The characterization of graphs with the largest Laplacian eigenvalue at most $\frac{5+\sqrt{13}}{2}$

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Abstract

In this paper connected graphs with the largest Laplacian eigenvalue at most $\frac{5+\sqrt{13}}{2}$ are characterized. Moreover, we prove that these graphs are determined by their Laplacian spectrum.

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1. Introduction

Let G be a finite simple graph with n vertices, m edges and the adjacency matrix $A(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the Laplacian eigenvalues (*L-eigenvalues*, for short) of G. The multiset of the eigenvalues of $L(G)$ are called the *Laplacian spectrum* (*L-spectrum*, for short) of G. We

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denote the characteristic polynomials of Laplacian and adjacency matrices of G by $Q_G(\lambda)$ and $\chi_G(\lambda)$, respectively.

There have been some attempts to characterize graphs having small number of L-eigenvalues exceeding a given value (see[2, 3]). In [6], Petrovic et al. consider a connected bipartite graph G with exactly two L-eigenvalues greater than two, and determined all those graphs. In [9], graphs with fourth L-eigenvalue less than two are characterized. Moreover connected bipartite graphs whose the third largest L-eigenvalue is less than three have been identified by Zhang [10]. Recently, all graphs with the largest L-eigenvalue at most 4 are characterized [5]. In this paper we identify all graphs with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$.

A graph is said to be *determined* (*DS* for short) *by its L-spectrum* if there is no other non-isomorphic graph with the same L-spectrum. Since the problem of characterizing DS graphs seems to be very difficult, finding any new infinite family of these graphs will be an interesting problem (see[7, 8]). Using the characterization of graphs with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$, we show that these graphs are DS with respect to the Laplacian matrix.

2. Graphs with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$

All connected graphs with the largest L-eigenvalue at most 4 are identified. In this section we determine connected graphs with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. So the characterization of graphs with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$ will be completed.

Theorem 1. *[5] The list of all connected graphs with the largest L-eigenvalue at most 4 includes precisely the following graphs: $P_n, C_n (n \geq 3), K_{1,3}, K_4, H_1$ and H_2. Where H_2 and H_1 are obtained from K_4 by deleting one edge and two adjacent edges, respectively.*

Lemma 1. *[2] Let G be a connected graph, and let H be a proper subgraph of G. Then $\mu_1(H) \leq \mu_1(G)$.*

Lemma 2. *[1, page 59] Let G be the graph obtained from the disjoint union*

$H_1 + H_2$ by adding an edge v_1v_2 joining the vertex v_1 of H_1 and v_2 of H_2, then $\chi_G(\lambda) = \chi_{H_1}(\lambda)\chi_{H_2}(\lambda) - \chi_{(H_1-v_1)}(\lambda)\chi_{(H_2-v_2)}(\lambda)$, where $H_i - v_i$ denotes the graph obtained from H_i by deleting the vertex v_i and the edges incident to it ($i = 1, 2$).

Lemma 3. *Let G be a connected graph of type T_1 with $n \geq 9$ vertices (See Fig. 2) and let x and y be the vertices of degree 3 of G. Then*

$$Q_G(\lambda) = (-1)^n(\chi_G(2-\lambda) + 2\chi_{G-x}(2-\lambda) + \chi_{G-x-y}(2-\lambda)).$$

Proof. Let P be the matrix of size $n \times n$ where $p_{11} = p_{22} = 1$ and $p_{ij} = 0$, otherwise. It is easy to see that for each matrix A of size $n \times n$, we have:

$$\det(P + A) = \det(A) + \det(B) + \det(C) + \det(E) \tag{1}$$

where B and C are of sizes $n-1 \times n-1$ and $n-2 \times n-2$, respectively, and they can be obtained by deleting the first row and the first column of A and B, respectively. The matrix E is of size $n-1 \times n-1$ and it can be obtained by deleting the second row and the second column of A. Now let A be the adjacency matrix of G. Without any loss of generality we can assume that $v_1 = x$ and $v_2 = y$ are the vertices of degree 3 of G. So $Q_G(\lambda) = \det(\lambda I_n - D + A) = \det(-P + (\lambda-2)I_n + A) = (-1)^n(\det(P + (2-\lambda)I_n - A))$. Since two graphs $G - v_1$ and $G - v_2$ are isomorphic they have the same A-characteristic polynomials. On the other hand the adjacency matrices of $G - v_1$ and $G - v_1 - v_2$ can be obtained by deleting the first row and the first column of A and the adjacency matrix of $G - v_1$, respectively. The adjacency matrix of $G - v_2$ can be obtained by deleting the second row and the second column of A. Hence by (1) we have:

$$Q_G(\lambda) = (-1)^n(\chi_G(2-\lambda) + 2\chi_{G-x}(2-\lambda) + \chi_{G-x-y}(2-\lambda)).$$

□

Lemma 4. *Let G be a connected graph of type T_1 with at least 9 vertices (See Fig. 2). Then $\mu_1(G) \leq \frac{5+\sqrt{13}}{2}$.*

Proof. Let $\lambda = 2 - \mu$ and let x and y be the vertices of degree 3 of G. By Lemma 3, we have $Q_G(\mu) = (-1)^n(\chi_G(\lambda) + 2\chi_{G-x}(\lambda) + \chi_{G-x-y}(\lambda))$.

Let P_l be the path on l vertices. Using Lemma 2, we have:

$$\chi_G(\lambda) = (\lambda+1)^2(\lambda-2)(\lambda(\lambda+1)^2(\lambda-2) - 2(\lambda^2-1))\chi_{P_{n-7}} +$$
$$((\lambda^2-1)^2 - (\lambda+1)^4(\lambda-2)^2)\chi_{P_{n-8}},$$
$$\chi_{G-x}(\lambda) = (\lambda^2-1)((\lambda+1)(\lambda^3-\lambda^2-3\lambda+1)\chi_{P_{n-7}} - (\lambda-2)(\lambda+1)^2\chi_{P_{n-8}}),$$
$$\chi_{G-x-y}(\lambda) = (\lambda^2-1)^2\chi_{P_{n-6}} = \lambda(\lambda^2-1)^2\chi_{P_{n-7}} - (\lambda^2-1)^2\chi_{P_{n-8}}.$$

Therefore:

$$Q_G(\mu) = (-1)^n(\lambda+1)^2(((\lambda+1)(\lambda-2)(\lambda^3-\lambda^2-4\lambda+2)+$$
$$(\lambda-1)(2\lambda^3-\lambda^2-7\lambda+2))\chi_{P_{n-7}}(\lambda) - (\lambda-2)(\lambda^3+2\lambda^2-3\lambda-4)\chi_{P_{n-8}}(\lambda)).$$

Moreover for $l > 2$ we have $\chi_{P_l}(\lambda) = \lambda\chi_{P_{l-1}}(\lambda) - \chi_{P_{l-2}}(\lambda)$. Hence $Q_G(\mu) = (-1)^n(\lambda+1)^2((\lambda\varphi(\lambda) - \psi(\lambda))\chi_{P_{n-9}}(\lambda) - \varphi(\lambda)\chi_{P_{n-10}}(\lambda))$ where

$$\varphi(\lambda) = \lambda((\lambda-2)(\lambda+1)(\lambda^3-\lambda^2-4\lambda+2) + (\lambda-1)(2\lambda^3-\lambda^2-7\lambda+2))$$
$$-(\lambda-2)(\lambda^3+2\lambda^2-3\lambda-4)$$

and

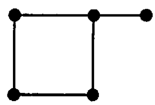
$$\psi(\lambda) = (\lambda+1)(\lambda-2)(\lambda^3-\lambda^2-4\lambda+2) + (\lambda-1)(2\lambda^3-\lambda^2-7\lambda+2).$$

On the other hand $\chi_{P_2}(\lambda) = \lambda^2 - 1$ and $\chi_{P_l}(\lambda) = \lambda\chi_{P_{l-1}}(\lambda) - \chi_{P_{l-2}}(\lambda)$. So we can see that $(-1)^l\chi_{P_l}(\lambda) > 0$ and $|\chi_{P_l}(\lambda)| > |\chi_{P_{l-1}}(\lambda)|$ for $\lambda \leq -\frac{3+\sqrt{13}}{2}$. It is an easy task to see that $(\lambda\varphi(\lambda) - \psi(\lambda)) < 0$ and $\varphi(\lambda) < 0$ or $|(\lambda\varphi(\lambda) - \psi(\lambda))| > |\varphi(\lambda)|$ for $\lambda < -\frac{3+\sqrt{13}}{2}$. Therefore if $n > 10$, then for each $\mu > \frac{5+\sqrt{13}}{2}$, we have $Q_G(\mu) > 0$ and so $\mu_1(G) < \frac{5+\sqrt{13}}{2}$. Moreover it is an easy task to see that for $n = 9, 10$, $\mu_1(G) \leq \frac{5+\sqrt{13}}{2}$. □

A tree is called *T-shape* if it has exactly one vertex of degree greater than two and the degree of this vertex is 3. We will denote by $T(l_1, l_2, l_3)$ the unique *T-shape* tree such that $T(l_1, l_2, l_3) - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where P_{l_i} is the path on l_i vertices $(i = 1, 2, 3)$, and v is the vertex of degree 3.

Lemma 5. *Let $G \neq T_8$ be a connected graph with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. Then G does not have C_n for $n \geq 4$ as a subgraph (See Fig. 2).*

Proof. Since $\mu_1(K_{1,4}) = 5$, by Lemma 1, the maximum degree of G is at most 3. The computer package newGRAPH [4], can be used to find eigenvalues. By newGRAPH, we have $\mu_1(T(2,2,2)) > \mu_1(T(1,2,3)) > \mu_1(T(1,2,2)) = \frac{5+\sqrt{13}}{2}$. So by Lemma 1, G does not have $T(1,2,3)$ or $T(2,2,2)$ as subgraphs. Moreover by newGRAPH we can see that every graph on 7 vertices with $T(1,2,2)$ as a proper subgraph has largest L-eigenvalue greater than $\frac{5+\sqrt{13}}{2}$. So if G has at least 7 vertices, then G does not have $T(1,2,2)$ as a proper subgraph. By the previously facts and using Lemma 1, G does not have C_n for $n \geq 7$ as a proper subgraph and if G has at least 7 vertices, then G does not have C_6 as a proper subgraph. Again by newGRAPH we can see that if G has 6 vertices, then G does not have C_6 as a proper subgraph. Moreover using newGRAPH, $\mu_1(T_0) > \frac{5+\sqrt{13}}{2}$ and so by Theorem 1 and Lemma 1, G does not have C_4 as a proper subgraph.



T0

Fig. 1

It is an easy task to see that the only graph G on 6 vertices with C_5 as a subgraph is T_8 and $\mu_1(T_8) = \frac{5+\sqrt{13}}{2}$. Since each graph G with at least 7 vertices does not have $T(1,2,2)$ as a proper subgraph, G has C_5 as a subgraph if and only if G is of type T_8. On the other hand $\mu_1(C_n) \leq 4$, so G is not a cycle. Hence if $G \neq T_8$, then G does not have C_n for $n > 3$ as a subgraph. □

Theorem 2. *The list of all connected graphs with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$ includes precisely the following graphs: T_i for $i = 1,2,3$ and $n \geq 9$, T_i for $i = 4,5$ and $n \geq 5$ and T_i for $i = 6,7,8,9$.*

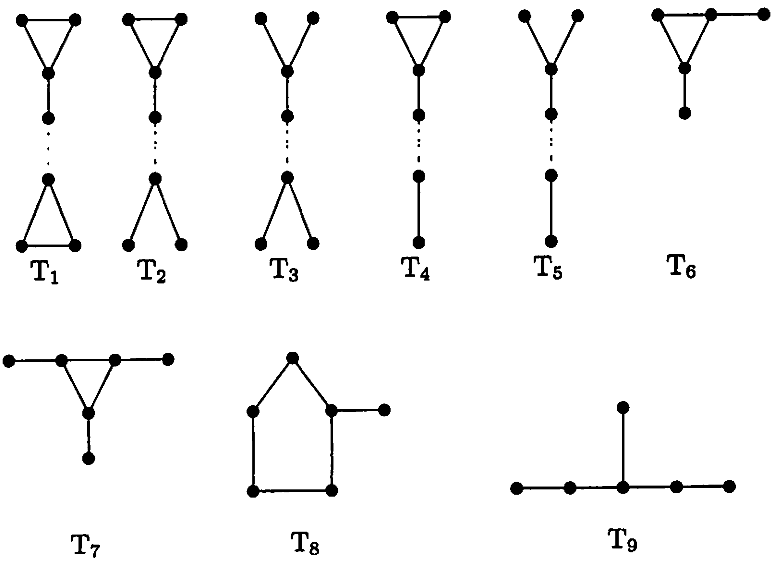


Fig. 2

Proof. Using newGRAPH, we can see that the largest L-eigenvalue of each T_i for $6 \le i \le 9$ is in the interval $(4, \frac{5+\sqrt{13}}{2}]$. On the other hand by Theorem 1 and Lemmas 1 and 4, the largest L-eigenvalue of each T_i for $1 \le i \le 5$ is in the interval $(4, \frac{5+\sqrt{13}}{2}]$. Hence all mentioned graphs in this theorem have largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. Now let G be a connected graph with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. We show that G is one of the previously mentioned graphs. By Lemma 5, if $G \neq T_8$, then G does not have C_n for $n > 3$ as a subgraph. Again using newGRAPH and this fact that the graph G on $n \ge 7$ vertices does not have $T(1,2,2)$ as a proper subgraph, G has at most 2 vertices of degree 3 and so G has at most 2 subgraph of type C_3. Moreover by newGRAPH, we have $\mu_1(L(T(2,2,3)) > \mu_1(L(T(1,2,3)) > \mu_1(T_6) = \mu_1(T_7) = \frac{5+\sqrt{13}}{2}$ and so G does not have T_7 as a proper subgraph. By newGRAPH, we can see that if G has T_6 as a proper subgraph, then G is of type T_7. Therefore if G has two C_3 as a subgraph, then G must be of type T_1. If G has one C_3 as a subgraph, then G must be of type T_i for $i \in \{2,4,6,7\}$. If G does not have C_3 as a proper subgraph, then by Lemma 5, G is either of type T_8 or a tree. Now let G be a tree. If G has 2 vertices of degree 3, then G must be of type T_3. If G has one vertex of degree 3, then G must be of type T_5 or T_9. Finally using newGRAPH, we can see that each graph T_i with $n < 9$ vertices for $i = 1, 2, 3$ and each graph T_i with $n < 5$ vertices for $i = 4, 5$ does not have the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. □

3. Spectral characterization of connected graphs with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$

Two graphs are said to be *L-cospectral* if they have the same L-spectrum. In this section, we show that each connected graph with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$ is determined by its L-spectrum.

Lemma 6. *[5] Any connected graph with the largest L-eigenvalue at most 4 is determined by its L-spectrum.*

Lemma 7. *[7] Let G be a graph. The following can be obtained from adjacency (respectively, Laplacian) spectrum:*
i) *The number of vertices,*
ii) *The number of edges.*
The L-spectrum determines:
iii) *The number of components,*
iv) *The sum of squares of degrees of vertices.*

Theorem 3. *Each connected graph with the largest L-eigenvalue at most $\frac{5+\sqrt{13}}{2}$ is determined by its L-spectrum.*

Proof. Using Lemma 6, it is sufficient to show that each connected graph with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$ is determined by its L-spectrum. Now let G be a connected graph with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$ and let H be L-cospectral to G. By (iii) of Lemma 7, H is a connected graph with the largest L-eigenvalue in the interval $(4, \frac{5+\sqrt{13}}{2}]$. If G is of type T_2, then by (i) and (ii) of Lemma 7, H and G have the same number of vertices and edges. So H is of type T_i for $i \in \{2, 4, 6, 7\}$. Again by (iv) of Lemma 7, the sum of squares of degrees of vertices are equal for H and G. So H is of type T_i for $i \in \{2, 6, \}$. By newGRAPH, we can see that T_6 is not cospectral to the other graph with 5 vertices. Hence H is of type T_2, and so H and G are isomorphic. Moreover it is easy to see that T_8 and T_9 are determined by their L-spectrum. Now let G be of type T_i for $i \in \{1,3,4,5,7\}$, then by (i) and (ii) of Lemma 7, H and G have the same number of vertices and edges. Again by (iv) of Lemma 7, the sum of squares of degrees of vertices of H and G are equal. Therefore H must be of type T_i and so they are isomorphic. □

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