Some Classes of Extended Directed Triple Systems and Numbers of Common Blocks

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Abstract

An extended directed triple system of the order v with a idempotent element (EDTS(v,a)) is a collection of triples of the type $[x,y,z]$, $[x,y,x]$ or $[x,x,x]$ chosen from a v-set, such that every ordered pair (not necessarily distinct) belongs to only one triple and there are a triples of the type $\{x,x,x\}$. If such a design with parameters v and a exist, then they will have $b_{v,a}$ blocks, where $b_{v,a} = (v^2+2a)/3$. A necessary and sufficient condition for the existence of EDTS$(v,0)$ and EDTS$(v,1)$ are $v \equiv 0 \pmod 3$ and $v \not\equiv 0 \pmod 3$, respectively. In this paper, we have constructed two EDTS(v,a)'s such that the number of common triples is in the set $\{0,1,2,\ldots,b_{v,a}-2,b_{v,a}\}$, for $a=0,1$.

1 Introduction

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A directed triple system of order v, DTS(v), is a pair (V, T), where T is a collection of transitive triples from a v-set V, such that every ordered pair of distinct elements of V is contained in exactly one transitive triple of T (The transitive triple $[a, b, c]$ contains the ordered pairs ab, bc, ac but not ab, bc, ca). This concept was introduced by Huang and Mendelsohn [11], who proved that a DTS(v) exists if and only if $v \not\equiv 2 \pmod 3$. In the same way, Steiner triple systems and Mendelsohn triple systems have been generalized to extended triple systems [2, 12] and extended Mendelsohn triple systems [1], respectively. The concept of such a system, similar to a DTS, is introduced in which a triple may have repeated elements. An extended directed triple system of order v, EDTS(v), is a pair (V, B), where B is a collection of ordered triples from a v-set V (each ordered triple may have repeated elements) such that every ordered pair of elements of V, not necessarily distinct, is contained in exactly one ordered triple of B. The elements of B are called blocks. There are five types of blocks: (1)$[a, b, c]$, (2)$[a, b, a]$, (3)$[a, a, b]$, (4)$[b, a, a]$ and (5)$[a, a, a]$ in which they are the set of ordered pairs $\{ab, bc, ac\}$, $\{ab, ba, aa\}$, $\{aa, ab\}$, $\{ba, aa\}$ and $\{aa\}$, respectively. For convenience, we call the transitive triple for type (1), 2-arc lollipop (2-lollipop for brevity) for type (2), 1-arc lollipop (1-lollipop for brevity) for type (3) or (4), and loop for type (5). Let b_3, b_2, b_1, and b_0 be used to denote the number of blocks of (V, B) that are of the type (1), (2), (3) or (4), and (5), respectively. A simple counting argument shows that if (V, B) is EDTS(v), then

$$b_3 = \frac{1}{3}(v(v-1) - 2b_2 - b_1) \qquad (1)$$

$$b_0 = v - b_2 - b_1 \qquad (2)$$

Evidently b_3 and b_0 are determined by b_2 and b_1. Let $\{v; b_2, b_1\}$ denote the class of EDTS(v) with parameters b_2 and b_1. We say that $\{v; b_2, b_1\}$ exists if there is a design with the specified parameters.

In [7], it was shown that the necessary and sufficient conditions for the existence of the class $\{v; b_2, b_1\}$ are $b_1 \neq 1$, $0 \leq b_2 + b_1 \leq v$ and

(1) $b_2 \equiv b_1 (mod\, 3)$ for $v \not\equiv 2 (mod\, 3)$;

(2) $b_2 \equiv b_1 + 1 (mod\, 3)$ for $v \equiv 2 (mod\, 3)$.

In graph notation, a DTS(v) is equivalent to the decomposition of the digraph D_v into transitive triples, where D_v is the complete symmetric digraph of order v. And an EDTS(v) is equivalent to the decomposition of the digraph D_v^+ into transitive triples, 2-lollipops, 1-lollipops and loops, where D_v^+ is the digraph obtained by attaching a loop to each vertex of D_v. In the following paragraphs, we consider the systems with $b_1 = 0$. An extended directed triple system of order v with a loops and $b_1 = 0$ will be denoted by EDTS(v, a). If (V, B) is an EDTS(v, a), then $|B| = b_{v,a} = (v^2 + 2a)/3$.

From the results of [7], the necessary and sufficient conditions for the existence of an EDTS(v, a), with $0 \le a \le v$, are:

(i) if $v \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(ii) if $v \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$.

Recently, some papers investigated the possible number of common blocks with two generalized triple systems with the same parameters, based on the same v-set. G. Lo Faro [14] considered this problem for extended triple systems without idempotent; W. C. Huang [6, 8] for extended triple systems; K. B. Huang, W. C. Huang, C. C. Hung and G. H. Wang [9, 10] for extended Mendelsohn triple systems; and C. M. Fu, Y. H. Gwo and F. C. Wu [4] for semi-symmetric latin squares.

In this paper, we have considered the intersection problems for the systems EDTS$(v, 0)$ and EDTS$(v, 1)$. Let $J[v, a]$ be the set of non-negative integers k such that there is a pair of EDTS(v, a) with k common blocks, let $I[v, a] = \{0, 1, 2, \ldots, b_{v,a} - 2, b_{v,a}\}$. Since the smallest possible mutually balanced subsets of an EDTS(v, a) are $\{[x, y, z], [z, y, x]\}$ (which can be changed to $\{y, x, z], [z, x, y]\}$), it follows that $J[v, a] \subseteq I[v, a]$.

Main Theorem $J[v, 0] = I[v, 0]$, for $v \equiv 0 \pmod 3$ and $v \ne 3$, and $J[v, 1] = I[v, 1]$, for $v \not\equiv 0 \pmod 3$.

Let A and B be two sets of integers and k a positive integer. We define $A + B = \{a + b \mid a \in A, b \in B\}$, $k + A = \{k\} + A$, and $kA = \{k \cdot a \mid a \in A\}$. For convenience, we denote the k-triple $\langle v_1, v_2, \ldots, v_k \rangle$ by $\{[v_1, v_2, v_1], [v_2, v_3, v_2], \ldots, [v_{k-1}, v_k, v_{k-1}], [v_k, v_k, v_k]\}$ where $v_i \ne v_j$ for all $i \ne j$. And $\langle v_1, v_2, \ldots, v_k, v_1 \rangle = \{[v_1, v_2, v_1], [v_2, v_3, v_2], \ldots, [v_{k-1}, v_k, v_{k-1}], [v_k, v_1, v_k]\}$.

2 Auxiliary constructions of EDTS

As usual, K_v is the complete graph on v vertices. An r-cycle is an elementary cycle of length r and is denoted by the sequence of its vertices $(x_1, x_2, \ldots, x_r)$. In [5], if v is even then K_v can be decomposed into $v - 1$ 1-factors and if v is odd then K_v can be decomposed into $(v - 1)/2$ edge-disjoint spanning cycles. In each case, we can construct transitive triples as follows:

Method 1. Let F be a 1-factor of K_v on V and u be any vertex not in V. $\mathcal{T}(F,u) = \{[x,u,y],[y,u,x] \mid \{x,y\} \in F\}$.

Method 2. Let $C = (c_1, c_2, \ldots, c_v)$ be a spanning cycle of K_v on V and a and b be any two different vertices not in V. $\mathcal{T}(C,a,b) = \{[c_1,a,c_2], [c_2,a,c_3], \ldots, [c_v,a,c_1], [c_v,b,c_{v-1}], [c_{v-1},b,c_{v-2}], \ldots, [c_1,b,c_v]\}$.

In order to count the number of common blocks of the two extended directed triple systems, we need some special embedding constructions. Let (V_1, B_1) be an EDTS(v,a), where $V_1 = \{a_1, a_2, \ldots, a_v\}$.

(1) v to $2v$, v even

Let $\mathcal{F} = \{F_i \mid i = 1,2,\ldots,v-1\}$ be a 1-factorization of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(F_i, a_i) \mid i = 1,2,\ldots,v-1\}$ and $L = \{[x,a_v,x] \mid x \in V_2\}$. Then (V,B) is an EDTS$(2v,a)$.

(2) v to $2v$, v odd

Let $\mathcal{C} = \{C_i \mid i = 1,2,\ldots,(v-1)/2\}$ be the edge-disjoint spanning cycles of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(C_i, a_{2i-1}, a_{2i}) \mid i = 1,2,\ldots,(v-1)/2\}$ and $L = \{[x,a_v,x] \mid x \in V_2\}$. Then (V,B) is an EDTS$(2v,a)$.

(3) v to $2v+3$, v even

Let $\mathcal{C} = \{C_i \mid i = 1,2,\ldots,v/2+1\}$ be the edge-disjoint spanning cycles of K_{v+3} on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(C_i, a_{2i-1}, a_{2i}) \mid i = 1,2,\ldots,v/2\}$ and $L = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_{v+3}}, x_{i_1}\rangle$ for the last spanning cycle $C_{v/2+1} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{v+3}})$. Then (V,B) is an EDTS$(2v+3,a)$.

Let $\mathcal{F} = \{F_i \mid i = 1,2,\ldots,2v-1\}$ be a 1-factorization of K_{2v} on $N = \{1,2,\ldots,2v\}$. If $F_a, F_b \in \mathcal{F}$, the notation $F_a \cdot F_b$ [14] will denote the following set of blocks: $\langle 1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, 1\rangle \cup \langle x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, x_{j_1}\rangle \cup \ldots \cup \langle x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_t}, x_{p_1}\rangle \cup \langle x_{q_1}, x_{q_2}, x_{q_3}, \ldots, x_{q_m}, x_{q_1}\rangle$ where $x_{j_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}), \ldots, x_{q_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, \ldots, x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_t}\})$; $F_a = \{1x_{i_2}, x_{i_3}x_{i_4}, \ldots, x_{i_{r-1}}x_{i_r}, x_{j_1}x_{j_2}, x_{j_3}x_{j_4}, \ldots, x_{j_{s-1}}x_{j_s}, \ldots, x_{p_1}x_{p_2}, x_{p_3}x_{p_4}, \ldots, x_{p_{t-1}}x_{p_t}, x_{q_1}x_{q_2}, x_{q_3}x_{q_4}, \ldots, x_{q_{m-1}}x_{q_m}\}$ and $F_b = \{x_{i_2}x_{i_3}, x_{i_4}x_{i_5}, \ldots, x_{i_r}1, x_{j_2}x_{j_3}, x_{j_4}x_{j_5}, \ldots,$

$x_{j_s}x_{j_1}, \ldots, x_{p_2}x_{p_3}, x_{p_4}x_{p_5}, \ldots, x_{p_t}x_{p_1}, x_{q_2}x_{q_3}, x_{q_4}x_{q_5}, \ldots, x_{q_m}x_{q_1}\}$. For example, let F_a and F_b be two 1-factors in K_{14}, where $F_a = \{\{1,3\},\{4,8\},\{2,14\},\{11,13\},\{5,6\},\{7,9\},\{10,12\}\}$ and $F_b = \{\{3,4\},\{8,1\},\{14,11\},\{13,2\},\{6,7\},\{9,10\},\{12,5\}\}$. Then $F_a \cdot F_b = \langle 1,3,4,8,1\rangle \cup \langle 2,14,11,13,2\rangle \cup \langle 5,6,7,9,10,12,5\rangle$.

(4) v to $2v+3$, v odd

Let $\mathcal{F} = \{F_i \mid i = 1,2,\ldots,v+2\}$ be a 1-factorization of K_{v+3} on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{T(F_i, a_i) \mid i = 1,2,\ldots,v\}$ and $L = F_{v+1} \cdot F_{v+2}$. Then (V, B) is an EDTS$(2v+3, a)$.

3 For the Class of EDTS$(v,0)$

Lemma 3.1 *If $J[v,0] = I[v,0]$ and v is an integer ≥ 9 then $J[2v,0] = I[2v,0]$.*

Proof. By using constructions 1 and 2, we can embed an EDTS$(v,0)$ in an EDTS$(2v,0)$. By replacing an EDTS$(v,0)$ and interchanging any two vertices of V_1 corresponding to different 1-factors or spanning cycles to form different transitive triples or lollipops, we obtain $J[2v,0] \supseteq J[v,0] + \{0, v, 2v, \ldots, (v-2)v, v^2\}$. If $v \geq 9$ and $J[v,0] = I[v,0]$ then $J[2v,0] \supseteq I[2v,0]$. Therefore $J[2v,0] = I[2v,0]$.∎

Lemma 3.2 *If $J[v,0] = I[v,0]$ and v is an integer ≥ 6 then $J[2v+3,0] = I[2v+3,0]$.*

Proof. By using constructions 3 and 4, we can embed an EDTS$(v,0)$ in an EDTS$(2v+3,0)$. By replacing an EDTS$(v,0)$ and interchanging any two vertices of V_1 corresponding to different 1-factors or spanning cycles to form different transitive triples or lollipops, we obtain $J[2v+3,0] \supseteq J[v,0] + \{0, v+3, 2(v+3), \ldots, (v-2)(v+3), v(v+3)\}$. If $v \geq 6$ and $J[v,0] = I[v,0]$ then $J[2v+3,0] \supseteq I[2v+3,0]$. Therefore $J[2v+3,0] = I[2v+3,0]$.∎

There are precisely two EDTS(3,0): $\langle 1,2,3,1\rangle$ and $\langle 1,3,2,1\rangle$. So, we have $J[3,0] = \{0,3\} \subset I[3,0] = \{0,1,3\}$.

Lemma 3.3 $J[v,0] = I[v,0]$, *for* $v = 6, 9, 12$.

Proof. For $v = 6$, using a similar argument to Lemma 3.1, we obtain $\{0,3,6,9,12\} \subseteq J[6,0]$. Let $T_1 = \langle 1,2,3,1\rangle \cup \{[4,1,4], [5,3,5], [6,2,6], [6,3,4], [4,3,6], [5,4,2], [2,4,5]\} \cup A$, where $A = \{[6,1,5], [5,1,6]\}$. Now, N_1 comes from T_1 by removing the blocks A and replacing them with $\{[6,5,1], [1,5,6]\}$. Then, $|T_1 \cap N_1| = 10$. Using the isomorphic designs obtained from T_1 by permuting elements in Table 1, we have $J[6,0] = I[6,0]$.

Table 1

Intersection	Size	Intersection	Size
$T_1 \cap (12)(456)T_1$	1	$T_1 \cap (34)T_1$	5
$T_1 \cap (256)(34)T_1$	2	$T_1 \cap (13)(45)T_1$	7
$T_1 \cap (123)(56)T_1$	4	$T_1 \cap (25)T_1$	8

For $v = 9$, using a similar argument to Lemma 3.2, we obtain $\{0,3,6,9,12,15,18,21,24,27\} \subseteq J[9,0]$. Let $T_1 = \langle 4,6,5,4\rangle \cup \langle 7,8,9,7\rangle \cup \{[3,5,7], [7,5,3], [3,4,8], [8,4,3]\} \cup A \cup B \cup C$, where $A = \langle 1,2,3,1\rangle \cup \{[2,5,8], [8,5,2], [2,6,7], [7,6,2]\}$, $B = \{[1,4,7], [7,4,1], [1,5,9], [9,5,1], [1,6,8], [8,6,1], [2,4,9], [9,4,2]$, and $C = \{[3,6,9], [9,6,3]\}$. $T_2 = \{[5,1,5], [6,2,6], [7,3,7], [8,4,8], [9,4,9], [1,3,6], [8,6,1], [1,7,9], [3,1,8], [9,7,1], [3,5,9], [5,8,3], [6,9,3], [9,5,6], [7,4,6], [6,7,8], [6,5,4], [8,5,7]\} \cup D \cup E$, where $D = \langle 1,2,3,4,1\rangle$ and $E = \{[2,4,5], [2,8,9], [4,2,7], [7,5,2], [9,8,2]\}$. Now, N_1 comes from T_1 by removing the blocks A and replacing them with $\langle 1,3,2,1\rangle \cup \{[7,2,6], [6,2,7], [8,2,5], [5,2,8]\}$. N_2 comes from T_1 by removing the blocks B and replacing them with $\{[7,1,4], [4,1,7], [5,1,9], [9,1,5], [8,6,1], [1,6,8], [4,2,9], [9,4,2]\}$. N_3 comes from T_1 by removing the blocks C and replacing them with $\{[6,3,9], [9,3,6]\}$. N_4 comes from T_2 by removing the blocks D and replacing them with $\langle 1,4,3,2,1\rangle$. N_5 comes from T_2 by removing the blocks E and replacing them with $\{[2,4,7], [4,5,2], [7,2,5], [8,2,9], [9,2,8]\}$. And N_6 comes from N_2 by removing the blocks C and replacing them with $\{[6,3,9], [9,3,6]\}$. From $|T_1 \cap N_6| = 17$, $|T_1 \cap N_2| = 19$, $|T_1 \cap N_1| = 20$, $|T_2 \cap N_5| = 22$, $|T_2 \cap N_4| = 23$, $|T_1 \cap N_3| = 25$ and Table 2, we have $J[9,0] = I[9,0]$.

Table 2

Intersection	Size	Intersection	Size
$T_1 \cap (67)(89)T_1$	1	$T_2 \cap (789)T_2$	10
$T_1 \cap (34)(6789)T_1$	2	$T_2 \cap (56)(789)T_2$	11
$T_1 \cap (67)(89)T_1$	4	$T_1 \cap (89)T_1$	13
$T_1 \cap (6789)T_1$	5	$T_2 \cap (59)T_2$	14
$T_1 \cap (56)(79)T_1$	7	$T_2 \cap (23)(56)(78)T_2$	16
$T_1 \cap (678)T_1$	8		

For $v = 12$, using a similar argument to Lemma 3.1, we obtain $J[12,0] \supseteq I[12,0] \setminus \{34,35,46\}$. Let $T_1 = \langle 1,5,9,1\rangle \cup \{[6,5,6], [7,6,7], [8,6,8], [10,9,10], [11,10,11], [12,10,12], [1,7,11], [11,7,1], [1,8,12], [12,8,1], [2,5,10], [10,5,2], [2,8,11], [11,8,2], [3,8,10], [10,8,3], [4,7,10], [10,7,4], [9,8,4], [4,8,9], [5,7,8], [8,7,5], [9,11,12], [12,11,9]\} \cup B \cup C \cup D \cup E$, where $B = \{[2,6,12], [12,6,2], [2,7,9], [9,7,2], [3,6,9], [9,6,3], [3,7,12], [12,7,3]\}$, $C = \{[3,5,11], [11,5,3], [4,5,12], [12,5,4], [4,6,11], [11,6,4]\}$, $D = \{[2,1,2], [3,2,3], [4,2,4], [1,3,4], [4,3,1]\}$ and $E = \{[1,6,10], [10,6,1]\}$. Now, N_1 comes from T_1 by removing the blocks $B \cup C$ and replacing them with $\{[2,6,9], [9,6,2], [2,7,12], [12,7,2], [3,5,12], [12,5,3], [3,6,11], [11,6,3], [3,7,9], [9,7,3], [4,5,11], [11,5,4], [4,6,12], [12,6,4]\}$. N_2 comes from T_1 by removing the blocks $B \cup D$ and replacing them with $\{[2,6,9], [9,6,2], [2,7,12], [12,7,2], [3,6,12], [12,6,3], [3,7,9], [9,7,3], [3,1,3], [2,3,2], [4,3,4], [1,2,4], [4,2,1]\}$. And N_3 comes from T_1 by removing the blocks E and replacing them with $\{[6,1,10], [10,1,6]\}$. From $|T_1 \cap N_1| = 34$, $|T_1 \cap N_2| = 35$ and $|T_1 \cap N_3| = 46$, we have $J[12,0] = I[12,0]$.∎

Combining the above Lemmas 3.1, 3.2 and 3.3, we obtained the following results:

Theorem 3.4 *$J[v,0] = I[v,0]$ for $v \equiv 0 \pmod 3$, $v > 3$ and $J[3,0] = \{0,3\}$.*

4 For the Class of EDTS$(v,1)$

Lindner and Wallis [13] and independently Fu [3] prove that there exist two DTS(v) intersecting in s triples if and only if $s \in S_v = \{0,1,2,\ldots,v(v-1)/3-2, v(v-1)/3\}$, for $v \not\equiv 2 \pmod 3$. So, if $v \not\equiv 0 \pmod 3$, there exist two DTS$(v-1)$, (V,B_1) and (V,B_2), with $|B_1 \cap B_2| = r \in S_{v-1}$, where $V = \{1,2,3,\ldots,v-1\}$. Let $V^* = V \cup \{v\}$, $B_1^* = B_1 \cup N$ and $B_2^* = B_2 \cup N$, where $N = \{[1,v,1],[2,v,2],[3,v,3],\ldots,[v-1,v,v-1],[v,v,v]\}$. Then (V^*,B_1^*) and (V^*,B_2^*) are two EDTS$(v,1)$ and they have $v+r$ common blocks. Therefore,

$$v + S_{v-1} = \{v, v+1, v+2, \ldots, b_{v,1}-2, b_{v,1}\} \subseteq J[v,1] \qquad (3)$$

The missing data $\{0,1,2,\ldots,v-1\}$ can be obtained by the following two Lemmas.

Lemma 4.1 *If $J[v,1] = I[v,1]$ and v is an integer ≥ 4 then $J[2v,1] = I[2v,1]$.*

Proof. From equation (3), the missing data are the set $\{0, 1, 2, \ldots, 2v - 1\}$. By using constructions 1 and 2, we can embed an EDTS(v,1) in an EDTS($2v$,1). By replacing an EDTS(v,1) and interchanging all (all but one) vertices of V_1 corresponding to different 1-factors or spanning cycles to form different transitive triples or lollipops, we obtain $J[2v, 1] \supseteq J[v, 1] + \{0, v\}$. Since and $J[v, 1] = I[v, 1]$, we have $J[2v, 1] \supseteq \{0, 1, 2, \ldots, b_{v,1} - 2, b_{v,1}\} + \{v, v + 1, v + 2, \ldots, v + b_{v,1} - 2, v + b_{v,1}\}$.

In order to solve the missing data, we have to estimate the smallest v satisfying

$$2v - 1 \leq v + b_{v,1} - 2 \tag{4}$$

$$v \leq b_{v,1} - 1 \leq v + b_{v,1} - 2 \tag{5}$$

which is equivalent to the system

$$\begin{cases} v^2 - 3v - 1 & \geq & 0 \\ v & \geq & 1. \end{cases}$$

The smallest positive integer v satisfied the above system is 4. Therefore, we have $J[2v, 1] = I[2v, 1]$, for $v \geq 4$. ∎

Lemma 4.2 *If* $J[v, 1] = I[v, 1]$ *and* v *is an integer* ≥ 5 *then* $J[2v + 3, 1] = I[2v + 3, 1]$.

Proof. By the same method as Lemma 4.1, we obtain $J[2v + 3, 1] \supseteq J[v, 1] + \{0, v + 3\}$. This implies that $J[2v, 1] \supseteq \{0, 1, 2, \ldots, b_{v,1} - 2, b_{v,1}\} + \{v + 3, v + 4, v + 5, \ldots, v + b_{v,1} + 1, v + b_{v,1} + 3\}$.

In order to solve the missing data, we have to estimate the smallest v satisfying

$$2v + 2 \leq v + b_{v,1} + 1 \tag{6}$$

$$v + 3 \leq b_{v,1} - 1 \leq v + b_{v,1} + 1 \tag{7}$$

which is equivalent to the system

$$\begin{cases} v^2 - 3v - 1 & \geq & 0 \\ v^2 - 3v - 10 & \geq & 0 \\ v & \geq & -2 \end{cases}$$

The smallest positive integer v satisfied the above system is 5. Therefore, we have $J[2v + 3, 1] = I[2v + 3, 1]$, for $v \geq 5$. ∎

Lemma 4.3 $J[v, 1] = I[v, 1]$, *for* $v =$ *4, 5, 7, 11.*

Proof. For $v = 4$, let $T_1 = \{[1,1,1],[2,1,2],[3,2,3],[4,2,4],[1,3,4],[4,3,1]\}$. Now, N_1 comes from T_1 by removing the blocks $\{[1,1,1],[2,1,2],[1,3,4],[4,3,1]\}$ and replacing them with $\{[2,2,2],[1,2,1],[1,4,3],[3,4,1]\}$. From $|T_1 \cap N_1| = 2$, $|T_1 \cap (12)T_1| = 0$, $|T_1 \cap (23)T_1| = 1$, $|T_1 \cap (14)T_1| = 3$ and $|T_1 \cap (34)T_1| = 4$, we have $J[4,1] = I[4,1]$.

For $v = 5$, let $T_1 = \{[4,4,4], [1,4,1], [2,3,5]\} \cup A$, where $A = \{[3,4,2], [2,1,2], [3,1,3], [5,2,4], [5,1,5], [4,5,3]\}$. Now, N_1 comes from T_1 by removing the blocks A and replacing them with $\{[2,4,2], [3,4,3], [5,4,5], [1,5,3], [3,2,1], [5,1,2]\}$. By $|T_1 \cap (12)(45)T_1| = 0$, $|T_1 \cap (12)(35)T_1| = 1$, $|T_1 \cap (45)T_1| = 2$, $|T_1 \cap N_1| = 3$ and $|T_1 \cap (234)T_1| = 4$, we have $J[5,1] = I[5,1]$.

For $v = 7$, let $T_1 = \{[1,1,1], [2,1,2], [3,2,3], [4,2,4], [5,2,5], [6,3,6], [7,4,7], [1,3,7], [3,4,1], [1,6,4], [6,5,1], [7,1,5], [2,6,7], [7,6,2], [4,3,5], [5,7,3], [5,4,6]\}$. From Table 3, we obtain $J[7,1] = I[7,1]$.

Table 3

Intersection	Size	Intersection	Size
$T_1 \cap (12)(45)(67)T_1$	0	$T_1 \cap (467)T_1$	4
$T_1 \cap (23)(67)T_1$	1	$T_1 \cap (45)(67)T_1$	5
$T_1 \cap (23)(57)T_1$	2	$T_1 \cap (47)T_1$	6
$T_1 \cap (46)(57)T_1$	3		

For $v = 11$, Using a similar argument to Lemma 4.2, we obtain $J[11,1] = I[11,1] \setminus \{5\}$. Let $T_1 = \{[9,9,9], [1,5,1], [2,11,2], [3,8,3], [4,7,4], [5,6,5], [6,4,6], [7,10,7], [8,2,8], [10,1,10], [11,3,11], [1,2,7], [7,2,1], [1,3,4], [4,3,1], [1,8,11], [11,8,1], [1,9,6], [6,9,1], [2,3,6], [6,3,2], [2,4,10], [10,4,2], [2,5,9], [9,5,2], [3,5,10], [10,5,3], [3,7,9], [9,7,3], [4,5,8], [8,5,4], [4,9,11], [11,9,4], [11,7,5], [5,7,11], [6,7,8], [8,7,6], [6,10,11], [11,10,6], [8,9,10], [10,9,8]\}$. Then $J[11,1] = I[11,1]$ follows by $|T_1 \cap (56)(78)(9t_1)T_1| = 5$, where $t_1 = 11$.∎

Combining the above Lemmas 4.1, 4.2 and 4.3, we obtained the following results:

Theorem 4.4 $J[v,1] = I[v,1]$, *for* $v \not\equiv 0 \pmod 3$*.*

5 Conclusions.

From Theorems 3.4 and 4.4, we obtained the following results:

Main Theorem $J[v,0] = I[v,0]$, for $v \equiv 0 \pmod 3$ and $v \neq 3$, and $J[v,1] = I[v,1]$, for $v \not\equiv 0 \pmod 3$.

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