Genus Distribution of Graph Amalgamations: Pasting at Root-Vertices

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Abstract

We pursue the problem of counting the imbeddings of a graph in each of the orientable surfaces. We demonstrate how to achieve this for an iterated amalgamation of arbitrarily many copies of any graph whose genus distribution is known and further analyzed into a *partitioned genus distribution*. We introduce the concept of *recombinant strands of face-boundary walks*, and we develop the use of multiple *production rules* for deriving simultaneous recurrences. These two ideas are central to a broadbased approach to calculating genus distributions for graphs synthesized from smaller graphs.

1 Introduction

Counting the imbeddings in orientable and non-orientable surfaces of the graphs from interesting families is a well-established endeavor with considerable recent activity. Past investigations have yielded various results for specific families of graphs, including formulas with exact numbers, recursions that are useful in constructing tables, and asymptotic formulas for lower bounds. In this paper, we give a method applicable to the sequence of open chains of copies of any graph whose genus distribution is known.

By focusing on amalgamations at 2-valent vertices, this paper and a companion paper ([Gr09a]) on self-amalgamations especially facilitate calculating genus distributions for 4-regular graphs, since any 4-regular graph can be obtained from a set of cycle graphs by a sequence of amalgamations and self-amalgamations at 2-valent vertices. Specific calculations generally require careful attention to the order of the amalgamations, as illustrated by [Gr09b], which uses the post-order of a plane tree.

Prior work on counting imbeddings of a graph in a minimum-genus surface includes [BGGS00], [GoRiSi07], [GrGr08], and [KoVo02]. Prior work on counting imbeddings in all orientable surfaces or in all surfaces includes [ChGrRi94], [ChLiWa06], [FGS89], [GrFu87], [GRT89], [KwLe93],

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[KwLe94], [KwSh02], [McG87], [Mu99], [St90], [St91a], [St91b], [Tesa00], [ViWi07], [WaLi06], and [WaLi08]. Complementary work on counting maps on a given surface is given by [CoDo01], [Ja87], [JaVi90], [JaVi01], and many others.

TERMINOLOGY A ***graph*** is connected and labeled, and an ***imbedding*** is cellular and orientable, unless it is evident from context that something else is intended. A graph may have self-loops and multiple adjacencies between two vertices. Two imbeddings are considered different if their sets of oriented face-boundary walks are different. The words ***degree*** and ***valence*** are used interchangeably. Each edge, even a self-loop, has two topological ***edge-ends***. The terminology used here is consistent with [GrTu87] and [BWGT09]. For additional background (with some terminological differences), see [BoLi95], [MoTh01], or [Wh01].

NOTATION The ***degree*** of a vertex y is denoted $deg(y)$. The ***genus*** of a surface S is denoted $\gamma(S)$. The ***number of imbeddings*** of a graph G in the surface S_i of genus i is denoted g_i.

ABBREVIATION We abbreviate face-boundary walk as ***fb-walk***.

The ***vertex-amalgamation*** of a pair of rooted graphs (G, t) and (H, u) is the graph obtained from their disjoint union by merging the roots t and u. An asterisk denotes the operation:

$$(G, t) * (H, u) \;=\; (X, w)$$

where w is the merged root. The sequence $\{g_i(G) \mid i \geq 0\}$ is called the ***genus distribution*** of the graph G.

Counting Consistent Rotation Systems

Each imbedding of an amalgamation $X = G * H$ ***induces*** a unique imbedding of G and a unique imbedding of H, by which we mean the imbeddings of G and H whose rotation systems are consistent (as cyclic permutations) with the rotation system of X. We also say that the pair of imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ ***induce*** the set of imbeddings of $X = G * H$, and that this set of imbeddings of X is the result of ***amalgamating the two imbeddings*** $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$.

Proposition 1.1 *For any two imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ of graphs into surfaces, the cardinality of the set of imbeddings of the amalgamated graph $(X, w) = (G, t) * (H, u)$ whose rotation systems are consistent with the imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ is*

$$deg(u) \cdot \binom{deg(t) + deg(u) - 1}{deg(u)} \tag{1.1}$$

Proof Let y be an edge-end at vertex u that we single out for this counting problem. Each imbedding of X whose rotation is consistent with $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ can be obtained by inserting the $deg(u)$ edge-ends incident at u within the $deg(t)$ corners formed at vertex t (a *corner* is the "gap" between two consecutive end-ends in a rotation) in the imbedding of G. We make $deg(u)$ choices-with-repetition-allowed from the set of $deg(t)$ corners, as locations where edge-ends at u will be inserted, which accounts for the binomial coefficient in Formula (1.1). There are $deg(u)$ choices of locations for the edge-end y, and the rest of the edge-ends at u must be inserted in rotational order. ◇

2 Induced Imbeddings and Production Rules

In the amalgamation $(G,t)*(H,u) = (X,w)$, when one of the roots t and u is 1-valent, the genus distribution of the resulting graph is easily derivable via bar-amalgamations (see [GrFu87]). In the case where $deg(t) = deg(u) = 2$, a pair of imbeddings

$$\iota_G : G \to S_G \quad \text{and} \quad \iota_H : H \to S_H$$

induce, in accordance with Formula (1.1), six imbeddings of the amalgamated graph X. Theorem 2.2 implies that for each such imbedding $\iota_X : X \to S_X$, we have

$$\gamma(S_X) = \begin{cases} \gamma(S_G) + \gamma(S_H) & \text{or} \\ \gamma(S_G) + \gamma(S_H) + 1 \end{cases}$$

TERMINOLOGY The difference $\gamma(S_X) - (\gamma(S_G) + \gamma(S_H))$ is called the ***genus increment of the amalgamation***, or more briefly, the ***genus increment*** or ***increment***.

Proposition 2.1 *In any vertex-amalgamation $(G,t) * (H,u) = (X,w)$ of two graphs, the increment of genus lies within these bounds:*

$$\left\lceil \frac{1 - deg(t) - deg(u)}{2} \right\rceil \leq \gamma(S_X) - (\gamma(S_G) + \gamma(S_H)) \leq \left\lfloor \frac{deg(t) + deg(u) - 2}{2} \right\rfloor$$

Proof In the respective imbeddings of G and H, the maximum number of faces incident on t and u are $deg(t)$ and $deg(u)$. This asserted upper bound corresponds to the case in which these $deg(t) + deg(u)$ faces are merged into a single face. The asserted (non-positive) general lower bound would be realized by amalgamands in which only one face is incident at vertex t and only one at vertex u, and a resulting imbedding in which there are $deg(t) + deg(u)$ faces incident at vertex w. ◇

Example 2.1 To see a negative increment, suppose that graphs G and H are both isomorphic to the bouquet B_2, and that they have respective rotation systems

$$t. \quad a\, b\, a^-\, b^- \qquad \text{and}$$
$$u. \quad c\, d\, c^-\, d^-$$

which both correspond to toroidal imbeddings. Then the rotation system

$$w. \quad a\, c\, d\, b\, c^-\, a^-\, b^-\, d^-$$

for $X = G * H$ is also toroidal. It is consistent with the rotation systems of G and H, and the increment is -1. Figure 2.1 illustrates this example.

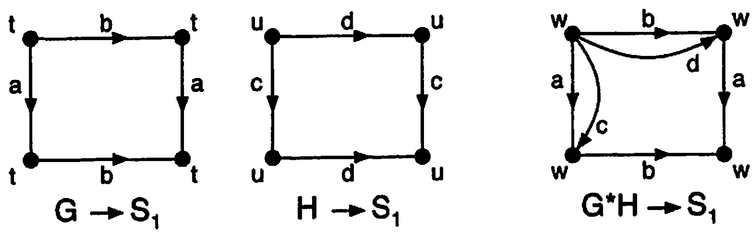

Figure 2.1: A genus increment of -1.

Partial Genus Distributions

In what follows, we shall suppose that the roots t and u are both 2-valent. The genus distribution of the six imbeddings of $(X, w) = (G, t) * (H, u)$ that are consistent with prescribed rotation systems for (G, t) and (H, u) depends only on $\gamma(S_G)$, $\gamma(S_H)$, and the respective numbers of faces in which the two vertices of amalgamations t and u lie. Accordingly, we partition the imbeddings of a single-rooted graph (G, t) with $deg(t) = 2$ in a surface of genus i into the subset of ***type-d_i imbeddings***, in which root t lies on two distinct fb-walks, and the subset of ***type-s_i imbeddings***, in which root t occurs twice on the same fb-walk. Moreover, we define

$$\begin{aligned} d_i(G,t) &= \text{the number of imbeddings of type-}d_i\text{, and} \\ s_i(G,t) &= \text{the number of imbeddings of type-}s_i. \end{aligned}$$

Thus,

$$g_i(G,t) \;=\; d_i(G,t) + s_i(G,t)$$

The numbers $d_i(G,t)$ and $s_i(G,t)$ are called ***single-root partials***. Each of the sequences $\{d_i(G,t) \mid i \geq 0\}$ and $\{s_i(G,t) \mid i \geq 0\}$ is called a ***partial genus distribution***. Together they are called a ***partitioned genus distribution***.

NOTATION We may simply write d_i and s_i, when it is clear from context to which graph they apply.

REMARK More generally, with a root of higher valence, there would be more partials, corresponding to a larger number of possible configurations of fb-walks at the root.

REMARK An incipient use of partials (but not by that name) appeared in [FGS89]. The present paper expands this approach considerably.

Verifying Productions by Recombinant Strands

A ***production rule*** for an amalgamation

$$(G,t) * (H,u) \;=\; (X,w)$$

of single-rooted graphs is an expression of the form

$$p_i(G,t) * q_j(H,u) \longrightarrow c_{i+j}\, g_{i+j}(G * H) + c_{i+j+1}\, g_{i+j+1}(G * H)$$

where, p_i and q_j are partials, and where c_{i+j} and c_{i+j+1} are integers. It means that amalgamation of a type-p_i imbedding of graph G and a type-q_j imbedding of graph H induces a set of c_{i+j} genus-$(i+j)$ imbeddings of $G * H$ and c_{i+j+1} genus-$(i+j+1)$ imbeddings of the graph X. We often write such a rule in the form

$$p_i * q_j \;\longrightarrow\; c_{i+j}\, g_{i+j} \;+\; c_{i+j+1}\, g_{i+j+1}$$

When an imbedding of graph (G,t) and an imbedding of graph (H,u) are amalgamated, the edge-ends of root vertex u break some or all of the fb-walks incident at root vertex t in the imbedding of G into ***strands***; conversely, the edge-ends at t break some or all of the face-walks incident at u in the imbedding of H into strands. This phenomenon holds for roots of arbitrary degree, but for the sake of simplicity, we confine our attention here to 2-valent roots. There are two cases. It is sufficient to describe the strands in the imbedding of G.

TERMINOLOGY In the absence of standard names for the various graphics that represent fb-walks in the figures, we assign names of colors to them, and we provide a legend.

Case d. Suppose that two different fb-walks of the imbedding of G are incident at t. Let's call them the *red* fb-walk and the *purple* f-b walk. If the two edge-ends at u are both placed on the red side (as in the leftmost and center imbeddings of Figure 2.2 below), which can be done in two ways, then the red fb-walk is broken into a single strand, and the purple walk remains closed. Similarly, if both edge-ends at u are placed on the purple side, which can also be done in two ways, then the purple walk is broken into a single strand, and the red walk remains closed. If one edge-end at u is on the red side of t and the other edge-end on the purple side (as in the rightmost imbedding of Figure 2.2), then both the red walk and the purple walk are broken into single strands. We assign colors *blue* and *brown* to the fb-walks at u, if there are two different fb-walks in the imbedding of H.

Case s. Suppose that one fb-walk of the imbedding of G is twice incident at t. Let's color it red. If both edge-ends at u are inserted at t so that they are contiguous, then the red walk is broken into a single strand (as on the left of Figure 2.3 below). However, if the edge-ends at u are placed so that they alternate with the edge-ends at t (as on the right of Figure 2.3), then the red walk is broken into two red strands. If there is only one fb-walk at u in the imbedding of H, it is colored blue.

In each of the six imbeddings resulting from the amalgamation, the strands from G and H recombine into fb-walks in $G * H$. Each end of each strand from one graph attaches to an end of a strand from the other graph.

Theorem 2.2 *Let (G, t) and (H, u) be single-rooted graphs with 2-valent roots. Then the following production rules, which cover all four possible cases, all hold.*

$$d_i * d_j \longrightarrow 4g_{i+j} + 2g_{i+j+1} \qquad (2.1)$$

$$s_i * d_j \longrightarrow 6g_{i+j} \qquad (2.2)$$

$$d_i * s_j \longrightarrow 6g_{i+j} \qquad (2.3)$$

$$s_i * s_j \longrightarrow 6g_{i+j} \qquad (2.4)$$

Proof This theorem is not difficult. Nonetheless, we develop our method of proof for production rules carefully, so that in our proofs of some more difficult production rules later, our method is well understood.

Although it is possible to derive production rules entirely with prose exposition, it is more concise and convenient to do face-tracing on *rotation projections* (see §3.2.5 and §3.2.6 of Gross and Tucker [GrTu87]). Figure 2.2 illustrates three of the six possible outcomes corresponding to Production Rule (2.1).

Case $d_i * d_j$. The leftmost drawing illustrates a placement of the imbedding of H so that both edge-ends of u are on the red side of vertex t and that

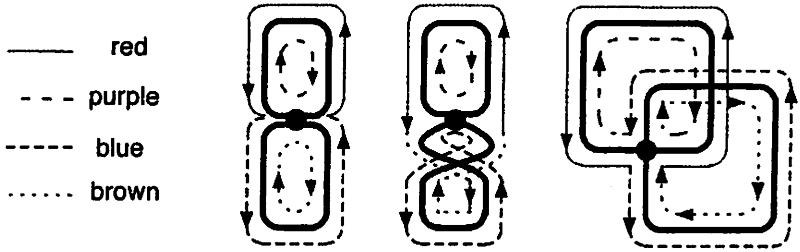

Figure 2.2: $d_i * d_j \longrightarrow 4g_{i+j} + 2g_{i+j+1}$.

it breaks the blue boundary walk at u. We see that the purple walk and the brown walk remain closed, and the single red strand and the single blue strand are recombined into a closed walk in the imbedding of $G * H$. Observe that the recombination preserves the direction on the strands. Euler characteristic considerations imply that the genus of the surface of resulting imbedding is the sum $i+j$ of the genera i and j of the respective imbedding surfaces of the amalgamand imbeddings.

The middle drawing illustrates the other way of placing both edge-ends of u on the red side of t. In this case the purple walk and the blue walk remain closed, while the red strand and the brown strand recombine into a single closed walk. Once again the genus of the resulting imbedding is $i+j$.

Similarly, there are two ways to paste both edge-ends of u on the purple side of t. Both ways lead to genus $i+j$.

The rightmost drawing illustrates one of the two ways to place the edge-ends of u on different sides of t. Either way breaks the face boundary-walks into four strands, red, purple, blue, and brown. In either way, the four strands are recombined into a single closed walk, and the resulting genus is $i+j+1$.

Case $s_i * d_j$. Figure 2.3 illustrates what can happen when the same fb-walk (in red) is twice incident on vertex t and there are two different face-boundaries incident at u.

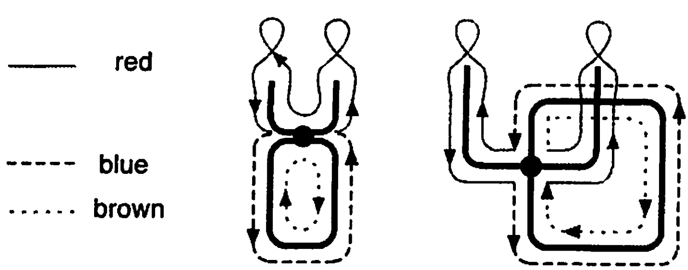

Figure 2.3: $s_i * d_j \longrightarrow 6g_{i+j}$.

In any of the four different ways in which both edge-ends at u can be inserted on the same side of t, there is a single red strand in G. It is recombined with a strand (in blue on the left) from H. The other fb-walk (brown) remains intact. The result is g_{i+j}. When the edge-ends of u are inserted on different sides of t (shown on the right), which can happen in two ways, they break the red fb-walk of G into two red strands, and in H, both the blue fb-walk and the brown fb-walk are broken into single strands. As illustrated, the blue strand joins with one red strand to form a fb-walk in $G * H$, and the brown strand joins with the other red strand to form another fb-walk in $G * H$. Thus, the total number of faces incident at the merged vertex of $G * H$ is 3 in all six imbeddings of $G * H$. They all have genus g_{i+j}.

Case $d_i * s_j$. By symmetry with Case $s_i * d_j$, the result of amalgamation is six imbeddings of genus g_{i+j}.

Case $s_i * s_j$. There are four ways in which both edge-ends of u are inserted on the same side of t. In each of them, the red fb-walk twice incident on t is broken into a single strand, and the blue fb-walk twice incident on u is also broken into a single strand. The blue and red strands are merged, as shown on the left in Figure 2.4, into a single fb-walk of $G * H$.

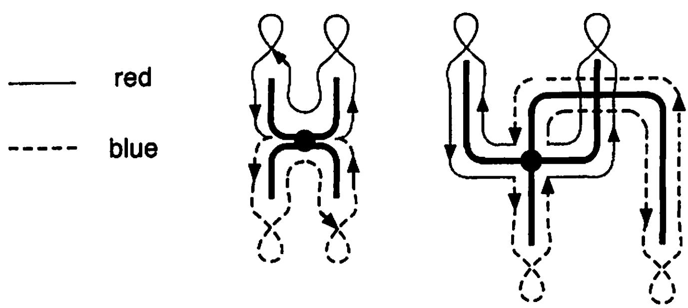

Figure 2.4: $s_i * s_j \longrightarrow 6g_{i+j}$.

When the end-ends of u alternate with the edge-ends of t, as shown on the right of Figure 2.4, the red fb-walk of G and the blue fb-walk of H are both broken into two strands. These four strands are recombined, as shown, into a single fb-walk in $G * H$. Thus, all six imbeddings induced in $G * H$ have genus g_{i+j}. ◇

Deriving a General Recursion

The four production rules of Theorem 2.2 lead to an equation involving five summations, one for each of the terms on the right of those production rules. The equation can be used to derive the genus distribution of an amalgamated graph from the genus distributions of the amalgamands.

Corollary 2.3 *Let (G,t) and (H,u) be single-rooted graphs with 2-valent roots, and let $(X,w) = (G,t) * (H,u)$. Then*

$$\begin{aligned} g_k(X) &= 4\sum_{i=0}^{k} d_i(G)d_{k-i}(H) + 2\sum_{i=0}^{k-1} d_i(G)d_{k-i-1}(H) + 6\sum_{i=0}^{k} d_i(G)s_{k-i}(H) \\ &\quad + 6\sum_{i=0}^{k} s_i(G)d_{k-i}(H) + 6\sum_{i=0}^{k} s_i(G)s_{k-i}(H) \end{aligned} \tag{2.5}$$

Proof We observe, for instance, that the pair of partials $d_i(G,t)$ and $d_{k-i}(H,u)$ contribute 4 to $g_k(X,w)$ and the pair $d_i(G,t)$ and $d_{k-i-1}(H,u)$ contribute 2 to $g_k(X,w)$. Eq (2.5) is simply the summation of these contributions. ◇

Example 2.2 Let $\dot{K}_4$ be the graph obtained by subdividing an edge of the complete graph K_4, and let X be the graph illustrated in Figure 2.5, obtained by amalgamating $\dot{K}_4$ to another copy of itself at the 2-valent vertices. The non-zero partials for the genus distribution of K_4 are

$$d_0 = 2 \quad d_1 = 8 \quad s_1 = 6$$

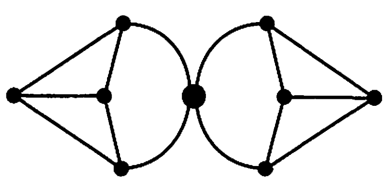

Figure 2.5: The graph X of Example 2.2.

By Corollary 2.3, after eliminating 0-valued summands, we have

$$\begin{aligned} g_0(X) &= 4d_0(\dot{K}_4)d_0(\dot{K}_4) = 4\cdot 2\cdot 2 = 16 \\ g_1(X) &= 4[d_0d_1 + d_1d_0] + 2d_0d_0 + 6d_0s_1 + 6s_1d_0 \\ &= 4[2\cdot 8 + 8\cdot 2] + 2[2\cdot 2] + 6[2\cdot 6] + 6[6\cdot 2] = 280 \\ g_2(X) &= 4[d_1d_1] + 2[d_0d_1 + d_1d_0] + 6d_1s_1 + 6s_1d_1 + 6s_1s_1 \\ &= 4[8\cdot 8] + 2[2\cdot 8 + 8\cdot 2] + 6[8\cdot 6] + 6[6\cdot 8] + 6[6\cdot 6] = 1112 \\ g_3(X) &= 2d_1d_1 = 2\cdot[8\cdot 8] = 128 \end{aligned}$$

3 Double-Root Partials

We have seen that the system of production rules of Theorem 2.2 can produce a genus distribution for the amalgamated graph from the partials for the amalgamands. However, Theorem 2.2 provides insufficient information for calculating the genus distributions for a sequence of graphs obtained by iterated amalgamation, because it produces no partials that could be used to continue the iteration. We now overcome this deficiency by amalgamating a single-rooted graph (G, t) to a doubly-rooted graph $H(u, v)$ and calculating the single-root partials for the result $(G * H, v)$. We choose v as the root for this amalgamated graph in anticipation of the next step of the iteration.

Toward this objective, we may need to know also whether one or both of the fb-walks incident on one co-root of H are incident on the other co-root of H. Whereas we partitioned $g_i(G, t)$ into the two parts $d_i(G, t)$ and $s_i(G, t)$, we now partition $g_i(H, u, v)$ into four ***double-root*** i^{th} ***partials***. Four basic double-root partials are given in Table 3.1 below.

Table 3.1: Defining some double-root partials of (H, u, v).

partial	*counts these imbeddings in* S_i
$dd_i(H, u, v)$	u and v both on two fb-walks
$ds_i(H, u, v)$	u on two fb-walk's and v on one fb-walk
$sd_i(H, u, v)$	u on one fb-walk and v on two fb-walk's
$ss_i(H, u, v)$	u on one fb-walk and v on one fb-walk

Sub-partials of $\mathbf{dd_i(H, u, v)}$

In the course of developing production rules for amalgamating a single-rooted graph (G, t) to a doubly-rooted graph (H, u, v), we shall discover that we sometimes need to refine a doubly-rooted partial into sub-partials. The following three numbers are the ***sub-partials of*** $dd_i(H, u, v)$:

$dd_i^0(H, u, v)$ = the number of imbeddings of type-dd_i such that neither fb-walk at u is incident on v;

$dd_i'(H, u, v)$ = the number of imbeddings of type-dd_i such that only one fb-walk at u is incident on v;

$dd_i''(H, u, v)$ = the number of imbeddings of type-dd_i such that both fb-walks at u are incident on v.

Theorem 3.1 *When an imbedding of type d_i of a single-rooted graph (G, t) is amalgamated to an imbedding of type dd_j of a doubly-rooted graph (H, u, v), with all roots 2-valent, the following production rules hold:*

$$d_i * dd_j^0 \longrightarrow 4d_{i+j} + 2d_{i+j+1} \qquad (3.1)$$

$$d_i * dd_j' \longrightarrow 4d_{i+j} + 2d_{i+j+1} \qquad (3.2)$$

$$d_i * dd_j'' \longrightarrow 4d_{i+j} + 2s_{i+j+1} \qquad (3.3)$$

Proof When both edge ends of u are inserted on the same side of vertex t, which can happen in four ways, it is clear (as in the leftmost rotation projection of Figure 2.2) that one of the fb-walks (e.g., red) at t and one of the fb-walks at u (e.g., blue) become single strands and are recombined into a single fb-walk of $G * H$. Clearly, at least one of the fb-walks at v (which becomes the root of $G*H$) is unaffected. This accounts for the term $4d_{i+j}$ in the results of all three production rules.

Figure 3.1 illustrates the rotation projection, in all three cases, of one of the two amalgamations in which the edge-ends of u alternate with the edge-ends of t. A heavier blue or brown fb-walk is taken to be a different walk from a lighter blue or brown walk. From left to right, the drawings correspond to dd^0, dd', and dd''. In all three drawings, four strands recombine into a single fb-walk of $G * H$, so the resulting genus is $i + j + 1$.

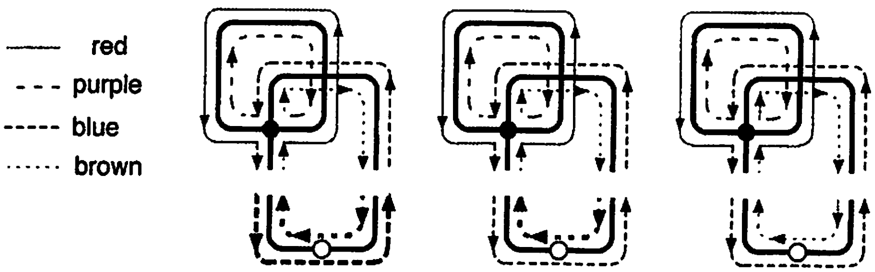

Figure 3.1: Alternating edge-ends in three subcases of $d * dd$.

In the drawing for dd^0 (leftmost), the heavy blue fb-walk and the heavy brown walk at vertex v remain unaffected, implying that the result is d_{i+j+1}. In the drawing for dd' (center), the light blue walk at v is part of the fb-walk through merged vertex w, while the heavy brown walk at v is unaffected; again, the result is d_{i+j+1}. In the drawing for dd'', the two fb-walks at v are both part of the strands that are merged into the single walk at w; this time, the result is s_{i+j+1}. ◇

Theorem 3.2 *When an imbedding of type s_i of a single-rooted graph (G, t) is amalgamated to an imbedding of type dd_j of a doubly-rooted graph (H, u, v), with all roots 2-valent, the following production rule holds:*

$$s_i * dd_j \longrightarrow 6d_{i+j} \qquad (3.4)$$

Proof When both edge-ends of u are inserted on the same side of vertex t, as in the left-hand drawing of Figure 2.3, the fb-walk (red) at t and one of the fb-walks at u become single strands and are recombined into a single fb-walk of $G * H$. At least one of the fb-walks at v is unaffected. This yields a contribution of $4d_{i+j}$ to the result of the production rule.

When the respective edge-ends of t and u alternate, the red fb-walk at t is broken into two red strands, as in the right-hand drawing of Figure 2.3; the single strand from one of the two fb-walks at u recombines with one of the red strands, and the single strand from the other fb-walk at u recombines with the other red strand. Thus, whether or not either or both of the fb-walks at u coincide with the walks at v, the resulting fb-walks at v in $G * H$ are distinct. This yields a contribution of $2d_{i+j}$ to the result of the production rule, raising the total to $6d_{i+j}$. ♢

Theorem 3.3 *When an imbedding of type d_i of a single-rooted graph (G, t) is amalgamated to an imbedding of type ds_j of a doubly-rooted graph (H, u, v), with all roots 2-valent, the following production rule holds:*

$$d_i * ds_j \longrightarrow 4s_{i+j} + 2s_{i+j+1} \tag{3.5}$$

Proof It follows from Production Rule (2.1) that four of the resulting imbeddings are of genus $i + j$ and two of genus $i + j + 1$. Since neither of the fb-walks at u is broken into more than one strand, it follows that both occurrences of v are on the same fb-walk or single strand. Thus, the net result is $4s_{i+j} + 2s_{i+j+1}$. ♢

Theorem 3.4 *When an imbedding of type s_i of a single-rooted graph (G, t) is amalgamated to an imbedding of type ds_j of a doubly-rooted graph (H, u, v), with all roots 2-valent, the following production rule holds:*

$$s_i * ds_j \longrightarrow 6s_{i+j} \tag{3.6}$$

Proof It follows from Production Rule (2.2) that all six resulting imbeddings have genus $i + j$. Since neither of the fb-walks at v is broken into more than one strand, all are of type s. ♢

Theorem 3.5 *When an imbedding of a single-rooted graph (G, t) is amalgamated to an imbedding of type sd_j of a doubly-rooted graph (H, u, v), with all roots 2-valent, the following two production rules hold:*

$$d_i * sd_j \longrightarrow 6d_{i+j} \tag{3.7}$$

$$s_i * sd_j \longrightarrow 6d_{i+j} \tag{3.8}$$

Proof Production Rules (2.3) and (2.4) imply that all six resulting imbeddings, in either case, have genus $i+j$. Since one of the two fb-walks at v is carried intact into $G * H$, all the resulting imbeddings are of type d. $\diamond$

Sub-partials of $ss_i(H, u, v)$

There are three ***sub-partials of*** $ss_i(H,u,v)$, based on incidence of the two strands formed when the single fb-walk at w is split twice at w:

$ss_i^0(H,u,v)$ = the number of imbeddings of type-ss_i such that the fb-walk at u is not incident on v;

$ss_i^1(H,u,v)$ = the number of imbeddings of type-ss_i such that one strand of the fb-walk at u is twice incident on v;

$ss_i^2(H,u,v)$ = the number of imbeddings of type-ss_i such that both strands of the fb-walk at u are incident on v.

Theorem 3.6 *When an imbedding of a single-rooted graph (G,t) is amalgamated to an imbedding of type ss_j of a doubly-rooted graph (H,u,v), with all roots 2-valent, the following three production rules hold:*

$$d_i * ss_j^0 \longrightarrow 6s_{i+j} \tag{3.9}$$

$$d_i * ss_j^1 \longrightarrow 6s_{i+j} \tag{3.10}$$

$$d_i * ss_j^2 \longrightarrow 4s_{i+j} + 2d_{i+j} \tag{3.11}$$

Proof Production Rule (2.3) implies for all three sub-partials, that all the imbeddings have genus $i+j$; in the four ways with both edge-ends at u placed on the same side of t, the result is of type s, yielding $4s_{i+j}$ for each sub-partial. Figure 3.2 illustrates alternating edge-ends for all three sub-partials.

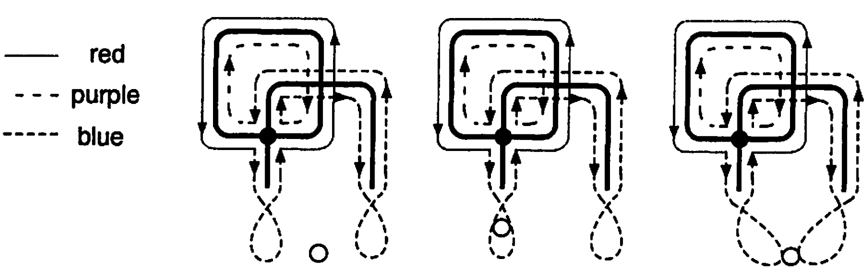

Figure 3.2: Alternating edge-ends in the three subcases of $d * ss$.

In case ss^0, depicted by the leftmost drawing, whatever fb-walk is twice incident at vertex v (and not incident on u) in the imbedding of H remains twice incident on v in the imbedding of $G * H$, yielding an additional $2s_j$. In case ss^1, shown in the middle, whichever strand was twice incident on v in the imbedding of H remains twice incident in the imbedding of $G * H$, also yielding $2s_j$. However, in case ss^2, shown at the right, both the strands that were incident on v remain incident on v, but one is recombined with the red strand and the other with the purple strand, yielding $2d_j$. ◇

Theorem 3.7 *When an imbedding of a single-rooted graph (G,t) is amalgamated to an imbedding of type ss_j of a doubly-rooted graph (H,u,v), with all roots 2-valent, the following three production rules hold:*

$$
\begin{aligned}
s_i * ss_j^0 &\longrightarrow 6s_{i+j}\\
s_i * ss_j^1 &\longrightarrow 6s_{i+j}\\
s_i * ss_j^2 &\longrightarrow 6s_{i+j}
\end{aligned}
$$

Proof Production Rule (2.4) implies for all three cases, that all six imbeddings have genus $i+j$ and that in the four ways with both edge-ends at u on the same side of t, the result is of type s, yielding $4s_{i+j}$ for each subpartial. Figure 3.3 illustrates alternating edge-ends for all three sub-partials. As in the analysis of $d * ss$, all three sub-partials ss^0 get an additional $2s_{i+j}$. ◇

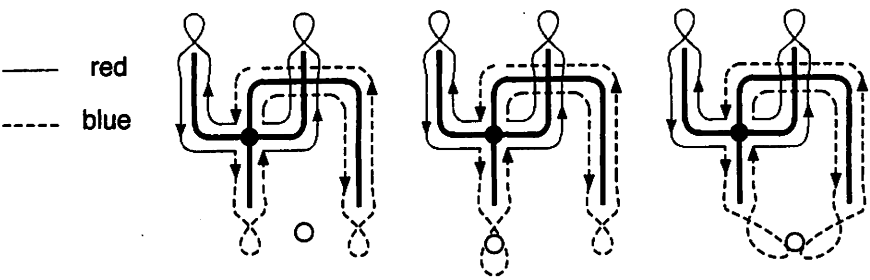

Figure 3.3: Alternating edge-ends in the three subcases of $s * ss$.

NOTATION We sometimes use $dd_j^\bullet$ for the sum $dd_j^0 + dd_j' + dd_j''$ and $ss_j^\bullet$ for $ss_j^0 + ss_j^1 + ss_j^2$. This makes it easier to follow subsequent calculations here.

We observe that Theorem 3.7 implies that

$$s_i * ss_j^\bullet \longrightarrow 6s_{i+j} \tag{3.12}$$

Corollary 3.8 *Let $(X,v) = (G,t) * (H,u,v)$, where (G,t) and (H,u,v) have 2-valent roots. Then we have*

$$d_k(X) = 4\sum_{i=0}^{k} d_{k-i}dd_i^0 + 2\sum_{i=0}^{k-1} d_{k-i-1}dd_i^0 + 4\sum_{i=0}^{k} d_{k-i}dd_i'$$
$$+2\sum_{i=0}^{k-1} d_{k-i-1}dd_i' + 4\sum_{i=0}^{k} d_{k-i}dd_i'' + 6\sum_{i=0}^{k} s_{k-i}dd_i^\bullet$$
$$+6\sum_{i=0}^{k} d_{k-i}sd_i + 6\sum_{i=0}^{k} s_{k-i}sd_i + 2\sum_{i=0}^{k} d_{k-i}ss_i^2 \quad (3.13)$$

$$s_k(X) = 2\sum_{i=0}^{k-1} d_{k-i-1}dd_i'' + 4\sum_{i=0}^{k} d_{k-i}ds_i + 2\sum_{i=0}^{k-1} d_{k-i-1}ds_i$$
$$+6\sum_{i=0}^{k} s_{k-i}ds_i 6\sum_{i=0}^{k} d_{k-i}ss_i^0 + 6\sum_{i=0}^{k} d_{k-i}ss_i^1$$
$$+4\sum_{i=0}^{k} d_{k-i}ss_i^2 + 6\sum_{i=0}^{k} s_{k-i}ss_i^\bullet \quad (3.14)$$

For convenience, we conclude this section by listing all the production rules derived in this section together in Table 3.2.

Table 3.2: The productions for an amalgamation $(G, t) * (H, u, v)$.

production			reference
$d_i * dd_j^0$	$\longrightarrow$	$4d_{i+j} + 2d_{i+j+1}$	(3.1)
$d_i * dd_j'$	$\longrightarrow$	$4d_{i+j} + 2d_{i+j+1}$	(3.2)
$d_i * dd_j''$	$\longrightarrow$	$4d_{i+j} + 2s_{i+j+1}$	(3.3)
$s_i * dd_j^\bullet$	$\longrightarrow$	$6d_{i+j}$	(3.4)
$d_i * ds_j$	$\longrightarrow$	$4s_{i+j} + 2s_{i+j+1}$	(3.5)
$s_i * ds_j$	$\longrightarrow$	$6s_{i+j}$	(3.6)
$d_i * sd_j$	$\longrightarrow$	$6d_{i+j}$	(3.7)
$s_i * sd_j$	$\longrightarrow$	$6d_{i+j}$	(3.8)
$d_i * ss_j^0$	$\longrightarrow$	$6s_{i+j}$	(3.9)
$d_i * ss_j^1$	$\longrightarrow$	$6s_{i+j}$	(3.10)
$d_i * ss_j^2$	$\longrightarrow$	$4s_{i+j} + 2d_{i+j}$	(3.11)
$s_i * ss_j^\bullet$	$\longrightarrow$	$6s_{i+j}$	(3.12)

REMARK In writing Recursions (3.13) and (3.14) above, we suppressed indication of graphs G and H as arguments, in order that they not occupy too many lines. In the examples to follow, we see how restriction of these recursions to particular genus distributions of interest greatly simplifies them. The reason for placing the index variable i of each sum with the second factor, rather than the first, also becomes clear in the applications.

4 Open Chains of Copies of a Graph

We can specify a sequence of *open chains* of copies of a doubly-rooted graph (G, u, v) recursively.

$$(X_1, t_1) = (G, v) \quad \text{(suppressing co-root } u) \tag{4.1}$$

$$(X_n, t_n) = (X_{n-1}, t_{n-1}) * (G, u, v) \quad \text{for } n \geq 1 \tag{4.2}$$

Theorem 4.1 *Let (G, u, v) be a double-rooted graph of known genus distribution and cycle rank $\beta > 0$. Then we can calculate the genus distribution of an open chain of n copies of (G, u, v) within time proportional to $\beta^2 n^2$.*

Proof For each amalgamation in the iteration, the number of applications of each of the 12 productions of Table 3.2 equals the number of possibly non-zero products of the respective partials for the left and right amalgamands. For the i^{th} amalgamation, each production has at most $i\beta$ possibly non-zero partials for its left amalgamand and at most β possibly non-zero partials for its right amalgamand. Summing $i\beta * \beta$ over i yields the result. ◇

REMARK By way of contrast with Theorem 4.1, we recall that calculating the minimum genus of a graph is NP-hard.

As an example, consider the graph $\ddot{K}_4$, which is obtained by inserting a midpoint on each of two non-adjacent edges of K_4. Its first few open chains are shown in Figure 4.1.

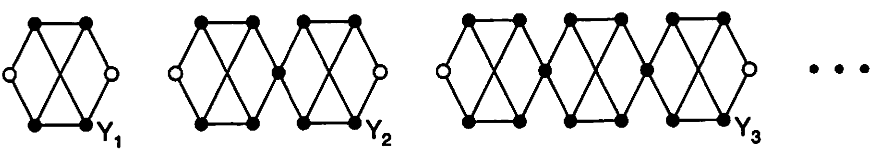

Figure 4.1: Y_n is an open chain of n copies of $\ddot{K}_4$.

The double-root and single-root partials of $\ddot{K}_4$, all derived by face-tracing, are given in Table 4.1.

Table 4.1: Double-root and single-root partials of $\ddot{K}_4$.

k	dd^0_k	dd'_k	dd''_k	ds_k	sd_k	ss^0_k	ss^1_k	ss^2_k	d_k	s_k	g_k
0	2	0	0	0	0	0	0	0	2	0	2
1	0	0	4	4	4	0	0	2	8	6	14

We define the sequence of graphs Y_n inductively, using Recursions (4.1) and (4.2). Using Recursions (3.13) and (3.14) and Table 4.1, we obtain the following recursions for the single-root partials of the graph Y_n.

$$\begin{aligned}
d_k(Y_n) &= 4d_k(Y_{n-1})dd_0^0(\ddot{K}_4) + 2d_{k-1}(Y_{n-1})dd_0^0(\ddot{K}_4) + 4\cdot 0 + 2\cdot 0 \\
&\quad +4d_{k-1}(Y_{n-1})dd_1''(\ddot{K}_4) + 6s_k(Y_{n-1})dd_0^\bullet(\ddot{K}_4) \\
&\quad +6s_{k-1}(Y_{n-1})dd_1''(\ddot{K}_4)] + 6d_{k-1}(Y_{n-1})sd_1(\ddot{K}_4) \\
&\quad +6s_{k-1}(Y_{n-1})sd_1(\ddot{K}_4) + 2d_{k-1}(Y_{n-1})ss_1^2(\ddot{K}_4) \\
&= 8d_k(Y_{n-1}) + 4d_{k-1}(Y_{n-1}) + 16d_{k-1}(Y_{n-1}) + 12s_k(Y_{n-1}) \\
&\quad +24s_{k-1}(Y_{n-1}) + 24d_{k-1}(Y_{n-1}) + 24s_{k-1}(Y_{n-1}) + 4d_{k-1}(Y_{n-1}) \\
&= 8d_k(Y_{n-1}) + 48d_{k-1}(Y_{n-1}) + 12s_k(Y_{n-1}) + 48s_{k-1}(Y_{n-1}) \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
s_k(Y_n) &= 2d_{k-2}(Y_{n-1})dd_1''(\ddot{K}_4) + 4d_{k-1}(Y_{n-1})ds_1(\ddot{K}_4) \\
&\quad +2d_{k-2}(Y_{n-1})ds_1(\ddot{K}_4) + 6s_{k-1}(Y_{n-1})ds_1(\ddot{K}_4) + 6\cdot 0 + 6\cdot 0 \\
&\quad +4d_{k-1}(Y_{n-1})ss_1^2(\ddot{K}_4) + 6s_{k-1}(Y_{n-1})ss_1^\bullet(\ddot{K}_4) \\
&= 8d_{k-2}(Y_{n-1}) + 16d_{k-1}(Y_{n-1}) + 8d_{k-2}(Y_{n-1}) \\
&\quad +24s_{k-1}(Y_{n-1}) + 8d_{k-1}(Y_{n-1}) + 12s_{k-1}(Y_{n-1}) \\
&= 24d_{k-1}(Y_{n-1}) + 16d_{k-2}(Y_{n-1}) + 36s_{k-1}(Y_{n-1}) \qquad (4.4)
\end{aligned}$$

Recursions (4.3) and (4.4) and Table 4.1 give the respective d_k- and s_k-partials of the genus distribution for graph Y_2. They are summarized in Table 4.2.

$$\begin{aligned}
d_0(Y_2) &= 8d_0(Y_1) + 0 + 12s_0(Y_1) + 0 \;=\; 8\cdot 2 + 12\cdot 0 \;=\; 16 \\
d_1(Y_2) &= 8d_1(Y_1) + 48d_0(Y_1) + 12s_1(Y_1) + 48s_0(Y_1) \\
&= 8\cdot 8 + 48\cdot 2 + 12\cdot 6 + 0 \;=\; 232 \\
d_2(Y_2) &= 8d_2(Y_1) + 48d_1(Y_1) + 12s_2(Y_1) + 48s_1(Y_1) \\
&= 0 + 48\cdot 8 + 0 + 48\cdot 6 \;=\; 672 \\
d_3(Y_2) &= 8d_3(Y_1) + 48d_2(Y_1) + 12s_3(Y_1) + 48s_2(Y_1) \;=\; 0
\end{aligned}$$

$$\begin{aligned}
s_0(Y_2) &= 0 + 0 + 0 \;=\; 0 \\
s_1(Y_2) &= 24d_0(Y_1) + 0 + 36s_0(Y_1) \;=\; 24\cdot 2 + 0 \;=\; 48 \\
s_2(Y_2) &= 24d_1(Y_1) + 16d_0(Y_1) + 36s_1(Y_1) \\
&= 24\cdot 8 + 16\cdot 2 + 36\cdot 6 \;=\; 440 \\
s_3(Y_2) &= 24d_2(Y_1) + 16d_1(Y_1) + 36s_2(Y_1) \;=\; 0 + 16\cdot 8 + 0 \;=\; 128
\end{aligned}$$

Table 4.2: Single-root partials of Y_2.

k	d_k	s_k	g_k
0	16	0	16
1	232	48	280
2	672	440	1112
3	0	128	128

5 Conclusions

The methods presented in this paper are sufficient to calculate detailed imbedding information, within quadratic time, for various graphs whose genus distributions were previously unknown, including the following:

- the genus distribution of the vertex amalgamation $(G, u) * (H, v)$ of any two graphs (G, u) and (H, v) with 2-valent roots whose single-root partitioned genus distributions are known, with arbitrarily large degrees at vertices of G and H other than at the roots;
- the single-root partitioned genus distributions for an open chain of copies of a graph (G, u, v) with a known double-root partitioned genus distribution, with arbitrarily large degrees at vertices of G other than at the roots;
- the single-root partitioned genus distributions for a periodic chain of copies of several different graphs of known genus distributions.

For instance, we could calculate the genus distribution of an open chain of copies of a complete graph K_8, a wheel graph W_7, or any of a variety of ladder graphs — whose double-rooted genus distribution is known (or calculable).

In combination with methods from [Gr09a], we can also calculate

- the double-root partitioned genus distribution of a graph K formed as an iterated amalgamation of doubly-rooted graphs

$$K = (H_1, u_1, v_1) * (H_2, u_2, v_2) * \ \dots \ * (H_n, u_n, v_n)$$

whose double-root partitioned genera are known, with arbitrarily large degrees in the amalgamands at vertices other than the roots;
- recurrences that specify the genus distributions for a sequence of graphs $G_n = G_{n-1} * G_1$ formed as a chain *or as a closed chain* of copies of G_1, where G_1 has arbitrarily large degrees anywhere except the roots;

These methods have the potential for calculating the genus distribution of various classes of 3-regular and 4-regular graphs, when applied in conjunction with auxiliary techniques for determining the order of the amalgamations. For instance, [Gr09b] shows how to use tree-traversal and root-splitting as auxiliary techniques, which are applied to deriving the genus distribution for any 3-regular outerplanar graph.

RESEARCH PROBLEM 1. Develop methods for solving simultaneous recursions of the form appearing in Corollary 3.8.

RESEARCH PROBLEM 2. Develop methods for characterizing the genus distributions that result from iterated amalgamations. For instance, are they unimodal? What are their asmyptotic properties?

Several recently prepared papers continue our concern with calculating genus distributions of sequences of graphs: [Gr09a], [PoKhGr09], [Gr09b], [Gr09c], and [KhPoGr09].

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