Critical problems of totally isotropic subspaces in finite orthogonal spaces *

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Abstract

As applications of the Anzahl theorems in finite orthogonal spaces, we study critical problem of totally isotropic subspaces, and obtain critical exponent.

Key words: Orthogonal space; totally isotropic subspace; critical problem.

1 Introduction

In this section we shall introduce the concepts of totally isotropic subspaces in orthogonal spaces. Notation and terminologiy will be adopted from Wan's book [5].

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. Denote by $\mathcal{K}_{2\nu+\delta}$ the set of all $(2\nu+\delta)\times(2\nu+\delta)$ alternate matrices over $\mathbb{F}_q$, where $\delta = 0, 1$ or 2. Two $(2\nu+\delta)\times(2\nu+\delta)$ matrices A and B over $\mathbb{F}_q$ are said to be *congruent* mod $\mathcal{K}_{2\nu+\delta}$, denoted by $A \equiv B \pmod{\mathcal{K}_{2\nu+\delta}}$, if $A - B \in \mathcal{K}_{2\nu+\delta}$. Clearly, $\equiv$ is an equivalence relation on the set of all $(2\nu+\delta)\times(2\nu+\delta)$ matrices. Let $[A]$ denote the equivalence class containing A. Two matrix classes $[A]$ and $[B]$ are said to be *cogredient* if there is a nonsingular $(2\nu+\delta)\times(2\nu+\delta)$ matrix Q over $\mathbb{F}_q$ such that $[QAQ^t] \equiv [B]$.

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For q being odd, let

$$S_{2s+\delta,\Delta} = \begin{pmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & \Delta \end{pmatrix}, \text{ where } \Delta = \begin{cases} \emptyset, & \text{if } \delta = 0, \\ (1) \text{ or } (z), & \text{if } \delta = 1, \\ \text{diag}(1, -z), & \text{if } \delta = 2, \end{cases}$$

where z is a fixed non-square element of $\mathbb{F}_q$. For q being even, let

$$S_{2s+\delta,\Delta} = \begin{pmatrix} 0 & I^{(s)} & \\ & 0 & \\ & & \Delta \end{pmatrix}, \text{ where } \Delta = \begin{cases} \emptyset, & \text{if } \delta = 0, \\ (1), & \text{if } \delta = 1, \\ \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}, & \text{if } \delta = 2, \end{cases}$$

where α is a fixed element of $\mathbb{F}_q$ such that $\alpha \notin \{x^2 + x \,|\, x \in \mathbb{F}_q\}$. The *orthogonal group* of degree $2\nu+\delta$ over $\mathbb{F}_q$ with respect to $S_{2\nu+\delta,\Delta}$, denoted by $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$, consists of all $(2\nu+\delta)\times(2\nu+\delta)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $[TS_{2\nu+\delta,\Delta}T^t] \equiv [S_{2\nu+\delta,\Delta}]$. There is an action of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ on $\mathbb{F}_q^{2\nu+\delta}$ defined as follows:

$$\begin{array}{rcl} \mathbb{F}_q^{2\nu+\delta} \times O_{2\nu+\delta,\Delta}(\mathbb{F}_q) & \to & \mathbb{F}_q^{2\nu+\delta} \\ ((x_1, x_2, \ldots, x_{2\nu+\delta}),\, T) & \mapsto & (x_1, x_2, \ldots, x_{2\nu+\delta})T. \end{array}$$

The row vector space $\mathbb{F}_q^{2\nu+\delta}$ together with the above group action of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ is called the $(2\nu+\delta)$-dimensional *orthogonal space* over $\mathbb{F}_q$. Let P be an m-dimensional subspace of $\mathbb{F}_q^{2\nu+\delta}$, denote also by P an $m \times (2\nu+\delta)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P. An m-dimensional subspace P in the $(2\nu+\delta)$-dimensional orthogonal space is a subspace of *type* $(m, 2s+\gamma, s, \Gamma)$ if $PS_{2\nu+\delta,\Delta}P^t$ is cogredient to $\text{diag}\,(S_{2s+\gamma,\Gamma}, 0^{(m-2s-\gamma)})$. In particular, subspaces of type $(m, 0, 0)$ are called m-dimensional *totally isotropic subspaces*, and ν-dimensional totally isotropic subspaces are called *maximal totally isotropic subspaces.* Let $\mathcal{M}(m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)$ denote the set of type $(m, 2s+\gamma, s, \Gamma)$. For q being odd, by [5, Theorem 6.3], subspaces of type $(m, 2s+\gamma, s, \Gamma)$ exist in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ if and only if

$$2s+\gamma \le m \le \begin{cases} \nu + s + \min\{\gamma, \delta\}, & \text{if } \gamma \ne \delta \text{ or } \gamma = \delta \text{ and } \Gamma = \Delta, \\ \nu + s, & \text{if } \gamma = \delta = 1 \text{ and } \Gamma \ne \Delta. \end{cases}$$

For q being even, by [5, Theorem 7.5], subspaces of type $(m, 2s+\gamma, s, \Gamma)$ exist in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ if and only if

$$2s+\gamma \le m \le \begin{cases} \nu + s + \min\{\gamma, \delta\}, & \text{if } \delta \ne 1, \text{ or } \gamma \ne 1, \text{ or } \gamma = \delta = 1 \\ & \text{and } \Gamma = 1, \\ \nu + s, & \text{if } \gamma = \delta = 1 \text{ and } \Gamma = 0. \end{cases}$$

By [5, Theorems 6.4 and 7.6], $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ acts transitively on $\mathcal{M}(m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)$.

Let $N(m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta) = |\mathcal{M}(m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)|$. Let P be a fixed subspace of type $(m, 2s+\gamma, s, \Gamma)$ in $\mathbb{F}_q^{2\nu+\delta}$. Denote by $\mathcal{M}(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m_1, 2s_1+\gamma_1, s_1, \Gamma_1)$ contained in P, and let $N(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta) = |\mathcal{M}(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)|$. Let P_1 be a fixed subspace of type $(m_1, 2s_1+\gamma_1, s_1, \Gamma_1)$ in $\mathbb{F}_q^{2\nu+\delta}$. Denote by $\mathcal{M}'(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m, 2s+\gamma, s, \Gamma)$ containing P_1, and let $N'(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta) = |\mathcal{M}'(m_1, 2s_1+\gamma_1, s_1, \Gamma_1; m, 2s+\gamma, s, \Gamma; 2\nu+\delta, \Delta)|$.

Critical problems are interest problems and have a vast literature (see [3, 6], for examples). The results on the critical problems of subspaces under finite classical groups can be found in Wan ([4]), Crapo and Rota ([1]), Kung ([2]). In [1], Crapo and Rota study the critical problems of finite vector spaces. In [2], Kung studies the critical problems of totally isotropic subspaces of finite symplectic spaces. In [4], Wan studies the critical problems non-isotropic subspaces of finite unitary spaces. Their researches stimulate us to consider the critical problems of orthogonal spaces. In this paper, we study the critical problems of totally isotropic spaces of orthogonal spaces.

2 Main results

In this section, we shall study the critical problems of totally isotropic spaces of orthogonal spaces. We begin with some useful lemmas.

Lemma 2.1. *Let $\nu \geq k \geq r$. The number of k-dimensional totally isotropic subspaces in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ containing a given r-dimensional totally isotropic subspace is*

$$N'(r,0,0;k,0,0;2\nu+\delta,\Delta) = \prod_{i=0}^{k-r-1} \frac{(q^{\nu-r-i}-1)(q^{\nu-r-i+\delta-1}+1)}{q^{k-r-i}-1}.$$

Proof. By [5, Corollaries 6.23 and 7.25],

$$N(r,0,0;2\nu+\delta,\Delta) = \frac{\prod_{i=\nu-r+1}^{\nu}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{r}(q^i-1)},$$

$$N(k,0,0;2\nu+\delta,\Delta) = \frac{\prod_{i=\nu-k+1}^{\nu}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{k}(q^i-1)}.$$

Clearly,

$$N(r,0,0;k,0,0;2\nu+\delta,\Delta) = \frac{\prod_{i=k-r+1}^{k}(q^i-1)}{\prod_{i=1}^{r}(q^i-1)}.$$

In order to compute $N'(r,0,0;k,0,0;2\nu+\delta,\Delta)$, we define M to be a binary matrix with row-indexed (resp. column-indexed) by $\mathcal{M}(r,0,0;2\nu+\delta,\Delta)$ (resp. $\mathcal{M}(k,0,0;2\nu+\delta,\Delta)$), whose (A,B) entry $M(A,B)=1$ if $A\subseteq B$, and 0 otherwise. Counting the number of 1's in the matrix by rows, we obtain

$$N(r,0,0;2\nu+\delta,\Delta)N'(r,0,0;k,0,0;2\nu+\delta,\Delta).$$

Counting the number of 1's in the matrix by columns, we obtain

$$N(k,0,0;2\nu+\delta,\Delta)N(r,0,0;k,0,0;2\nu+\delta,\Delta).$$

Therefore,

$$\begin{aligned}
& N'(r,0,0;k,0,0;2\nu+\delta,\Delta)\\
= & \frac{N(k,0,0;2\nu+\delta,\Delta)N(r,0,0;k,0,0;2\nu+\delta,\Delta)}{N(r,0,0;2\nu+\delta,\Delta)}\\
= & \frac{\prod_{i=\nu-k+1}^{\nu-r}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{k-r}(q^i-1)}\\
= & \prod_{i=0}^{k-r-1}\frac{(q^{\nu-r-i}-1)(q^{\nu-r-i+\delta-1}+1)}{q^{k-r-i}-1}.
\end{aligned}$$

□

Corollary 2.2. *The number of maximal totally isotropic subspaces in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ containing a given r-dimensional totally isotropic subspace is*

$$N'(r,0,0;\nu,0,0;2\nu+\delta,\Delta) = \prod_{i=0}^{\nu-r-1} (q^{\nu-r-i+\delta-1}+1).$$

A set X of vectors over $\mathbb{F}_q^{2\nu+\delta}$ is said to be *isotropic set*, if $uSv' = 0$, for all $u, v \in X$. Let X be a non-empty set of vectors of $\mathbb{F}_q^{2\nu+\delta}$. Denote by $\langle X \rangle$ the subspace spanned by X. Clearly, $\langle X \rangle$ is a totally isotropic subspace if X is an isotropic set of vectors. The *rank* of X, denoted by $r(X)$, is defined as $\dim\langle X \rangle$. If $X = \emptyset$, we agree that $\langle X \rangle = \emptyset$ and $r(X) = 0$.

Let S be a set of non-zero vectors in orthogonal space $\mathbb{F}_q^{2\nu+\delta}$. A totally isotropic subspace P is said to *distinguish* S if $P \cap S = \emptyset$. The *orthogonal critical exponent* of S, denoted by $c_{\text{or}}(S, \mathbb{F}_q^{2\nu+\delta})$ is defined as the minimum positive integer $\lambda \leq \nu + 1$ such that there exists a $(\nu + 1 - \lambda)$-dimensional totally isotropic subspace distinguish S. Let $P_1, P_2, \ldots, P_\lambda$ be λ maximal totally isotropic subspaces. $(P_1, P_2, \ldots, P_\lambda)$ is called the λ-*tuple* of maximal totally isotropic subspaces, if

$$\dim(P_1 \cap P_2) = \nu - 1, \ \dim(P_1 \cap P_2 \cap P_3) = \nu - 2,$$
$$\ldots, \ \dim(P_1 \cap P_2 \cap \cdots \cap P_\lambda) = \nu + 1 - \lambda.$$

Since any $(\nu+1-\lambda)$-dimensional totally isotropic subspace is an intersection of a λ-tuple of maximal totally isotropic subspaces, the orthogonal critical exponent of S can also be defined as minimum positive integer λ such that there exist a λ-tuple of maximal totally isotropic subspaces $(P_1, P_2, \ldots, P_\lambda)$ such that $\bigcap_{i=1}^{\lambda} P_i$ distinguishes S, i.e. $(\bigcap_{i=1}^{\lambda} P_i) \cap S = \emptyset$. By convention we also regard the intersection of totally isotropic subspace of 0-tuple as $\mathbb{F}_q^{2\nu+\delta}$. The equivalence of the two definitions is clear.

Let S be a set of non-zero vectors in $\mathbb{F}_q^{2\nu+\delta}$, $M(S)$ be the *matroid* on S defined by linear independence of vectors, and $L(M(S))$ be the *lattice* of flats of $M(S)$. An *isotropic flat* is a flat which is also an isotropic set of vectors. Clearly, subsets of isotropic set of vectors are also isotropic. It follows that the collection of isotropic flats forms an *ideal* in the lattice $L(M(S))$ and this ideal will be denoted by $L_I(M(S))$.

Lemma 2.3. *Let $\nu \geq k \geq r$. The number of k-dimensional totally isotropic subspaces in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ containing a given rank-r isotropic set of vectors is*

$$N'(r,0,0;k,0,0;2\nu+\delta,\Delta) = \prod_{i=0}^{k-r-1} \frac{(q^{\nu-r-i}-1)(q^{\nu-r-i+\delta-1}+1)}{q^{k-r-i}-1}.$$

Proof. When $r = 0$ our lemma follows from [5, Corollaries 6.23 and 7.25]. Now assume that $r > 0$. Let P be a k-dimensional totally isotropic subspace. The $P \supseteq X$ if and only if $P \supseteq \langle X \rangle$. Therefore by Lemma 2.1 the number of k-dimensional totally isotropic subspaces in the $(2\nu + \delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ containing a given rank-r isotropic set of vectors is

$$N'(r, 0, 0; k, 0, 0; 2\nu + \delta, \Delta) = \prod_{i=0}^{k-r-1} \frac{(q^{\nu-r-i} - 1)(q^{\nu-r-i+\delta-1} + 1)}{q^{k-r-i} - 1}.$$

□

Corollary 2.4. *The number of maximal totally isotropic subspaces in the $(2\nu + \delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$ containing a given rank-r isotropic set of vectors is*

$$N'(r, 0, 0; \nu, 0, 0; 2\nu + \delta, \Delta) = \prod_{i=0}^{\nu-r-1} (q^{\nu-r-i+\delta-1} + 1).$$

Theorem 2.5. *Let S be a set of non-zero vectors in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$, $M(S)$ be the matroid on S defined by linear independence of vectors, $L(M(S))$ be the lattice of flats of the matroid $M(S)$, $L_I(M(S))$ be the ideal of isotropic flats in the lattice $L(M(S))$, and μ be the Möbius function on $L(M(S))$. Then for any positive integer $\lambda \leq \nu + 1$, the number of $(\nu + 1 - \lambda)$-dimensional totally isotropic subspaces distinguishing S is equal to*

$$\sum_{\substack{X \in L_I(M(S)): \\ r(X) \leq \nu+1-\lambda}} \mu(\emptyset, X) \prod_{i=0}^{\nu-\lambda-r(X)} \frac{(q^{\nu-r(X)-i} - 1)(q^{\nu-r(X)-i+\delta-1} + 1)}{q^{\nu+1-\lambda-r(X)-i} - 1}.$$

Proof. Let X be a flat of $M(S)$. Denote by $g(\lambda, X)$ the number of $(\nu+1-\lambda)$-dimensional totally isotropic subspaces containing X. If X is isotropic and $r(X) \leq \nu + 1 - \lambda$, then by Lemma 2.3

$$g(\lambda, X) = \prod_{i=0}^{\nu-\lambda-r(X)} \frac{(q^{\nu-r(X)-i} - 1)(q^{\nu-r(X)-i+\delta-1} + 1)}{q^{\nu+1-\lambda-r(X)-i} - 1}.$$

If X is not isotropic or X is isotropic with $r(X) > \nu+1-\lambda$, then $g(\lambda, X) = 0$. Denote by $f(\lambda, X)$ the number of $(\nu+1-\lambda)$-dimensional totally isotropic subspaces P such that $P \cap S = X$. Then

$$g(\lambda, X) = \sum_{Y \in L_I(M(S)): Y \supseteq X} f(\lambda, Y).$$

By Möbius inversion

$$
\begin{aligned}
& f(\lambda, Y) \\
= & \sum_{X \in L_I(M(S)):\, X \supseteq Y} \mu(Y,X) g(\lambda, X) \\
= & \sum_{\substack{X \in L_I(M(S)):\, X \supseteq Y \\ \text{and } r(X) \leq \nu+1-\lambda}} \mu(Y,X) \prod_{i=0}^{\nu-\lambda-r(X)} \frac{(q^{\nu-r(X)-i}-1)(q^{\nu-r(X)-i+\delta-1}+1)}{q^{\nu+1-\lambda-r(X)-i}-1}.
\end{aligned}
$$

For $Y = \emptyset$, $f(\lambda, \emptyset)$ is the number of $(\nu+1-\lambda)$-dimensional totally isotropic subspaces distinguishing S. The theorem is proved. □

Corollary 2.6. *Let S be a set of non-zero vectors in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$, $M(S), L(M(S)), L_I(M(S)), \mu$ be as in Theorem 2.4. Then*

$$
\begin{aligned}
& c_{\mathrm{or}}(S, \mathbb{F}_q^{2\nu+\delta}) \\
= & \min\left\{ \lambda \,\middle|\, \sum_{\substack{X \in L_I(M(S)) \\ \text{with } r(X) \leq \nu+1-\lambda}} \mu(Y,X) \right. \\
& \left. \times \prod_{i=0}^{\nu-\lambda-r(X)} \frac{(q^{\nu-r(X)-i}-1)(q^{\nu-r(X)-i+\delta-1}+1)}{q^{\nu+1-\lambda-r(X)-i}-1} \neq 0 \right\}.
\end{aligned}
$$

In a similar way from Corollary 2.4 we deduce

Theorem 2.7. *Let S be a set of non-zero vectors in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$. Then the number of λ-tuples of maximal totally isotropic subspaces distinguishing S is*

$$
\sum_{X \in L_I(M(S))} \mu(\emptyset, X) \prod_{i=0}^{\nu-r(X)-1} (q^{\nu-r(X)-i+\delta-1}+1)^{\lambda}.
$$

Corollary 2.8. *Let S be a set of non-zero vectors in the $(2\nu+\delta)$-dimensional orthogonal space $\mathbb{F}_q^{2\nu+\delta}$. Then*

$$
\begin{aligned}
& c_{\mathrm{or}}(S, \mathbb{F}_q^{2\nu+\delta}) \\
= & \min\left\{ \lambda \,\middle|\, \sum_{X \in L_I(M(S))} \mu(\emptyset, X) \prod_{i=1}^{\nu-r(X)} (q^{\nu-r(X)-i+\delta-1}+1)^{\lambda} \neq 0 \right\}.
\end{aligned}
$$

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