suppose $s \geq 1$ and $p \geq s+2$, then $T_{1,2}^{-sK_2}$ is χ-unique for $t_1 = 1$.*

Proof. Suppose $H \sim G = T_{1,2}^{-sK_2}$ with $t_1 = 1$. By Lemma 2.6, the theorem holds for $t = 2$. So, we assume $t \geq 3$. We shall show that $H \cong G$.

By Lemma 2.9, $H \in \mathcal{T}^{-s}$ with $t_1 = 1$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G) = s$. Hence, $s_1(G) = s_3(G) = t_2(G) = s_1(H) = 0$, and so $s = s_2(G)$. By Lemma 2.10, we have that $s = s_2(H)+2s_3(H)+t_2(H)$. We assert that $s_3(H) = t_2(H) = 0$. Otherwise, $s_2(H) + 2s_3(H) + t_2(H) > s$, a contradiction.

It follows that for H each deleted edge in S must have one end-vertex in V_1, and another end-vertex in V_j for $2 \leq j \leq t$. Hence, each edge in S must be in one of S_{1j} $(2 \leq j \leq t)$. Moreover, all the edges in S must induce a matching in F. Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$. Clearly, $H \cong G$ if S is ideal. Otherwise, we may assume that $S_{1j} \neq \emptyset$ for $2 \leq j \leq k$ and $k \geq 3$. Observe that each induced C_4 in G and H is of Type 1 or 2. It is easy to see that

$$Q(G) = Q(F) - s(p-1)p + \binom{s}{2} + s(t-2)\binom{p+1}{2}.$$

By Lemma 3.1, we have

$$\begin{aligned} Q(H) &= Q(F) - s(p-1)p + \binom{s}{2} - \sum_{2 \leq j < l \leq k} s_{1j}s_{1l} + s(t-2)\binom{p+1}{2} \\ &= Q(G) - \sum_{2 \leq j < l \leq k} s_{1j}s_{1l}, \end{aligned}$$

and

$$\begin{aligned} K(H) &= K(F) - \binom{t-2}{2}(p+1)^2 \sum_{2\le j\le k} s_{1j} \\ &= K(F) - \binom{t-2}{2} s(p+1)^2 \\ &= K(G). \end{aligned}$$

Hence, $2K(G) - Q(G) \neq 2K(H) - Q(H)$. This contradicts Lemma 2.1. Therefore, G is χ-unique and the proof is now complete. □

Theorem 3.2. *Suppose $s \geq 1$ and $p \geq s+2$, then $T_{1,2}^{-sK_2}$ is χ-unique for $t_1 \geq 2$.*

Proof. Suppose $H \sim G = K_{1,2}^{-sK_2}$ with $t_1 \geq 2$. We shall show that $H \cong G$. By Lemma 2.9, $H \in \mathcal{T}^{-s}$ with $t_1 \geq 2$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G) = s$. Hence, $s_1(G) = s$ and $s_2(G) = s_3(G) = t_2(G) = 0$. By Lemma 2.10, we have that $s_2(H) + 2s_3(H) + t_2(H) = 0$. It follows immediately that $s_2(H) = s_3(H) = t_2(H) = 0$ and so $s_1(H) = s$.

Therefore, relatively to H each deleted edge in S must have one end-vertex in V_i and another end-vertex in V_j for $1 \leq i < j \leq t_1$. Hence, each edge in S must be in one of S_{ij} $(1 \leq i < j \leq t_1)$. Moreover, all the edges in S must induce a matching in F. Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$.

Clearly, $H \cong G$ if S is ideal. Otherwise, there exist i, j, k and l such that S_{ij} $(1 \leq i < j \leq t_1)$ and S_{kl} $(1 \leq k < l \leq t_1)$ are two disjoint non-empty subsets of S. Observe that each induced C_4 in G (respectively H) is of Type 1 or 2 (respectively Type 1, 2 or 3). It is easy to see that

$$Q(G) = Q(F) - s(p-1)^2 + \binom{s}{2} + s\left[(t_1-2)\binom{p}{2} + t_2\binom{p+1}{2}\right],$$

By Lemma 3.1, we have

$$\begin{aligned} Q(H) &= Q(F) - s(p-1)^2 + \binom{s}{2} - \sum_{1\le i<j<l\le t_1} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1\le i<j\le t_1 \\ 1\le k<l\le t_1 \\ i<k}} s_{ij}s_{kl} + s\left[(t_1-2)\binom{p}{2} + t_2\binom{p+1}{2}\right] + \\ &\quad \sum_{\substack{1\le i<j\le t_1 \\ 1\le i<k<l\le t_1 \\ j\notin\{k,l\}}} s_{ij}s_{kl} \end{aligned}$$

$$= Q(G) - \sum_{1\le i<j<l\le t_1} s_{ij}s_{il} - \sum_{\substack{1\le i<j\le t_1\\ 1\le k<l\le t_1\\ i<k}} s_{ij}s_{kl} +$$

$$\sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k.l\}}} s_{ij}s_{kl},$$

where $\sum_{1\le i<j<l\le t_1} s_{ij}s_{il} + \sum_{\substack{1\le i<j\le t_1\\ 1\le k<l\le t_1\\ i<k}} s_{ij}s_{kl} > 0$, and $\sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k,l\}}} s_{ij}s_{kl} \ge 0$. Moreover,

$$\begin{aligned} K(H) &= K(F) - \left[\binom{t_1-2}{2}p^2 + (t_1-2)t_2 \times p(p+1) + \binom{t_2}{2}(p+1)^2\right] \times \\ &\quad \sum_{1\le i<j\le t_1} s_{ij} + \sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k,l\}}} s_{ij}s_{kl} \\ &= K(F) - s\left[\binom{t_1-2}{2}p^2 + (t_1-2)t_2 \times p(p+1) + \right. \\ &\quad \left. \binom{t_2}{2}(p+1)^2\right] + \sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k,l\}}} s_{ij}s_{kl} \\ &= K(G) + \sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k,l\}}} s_{ij}s_{kl}. \end{aligned}$$

Clearly,

$$\begin{aligned} 2K(H) - Q(H) &= 2K(G) - Q(G) + \sum_{1\le i<j<l\le t_1} s_{ij}s_{il} + \\ &\quad \sum_{\substack{1\le i<j\le t_1\\ 1\le k<l\le t_1\\ i<k}} s_{ij}s_{kl} + \sum_{\substack{1\le i<j\le t_1\\ 1\le i<k<l\le t_1\\ j\notin\{k,l\}}} s_{ij}s_{kl}, \end{aligned}$$

contradicting Lemma 2.1.

The proof is now complete. □

Note that Lemma 2.8(ii) now follows directly from Theorem 3.2.

A matching subgraph sK_2 of $K(p_1, p_2, \ldots, p_t)$ is called a *special* sK_2 if no two vertices of sK_2 are in the same partite set of $K(p_1, p_2, \ldots, p_t)$. Let $G^= \in \mathcal{T}^{-s}$ such that $\langle S\rangle$ is a special sK_2 with each vertex belongs to a partite set of size p. For $m \leq n$, denote by $K_m \cdot K_n$ the graph obtained from $K_m + K_n$ by joining by an edge a vertex of K_m and a vertex of K_n. We now prove the chromatic uniqueness of the graph $G^=$ in the following theorem.

Theorem 3.3. *Suppose* $t_1 \geq 2s \geq 2$ *and* $p > 2 + \log_2 s$, *then* $G^=$ *is* χ*-unique.*

Proof. Suppose $H \sim G^=$. We shall show that $H \cong G$.

By Lemma 2.9, $H \in \mathcal{T}^{-s}$, and $\alpha_{t+1}(H) = \alpha_{t+1}(G^=) = s$. Hence, $s_1(G^=) = s$ and $s_2(G^=) = s_3(G^=) = t_2(G^=) = 0$. By an argument similar to that in Theorem 3.2, we conclude that $s_1(H) = s$ and $s_2(H) = s_3(H) = t_2(H) = 0$. Hence, each deleted edge in S must have one end-vertex in V_i and another end-vertex in V_j for $1 \leq i < j \leq t_1$. Moreover, all the edges in S must induce a matching. Otherwise, either $t_2(H) > 0$ or $\alpha_{t+1}(H) > s$.

Claim. $\langle S\rangle$ is a special sK_2.

Proof of the claim. Suppose the claim is not true. We shall show that $\beta(\overline{G^=}) > \beta(\overline{H})$, contradicting $\beta(\overline{G^=}) = \beta(\overline{H})$.

By Lemma 2.2, we have $h(\overline{G^=}, x) = [h(K_p \cdot K_p, x)]^s [h(K_p, x)]^{t_1 - 2s} \times [h(K_{p+1}, x)]^{t_2}$ and $h(\overline{H}, x) = h(\overline{H'}, x)[h(K_{p+1}, x)]^{t_2}$. So, $h(\overline{G^=}, x) = h(\overline{H}, x)$ implies that $h(\overline{G'}, x) = h(\overline{H'}, x)$ for $h(\overline{G'}, x) = [h(K_p \cdot K_p, x)]^s \times [h(K_p, x)]^{t_1 - 2s}$. Clearly, $\overline{H'}$ is a graph that contains a $K_p \cdot K_p$ as a proper subgraph. Hence, by Lemma 2.4, $\beta(\overline{G'}) = \beta(K_p \cdot K_p) > \beta(\overline{H'})$, a contradiction.

By the above claim, we have that $H \cong G^=$ and the proof is now complete. □

4 Turán graph with a star deleted

A star subgraph $K(1, s)$ of $K(p_1, p_2, \ldots, p_t)$ is called a *special* $K(1, s)$ if no two end-vertices of $K(1, s)$ are in the same partite set of G. Let $G_1^\star \in \mathcal{T}^{-s}$ such that $\langle S\rangle$ is a special $K(1, s)$ with each vertex belongs to a partite set of size $p+1$, and for $t_2 = 1$, let $G_2^\star \in \mathcal{T}^{-s}$ such that $\langle S\rangle$ is a special $K(1, s)$ with central vertex belongs to the only partite set of size $p + 1$. We now prove the chromatic uniqueness of the graphs $G_1^\star$ and $G_2^\star$ in the following two theorems.

Theorem 4.1. *Suppose $1 \le s \le t_2 - 1$ and $p > 2 + \log_2 s$, then G_1^* is χ-unique.*

Proof. Suppose $H \sim G_1^*$. We shall show that $H \cong G_1^*$.

By Lemma 2.9, $H \in \mathcal{T}^{-s}$ and $\alpha_{t+1}(H) = \alpha_{t+1}(G_1^*) = s$. If $s = 1$, then Lemma 2.10 implies that $s_3(H) = 1$. So, $H \cong G_1^*$. Hence, we may assume $s \ge 2$. Now, $s_1(G_1^*) = s_2(G_1^*) = 0$, $t_2(G_1^*) = \binom{s}{2}$ and $s = s_3(G_1^*)$. By Lemma 2.10, we have $2s + \binom{s}{2} = s_2(H) + 2s_3(H) + t_2(H)$. Note that $t_2(H) \le \binom{s}{2}$. We assert that $s_1(H) = s_2(H) = 0$.

If $s_1(H) \ne 0$, then $s_2(H) + s_3(H) < s$. Hence $s_2(H) + 2s_3(H) + t_2(H) < 2s + \binom{s}{2}$, a contradiction. Therefore, $s_1(H) = 0$. Consequently, if $s_2(H) \ne 0$, we also have a similar contradiction. Hence, $s_3(H) = s$ and $t_2(H) = \binom{s}{2}$.

Claim. *$\langle S \rangle$ is a special $K(1, s)$.*

Proof of the claim. Suppose the claim does not hold. We now show that $t_2(H) < \binom{s}{2}$, which is a contradiction. We proceed by induction on s. If $s = 2$ and $\langle S \rangle$ is not a special $K(1, 2)$, then $t_2(H) = 0 < \binom{2}{2}$. Hence, the claim holds for $s = 2$. Assume that the claim holds for $s = n \ge 3$. Let $s = n + 1$ and $\langle S \rangle$ is not a special $K(1, n + 1)$. Suppose e is an edge in S, then $S - \{e\}$ has n edges. Let t_2' (respectively t_2'') be the number of triangles in $K(p_1, p_2, \ldots, p_t)$ that contain two edges in S and do not contain (respectively contain) the edge e. We consider two cases.

Case 1. $\langle S - \{e\} \rangle$ is a special $K(1, n)$. In this case, edge e is not adjacent to the central vertex of $K(1, n)$. Therefore, $t_2' = \binom{n}{2}$ and $t_2'' \le 1$. Hence, $t_2(H) \le \binom{n}{2} + 1 < \binom{n+1}{2}$.

Case 2. $\langle S - \{e\} \rangle$ is not a special $K(1, n)$. In this case, $\max\{t_2(H)\}$ is attained if the edge e is adjacent to each edge in $\langle S - \{e\} \rangle$. By induction hypotheses, $t_2' < \binom{n}{2}$, whereas $t_2'' \le n$. Hence, $t_2(H) < \binom{n}{2} + n = \binom{n+1}{2}$.

Therefore, the claim holds. Consequently, we have $H \cong G_1^*$.

The proof is now complete. □

Corollary 4.1. *Suppose $1 \le s \le t - 1$ and $p > 2 + \log_2 s$, then $K^{-s}(\underbrace{p, \ldots, p}_{t})$ is χ-unique if $\langle S \rangle$ is a special $K(1, s)$.*

Theorem 4.2. *Suppose $1 \le s \le t - 1$ and $p > 2 + \log_2 s$, then G_2^* is χ-unique.*

Proof. Suppose $H \sim G_2^\star$. We shall show that $H \cong G_2^\star$.

By Lemma 2.9, $H \in \mathcal{K}^{-s}(\underbrace{p, \ldots, p}_{t-1}, p+1)$ and $\alpha_{t+1}(H) = \alpha_{t+1}(G_2^\star) = s$. If $s = 1$, then Lemma 2.10 implies that $s_2(H) = 1$. So, $H \cong G_2^\star$. Hence, we may assume $s \geq 2$. Now, $s_1(G_2^\star) = s_3(G_2^\star) = s_3(H) = 0$, $s_2(G_2^\star) = s$ and $t_2(G_2^\star) = \binom{s}{2}$. By Lemma 2.10, we have $s + \binom{s}{2} = s_2(H) + t_2(H)$.

Note that $t_2(H) \leq \binom{s}{2}$. We assert that $s_1(H) = 0$. Suppose $s_1(H) \neq 0$, then $s_2(H) < s$. This means $s_2(H) + t_2(H) < s + \binom{s}{2}$, a contradiction. Hence, $s_2(H) = s$ and $t_2(H) = \binom{s}{2}$. From the claim in the proof of Theorem 4.1, we know that $\langle S \rangle$ is a special star with central vertex in V_t. Hence, $H \cong G_2^\star$.

The proof is now complete. □

Remark. Theorem 4.2 is best possible in the sense that if $G_2^\star$ contains more than one partite set of size $p+1$, then it is not χ-unique. We can see this as follows. For $t_1, t_2 \geq s \geq 2$, let G' (respectively G'') be a graph in $\mathcal{T}^{-s}$ such that $\langle S \rangle$ is a special $K(1, s)$, $s \geq 2$, where the central vertex belongs to partite set of size p (respectively size $p+1$) and the end-vertices belong to partite sets of size $p+1$ (respectively size p). It is clear that $G' \not\cong G''$. By Lemma 2.3 and mathematical induction on s, it is easy to show that $h(\overline{G'}, x) = h(\overline{G''}, x)$ for all $p \geq 1$. Hence, G' and G'' are chromatically equivalent. It follows immediately that for integers $k \geq 1$, $\underbrace{G' \vee G' \vee \ldots \vee G'}_{k}$ is chromatically equivalent to $\underbrace{G'' \vee G'' \vee \ldots \vee G''}_{k}$ but they are not isomorphic.

Note that for G' with $t_1 < s$ and G'' with $t_2 < s$, we have $G' \not\sim G''$. We can show that G' (respectively, G'') is χ-unique if $t_1 < s$ (respectively, $t_2 < s$) for $s = 2, 3$. We end this paper with the following conjecture and problem.

Conjecture 4.1. *The graphs G' with $t_1 < s$ and G'' with $t_2 < s$ are χ-unique.*

Problem 4.1. *Study the chromaticity of Turán graphs with a matching or a star deleted.*

Acknowledgments

The authors would like to thank the referee for the valuable comments.

References

1. Brenti, F.: Expansions of chromatic polynomial and log-loncavity, Trans. Amer. Math. Soc. **332** (1992) 729–756.

2. Chao, C.Y., Novacky Jr., G.A.: On maximally saturated graphs, *Discrete Math.* **41** (1982), 139–143.

3. Dong, F.M., Koh, K.M., Teo, K.L., Little, C.H.C., Hendy, M.D.: Sharp bounds for the number of 3-independent partitions and the chromaticity of bipartite graphs, *J. Graph Theory* **37** (2001), 48–77.

4. Koh K.M., Teo, K.L.: The search for chromatically unique graphs, *Graphs and Combinatorics* **6** (1990) 259–285.

5. Koh K.M., Teo, K.L.: The search for chromatically unique graphs - II, *Discrete Math.* **172** (1997) 59–78.

6. Liu, R.Y.: A new method to find the chromatic polynomial of a graph and its applications, *Kexue Tongbao* **32** (1987), 1508 – 1509.

7. Liu, R.Y.: Adjoint polynomials of graphs (Chinese), *J. Qinghai Normal Univ. (Natur. Sci.)* (1990), **No. 1**, 1–9.

8. Read, R.C., Tutte, W.T.: Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, **Vol. 3**, Academic Press, New York, 1988, 15–42.

9. West, D.B.: Introduction to Graph Theory, second ed., Prentice Hall, New Jersey, 2001.

10. Zhao, H.X.: Chromaticity and adjoint polynomials of graphs, Ph.D. *thesis* (2005) University of Twente, Netherland.