The Mod Sum Number of Even Fans and Symmetric Complete Bipartite Graphs

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Abstract

A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer z and a labelling, λ, of the vertices of G with distinct elements from $\{1, 2, \ldots, z-1\}$ so that $uv \in E$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. The mod sum number $\rho(G)$ of a connected graph G is the smallest nonnegative m such that $G \cup mK_1$, the union of G and m isolated vertices, is a mod sum graph. In section 2, we prove that fan(F_n) is not a mod sum graph and give the mod sum number of F_n ($n \geq 6$ is even). In section 3 we give the mod sum number of symmetric complete graph.

Keywords: Mod sum graph; Fan; Complete bipartite graph

1. Introduction

All graphs in this paper are finite with no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of [3] unless otherwise specified.

Harary [2] introduced the idea of sum graphs and integral sum graphs. At first, let N denote the set of positive integers. The sum

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graph $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is said to be a sum graph if it is isomorphic to the sum graph of some $S \subset N$. It is obvious that a sum graph cannot be connected. There must always be at least one isolated vertex, namely the vertex with the highest label. The sum number $\sigma(G)$ of a connected graph G is the smallest nonnegative m such that $G \cup mK_1$, the union of G and m isolated vertices, is a sum graph.

Mod sum graph was introduced by Bolland, Laskar, Turner and Domke [1] as a generalization of sum graph. A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer z and a labelling, λ, of the vertices of G with distinct elements from $\{1, 2, \ldots, z-1\}$ so that $uv \in E$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. The mod sum number $\rho(G)$ of a connected graph G is the smallest nonnegative m such that $G \cup mK_1$, the union of G and m isolated vertices, is a mod sum graph. Any sum graph can be considered as a mod sum graph by choosing a sufficiently large modulus z. The converse is not true. Thus we have $\rho(G) \leq \sigma(G)$.

A fan F_n is a graph $G = (V, E)$ with a vertex set $V = \{v_c, v_1, v_2, \ldots, v$ such that $v_c v_i \in E$ for $i = 1, 2, \ldots, n$, $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$. The vertex v_c is called center, and the other n vertices $v_1, v_2, \ldots, v_n$ are called the rim vertices. An edge incident on the center and a rim vertex is called a spoke, and an edge incident on two rim vertices is called a rim edge.

Although some results on mod sum graphs were solved, a considerable number of unsolved problems remain. In section 2 we prove that fan(F_n) is not a mod sum graph and give the mod sum number of F_n ($n \geq 6$ is even) that is a open problem in [4]. In section 3 we give the mod sum number of symmetric complete bipartite graph that is a open problem in [5]. Both problems have a research history. Draganova [6] has shown that for $n \geq 5$ and n odd, $\rho(F_n) = n$. Sutton, Miller, Ryan and Slamin [5] showed that for $n \geq 3$, $K_{n,n}$ is not mod sum graph. Wallace [7] also proved that $K_{m,n}$ is mod sum graph when n is even and $n \geq 2m$ or when n is odd and that $\rho(K_{m,n}) = m$ when $3 \leq m \leq n \leq 2m$.

To simplify notations, throughout this paper, we assume that the vertices of G are already identified by their labels.

2. Fans

It is easy to verify that F_2, F_3 and F_4 are not mod sum graphs. Next we prove that F_n is not a mod sum graph when $n \geq 5$. In this section we let v_c be the center and $\{v_1, v_2, \ldots, v_n\}$ be the rim vertices of F_n. We shall use the term edge sum,written as $[a, b]$ to mean the sum of the labels of the two vertices incident on the edge so that $[a, b] \in V$ is the same as $a + b \in V$ (Since all arithmetic is performed modulo m, strictly speaking we should say $a + b \ mod \ m \in V$, m is some suitable modulo). We suppose F_n is a mod sum graph modulo some suitable m. Thus there are some properties of F_n.

Property 1. $\{v_1, v_2, \ldots, v_n\} = \{v_c + v_1, v_c + v_2, \ldots, v_c + v_n\}$. □

Property 2. *The labels of the rim vertices can be partitioned into sets of equal size t such that the elements of each set from a t-cycle under addition of v_c.*

By Property 1, we may achieve the following manner. Select the smallest(mod m)label of V, say v_i, remove it from V and place it as the first label of a new set representing the first partition. There must exist a rim vertex v_j with a label equal to $v_i + v_c$. So remove this label from the set V and place in the first partition. Repeat the process until no label in the series can not be found in V. This completes the selection of the first partition set. If there are labels still remaining in V, then the process is repeated for a second partition set until the required label is not found in V. It is easy to verify that the labels from the first partition set differ the label from the second partition set. This completes the second partition set.

Repeat the selection process until no label in V. Since the consecutive elements of each cycle differ by same quantity v_c. It is obvious that all cycles must be the same size, that is, $v_i + v_c = v_{i+1}$, $i = 1, 2, \ldots, t - 1$ and $v_t + v_c = v_1$ where t is the size of the sets. □

In this section, we call these partition sets as t-cycles. We denote the vertices whose labels from t-cycle C_i by $v_{i1}, v_{i2}, \ldots, v_{it}$.

Property 3. *If $v_{ij}v_{ik} \in E$, then $v_{ij} + v_{ik} \notin V(C_i)$, $j,k = \{1,2,\ldots,t\}, j \neq k$.*

Proof. Suppose that $v_{ij}+v_{ik} = v_{il}$ for some $j,k,l = \{1,2,\ldots,t\}, j \neq k$. So

$$v_{i1} + (j-1)v_c + v_{i1} + (k-1)v_c = v_{i1} + (l-1)v_c$$

then $v_{i1} = pv_c$ where $p \in \{1,2,\ldots,t\}$. Hence for some $j \in \{1,2,\ldots,t\}$ we have $v_{ij} = v_c$, a contradiction since all labels must be distinct. □

By Property 2 and 3, it is easy to see.

Property 4. $2 \leq t \leq \frac{n}{2}$. □

Property 5. *If $v_{ij}v_{ik} \in E$, then $v_{ij}+v_{ik} \neq v_c$, $j,k = \{1,2,\ldots,t\}, j \neq k$.*

Proof. Suppose that $v_{ij} + v_{ik} = v_c$ then there is a vertex $v_{il} = 0$, a contradiction. □

Property 6. *If $v_{ik}v_{jl} \in E$, then $v_{ik} + v_{jl} \notin V(C_i \cup C_j)$, $k,l = \{1,2,\ldots,t\}$.*

Proof. Suppose that $v_{ik}v_{jl} \in E$, $v_{ik} + v_{jl} = v_{ip}$. Then $v_{jl} = v_{ip} - v_{ik} = qv_c$. Thus ,there exists $s \in \{1,2,\ldots,t\}$ such that $v_{js} = v_c$, a contradiction. □

Property 7. *At least one t-cycle contains a pair of adjacent vertices.*

Proof. Suppose that all adjacent vertices have labels from different t-cycle if $v_{xi}v_{yj} \in E$ then $x \neq y$, $\forall\ x,y = 1,2,\ldots,\frac{n}{t}$, and $i,j =$

$1, 2, \ldots, t$. Without loss of generality, we can suppose that $v_{1i}v_{2j} \in E$. By property 6, the vertex $v_{1i}v_{2j}$ cannot be in C_1 or C_2. There are two cases to consider.

Case 1. Suppose for adjacent vertices from cycles C_1 and C_2 that $v_{1i} + v_{2j} = v_{3k}$. If this is true for any particular pair of vertices (one from C_1 and the other from C_2) then it will be true for every pair of vertices, since the labels in all t-cycles differ by an integral number of v_c. This implies that every vertex in C_1 is adjacent to every vertex vertex in C_2, say $v_{1i}v_{2j} \in E$, $\forall\ i, j = 1, 2, \ldots, t$. This is impossible in a fan if the rim vertex labels partition into three or more cycles.

Case 2. If $v_{1i} + v_{2j} = v_c$ then consider the other rim vertex adjacent to v_{1i} or v_{2j}, without loss of generality, say v_{xk} is adjacent to v_{1i} with $[v_{xk}, v_{1i}] \in E$. Obviously, $v_{1i} + v_{xk} \neq v_c$ since $v_{2j} \neq v_{xk}$ and so we have a special example of case 1 which is impossible in a fan if the vertex labels partition into three of more t-cycles.

Note that there are exactly two t-cycles of labels partition. If $v_{1i} + v_{xk} \neq v_c$ then $v_{xk} = v_{2k}$ and $v_{1i} + v_{2k} = v_r$, for some rim vertex v_r. This implies that one of the labels in C_2 is v_c, a contradiction. □

Property 8. *The labels of the rim vertices of any mod sum labelling of F_n can be partitioned into 2-cycle with respect to addition of v_c, and $v_c = \frac{m}{2}$*

Proof. Suppose $t \geq 3$. By Property 7,there exist two adjacent vertices with labels from the same t-cycle. Without loss of generality, we may suppose $v_{1i}v_{1j} \in E$. By Property 5, $v_{1i} + v_{1j} \neq v_c$. By Property 3, $v_{1i} + v_{1j} \neq v_{1k}$. So that $v_{1i} + v_{1j} = v_{xk}$ where $x \neq 1$,thus the sum of every pair of the labels in C_1 be a label in C_x. Hence every vertex in C_1 is adjacent to every other vertex in C_1 and the vertices of C_1 induce a complete graph K_t. This is true in fan when $t = 2$. Hence $v_c = \frac{m}{2}$ (m is modulo). □

Corollary 1. *When n is odd, there at least exists one isolate $[v_c, v_i]$, $i \in \{1, 2, \ldots, n\}$.* □

Theorem 1. *F_n is not a mod sum graph.*

Proof. We only need to verify that F_n is not a mod sum graph when $n \geq 5$. There are two cases to be considered.

Case 1. n is odd. By Corollary 1, we can see that F_n is not a mod sum graph.

Case 2. n is even. Since $n \geq 6$ then suppose that $C_1 = \{v_{11}, v_{12}\}$, $C_2 = \{v_{21}, v_{22}\}$ and $v_{21}, v_{22} \neq v_1, v_n$, without loss of generality, suppose that v_{11} is adjacent to v_{21}, so that $v_{11} + v_{21} = w$ where $w \in V$.

By Property 6, w is not in C_1 or C_2. Suppose $w \in C_x$, $C_x \neq C_1, C_2$, This implies that every vertex in C_1 is adjacent to every vertex in C_2, a contradiction. So $w = v_c$ and $v_{11} + v_{21} = v_c$. Since $v_{11} + v_c = v_{12}$ and $v_{21} + v_c = v_{22}$, we also have $v_{12} + v_{22} = v_c$. By Property 7, we suppose that v_{21} is adjacent to v_{22}. So that $v_{21} + v_{22} = u$. Obviously $u \neq v_c$, because $v_{22} \neq v_{11}$. We now suppose that $u \in C_y$, $C_y \neq C_1, C_2$ and let $u = v_{y1}$. Then $v_{21} + v_{22} = v_{y1}$. But we have $v_{12} + v_{22} = v_c$, thus $v_{y1} + v_{12} = v_{12} + v_{21} + v_{22} = v_{22}$. So $v_c + v_{y1} + v_{12} = v_{22} + v_c$ then $v_{y2} + v_{12} = v_{21}$. This implies that v_{12} is adjacent to both v_{y1} and v_{y2}. This is a contradiction in F_n when $n \geq 6$.

Hence F_n is not a mod sum graph. □

Lemma 1. *In a mod sum graph labelling of $F_n \cup sK_1$, $n \geq 5$, $s = \rho(F_n)$; $v_c \neq v_i + v_{i+1}$, $(i = 2, 3, \ldots, n-2)$, $v_c \neq v_c + v_i$, $(i = 1, 2, \ldots, n)$.*

Proof. The center v_c cannot be the edge sum of a spoke $v_c v_i (i = 1, 2, \ldots, n)$, as this would imply that $v_c = v_c + v_i$, so $v_i = 0$, a contradiction, because no vertex of the graph has a label of zero.

To show $v_c \neq v_i + v_{i+1} (i = 2, 3, \ldots, n-2)$, we suppose the contrary so that v_i and v_{i+1} are two adjacent rim vertices and $v_c = v_i + v_{i+1}$. Let v_{i-1}, v_{i+2} be the second rim vertices adjacent to v_i, v_{i+1} respectably. So $v_{i-1}, v_i, v_{i+1}, v_{i+2}$ are four consecutive rim vertices. The edge sum of the spoke $[v_c, v_{i-1}] = v_{i-1} + v_i + v_{i+1}$, so $v_{i-1} + v_i + v_{i+1}$ must be a vertex of the graph. Since the vertex v_{i+1} and the rim edge $v_{i-1} + v_i$ are both vertices of the graph, hence $v_{i+1} = v_{i-1} + v_i$ or $[v_{i+1}, (v_{i-1} + v_i)]$ exists. Similarly the spoke $[v_c, v_{i+2}] = v_i + v_{i+1} + v_{i+2}$ implied that either $v_i = v_{i+1} + v_{i+2}$ or $[v_i, (v_{i+1} + v_{i+2})]$ exists.

Allowing for symmetry, there are three cases to be considered.

Case 1. Both $[v_{i+1},(v_{i-1}+v_i)]$ and $[v_i,(v_{i+1}+v_{i+2})]$ exist.

When $[v_i,(v_{i+1}+v_{i+2})]$ exists, the vertex $v_{i+1}+v_{i+2}$ must be equal to one of the three vertices to v_i. Obviously $v_{i+1}+v_{i+2}\neq v_{i+1}$ and $v_{i+1}+v_{i+2}\neq v_i+v_{i+1}$ so that $v_{i+1}+v_{i+2}=v_{i-1}$. Similarly $v_i+v_{i-1}=v_{i+2}$. So $v_{i-1}=v_{i+1}+v_{i+2}=v_{i+1}+(v_{i-1}+v_i)=v_{i+1}+v_{i-1}+v_i\Longrightarrow v_i+v_{i+1}=0$ a contradiction.

Case 2. $v_{i+1}=v_{i-1}+v_i$ and $v_i=v_{i+1}+v_{i+2}$.

We note that $v_i=v_{i+1}+v_{i+2}=(v_{i-1}+v_i)+v_{i+2}\Longrightarrow v_{i-1}+v_{i+2}=0$. We write the four consecutive rim vertices and the center as $v_{i-1},v_i,v_i+v_{i-1},v_{i+2},2v_i+v_{i-1}$. The spoke $[v_c,v_i]=v_c+v_i=(2v_i+v_{i-1})+v_i=3v_i+v_{i-1}$ and the spoke $[v_c,v_{i+2}]=v_c+v_{i+2}=(2v_i+v_{i-1})+v_{i+2}=2v_i+v_{i-1}+v_{i+2}=2v_i$ imply that $[2v_i,(v_i+v_{i-1})]$ exists or $2v_i=v_i+v_{i-1}$. If $2v_i=v_i+v_{i-1}$ then $v_i=v_{i-1}$, a contradiction, hence $2v_i$ must be one of the three vertices adjacent to v_i+v_{i-1}. Obviously $2v_i\neq v_i,2v_i+v_{i-1}$, so $2v_i=v_{i+2}$. But $2v_i+v_{i-1}$ implies $2v_i$ is adjacent to v_{i-1}, namely v_{i+2} is adjacent to v_{i-1}, a contradiction .

Case 3. $[v_i,(v_{i+1}+v_{i+2})]$ exists and $v_{i+1}=v_{i-1}+v_i$

As in Case 1, we have $v_{i+1}+v_{i+2}=v_{i-1}$ and note that $v_{i+1}=v_i+v_{i-1}=v_i+(v_{i+1}+v_{i+2})\Longrightarrow v_i+v_{i+2}=0$. We write the four consecutive rim vertices and the center as $v_{i-1},v_i,v_i+v_{i-1},v_{i+2},2v_i+v_{i-1}$. The spoke $[v_c,v_{i-1}]=(2v_i+v_{i-1})+v_{i-1}=2v_i+2v_{i-1}$ and the spoke $[v_c,(v_{i-1}v_i)]=(2v_i+v_{i-1})+(v_{i-1}+v_i)=3v_i+2v_{i-1}$ that imply either $[v_i,(2v_i+2v_{i-1})]$ exists or $v_i=2v_i+2v_{i-1}$. We can see that $2v_i+2v_{i-1}\neq 2v_i+v_{i-1}$ otherwise $v_{i-1}=0$, a contradiction. Similarly we have $2v_i+2v_{i-1}\neq v_{i-1}$ and $2v_i+2v_{i-1}\neq v_i+v_{i-1}$. Hence $v_i=2v_i+2v_{i-1}$, so $v_i+2v_{i-1}=0$. The spoke $[v_c,v_i]=(2v_i+v_{i-1})+v_i=3v_i+v_{i-1}$ and the spoke $[v_c,(v_{i-1}v_i)]=(2v_i+v_{i-1})+(v_{i-1}+v_i)=3v_i+2v_{i-1}$ that imply either $[2v_i,(v_i+v_{i-1})]$ exists or $2v_i=v_i+v_{i-1}$. We can see $2v_i\neq v_i+v_{i-1}$ otherwise $v_i=v_{i-1}$. Obviously $2v_i\neq v_i$ and $2v_i\neq 2v_i+v_{i-1}$, so $2v_i=v_{i+2}$. But $2v_i+v_{i-1}$ implies $2v_i$ is adjacent to v_{i-1}, a contradiction. □

Lemma 2. *In a mod sum graph labelling of $F_n\cup sK_1$, $n\geq 5$,$s=\rho(F_n)$; If the spoke $[v_c,v_i]$ is an isolated, then $v_i+v_{i+1},(i=$*

$2, 3, \ldots, n-2)$ *are isolated.*

Proof. Suppose the contrary, if $\exists\ i \in \{2, 3, \ldots, n-2\}$, the spoke $[v_c, v_i]$ is an isolate, and rim edge $[v_i, v_{i+1}]$ is not an isolate. By Lemma 1, $v_i + v_{i+1} \neq v_c$. The spoke $[v_c, (v_i + v_{i+1})] = (v_c + v_i) + v_{i+1}$ implies that either $[(v_c + v_i), v_{i+1}]$ exists or $v_c + v_i = v_{i+1}$. Neither case can happen since v_{i+1} is a vertex of the fan where $v_c + v_i$ is an isolate. □

Lemma 3. *In a mod sum graph labelling of $F_n \cup sK_1$, $n \geq 5$,$s = \rho(F_n)$; If the spoke $[v_c, v_i], (i = 1, n)$ is an isolated vertex, then there are at least two isolated vertives.*

Proof. We only prove that if $v_c + v_1$ is isolated, then there are at least two isolates. The case of $v_c + v_n$ is similar.

Suppose to the contrary that, there is a single isolate. $v_1 + v_2$ must be a rim vertex or the center. If $v_1 + v_2$ is a rim vertex, then v_c must be adjacent to $v_1 + v_2$. Thus $v_c + v_1 + v_2 = (v_c + v_1) + v_2$ exists. Either $v_c + v_1 = v_2$ or $v_c + v_1$ is adjacent to v_2, there is a contradiction, since $v_c + v_1$ is an isolate.

If $v_c = v_1 + v_2$, then $v_c + v_3 = (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$. By Lemma 1, we know $v_2 + v_3$ is a rim vertex. If $(v_2 + v_3) \in V - \{v_1, v_2, v_3\}$, without loss of generality, we suppose $v_2 + v_3 = v_4$, that implies v_1 is adjacent to v_4. This is a contradiction in fan. If $v_1 = v_2 + v_3$, then the isolated $x = 3v_2 + 2v_3$. The label of spoke $[v_c, v_3] = (2v_2 + v_3) + v_3 = 2v_2 + 2v_3$ exists. Now $2v_2 + 2v_3 \neq v_3$ otherwise $2v_2 + v_3 = 0$ which is not true since $2v_2 + v_3$ is a label of graph. Similarly $2v_2+2v_3 \neq v_2, v_1(v_2+v_3), v_c(2v_2+v_3), x(3v_2+2v_3)$, that implies $v_c+v_3(2v_2+2v_3)$ is adjacent to v_2. This is a contradiction in fan. □

By Lemma 2 and 3, it is easy to see.

Corollary 2. *In a mod sum graph labelling of $F_n \cup sK_1$, $n \geq 5$,$s = \rho(F_n)$; If the spoke $v_c v_i, (i = 1, 2, \ldots, n)$ is an isolated vertex, then there are at least two isolated vertices.*

Lemma 4. $\rho(F_n) \geq 2$ *for* $n \geq 5$.

Proof. For $n \geq 5$, F_n is not a mod sum graph by theorem 1, that is $\rho(F_n) \geq 1$. We suppose that $\rho(F_n) = 1$ and show that this assumption leads to a contradiction. Let x be the only isolate. It is obviously that x is not the edge sum of the spoke by Corollary 2. So x is the edge sum one or more rim edges. When n is odd, $\rho(F_n) \geq 2$ by Corollary 1 and Corollary 2. Thus we only consider the case n is even.

In the following for symmetry, there are four cases to consider.

Case 1. $v_1 + v_2 = x$ or $v_n + v_{n-1} = x$.

We only consider the case $v_1 + v_2 = x$. Let v_3 be the second rim vertex that adjacent to v_2. Obviously, $v_2+v_3 \neq x$ and so v_2+v_3 must be a rim vertex. Both the spokes $[v_c, v_2] = v_c+v_2$ and $[v_c, v_3] = v_c+v_3$ exist so that the spoke $v_c+(v_2+v_3)$ implies that both of the following conditions hold.

(i) Either $v_2 = v_c + v_3$ or $[v_2, (v_c + v_3)]$ exists; and

(ii) Either $v_3 = v_c + v_2$ or $[v_3, (v_c + v_2)]$ exists.

Subcase 1. Either $v_2 \neq v_c + v_3$ or $v_3 \neq v_c + v_2$.

The rim vertices $v_2 + v_c$ and $v_3 + v_c$ can only be adjacent when $n \geq 6$ if both $v_2 = v_c + v_3$ and $v_3 = v_c + v_2$ so that if $v_2 \neq v_c + v_3$ and(or) $v_3 \neq v_c + v_2$ then $v_2 + v_c$ and $v_3 + v_c$ are not adjacent. The spoke $[v_c, (v_2 + v_3)] = v_c + v_2 + v_3$ must be an isolate otherwise the spoke $[v_c, (v_c + v_2 + v_3)]$ implies $[(v_c + v_2), (v_c + v_3)]$ exists. By Corollary 2, $\rho(F_n) \geq 2$ when any spoke is an isolate.

Subcase 2. $v_2 + v_c = v_3$ and $v_3 + v_c = v_2$.

When $v_c = \frac{m}{2}$ the equations $v_2 + v_c = v_3$ and $v_3 + v_c = v_2$ are both true. So $v_2 = v_3 + \frac{m}{2}$, $v_3 = v_2 + \frac{m}{2}$. Thus we have the center $v_c = \frac{m}{2}$, the rim vertices v_1, $v_3+\frac{m}{2}(v_2)$, v_3 and the isolate $v_1+v_3+\frac{m}{2}$. There exists a rim vertex $v_1 + \frac{m}{2}$ that adjacent to v_3 by the isolate. Since $n \geq 5$, we consider the other rim vertex e that adjacent to $v_1 + \frac{m}{2}$. We retain the label e that adjacent to $v_1 + \frac{m}{2}$. Then the spoke $[v_c, e] = v_c + e$ is a sixth distinct rim vertex that adjacent to v_1 by rim edge $[(v_1 + \frac{m}{2}), e] = v_1 + (e + \frac{m}{2})$. This is a contradiction in F_n when $n \geq 6$.

Case 2. $v_2 + v_3 = x$ or $v_{n-1} + v_{n-2} = x$.

We only consider the case $v_2 + v_3 = x$. Let v_4 be the second rim vertex that adjacent to v_3. Similar to Case 1, we have the following.

(i) Either $v_3 = v_c + v_4$ or $[v_3, (v_c + v_4)]$ exists; and

(ii) Either $v_4 = v_c + v_3$ or $[v_4, (v_c + v_3)]$ exists.

Subcase 1. Either $v_3 \neq v_c + v_4$ or $v_4 \neq v_c + v_3$.

The rim vertices $v_3 + v_c$ and $v_4 + v_c$ can only be adjacent when $n \geq 6$ if both $v_3 = v_c + v_4$ and $v_4 = v_c + v_3$ so that if $v_3 \neq v_c + v_4$ and(or) $v_4 \neq v_c + v_3$ then $v_3 + v_c$ and $v_4 + v_c$ are not adjacent. The spoke $[v_c, (v_3 + v_4)] = v_c + v_3 + v_4$ must be an isolate otherwise the spoke $[v_c, (v_c + v_3 + v_4)]$ implies $[(v_c + v_3), (v_c + v_4)]$ exists. By Corollary 2, $\rho(F_n) \geq 2$ when any spoke is an isolate.

Subcase 2. $v_3 + v_c = v_4$ and $v_4 + v_c = v_3$.

Thus we have the center $v_c = \frac{m}{2}$, the rim vertices v_2, $v_4 + \frac{m}{2}(v_3)$, v_4 and the isolate $v_2 + v_4 + \frac{m}{2}$. There exists a rim vertex $v_2 + \frac{m}{2}$ which is adjacent to v_4 by the isolate. Since $n \geq 6$, we consider the other rim vertex e that adjacent to $v_1 + \frac{m}{2}$. We retain the label e that adjacent to $v_2 + \frac{m}{2}$. Then the spoke $[v_c, e] = v_c + e$ is a sixth distinct rim vertex that adjacent to v_2 by the spoke $[(v_2 + \frac{m}{2}), e] = v_2 + (e + \frac{m}{2})$. $[v_2 + \frac{m}{2}, e]$ must be an isolate, otherwise the spoke $[v_c(v_2 + e + \frac{m}{2})] = \frac{m}{2} + (v_2 + e + \frac{m}{2}) = v_2 + e$ implies v_2 is adjacent to e, which is impossible in F_n. Obviously the isolates $v_2 + v_4 + \frac{m}{2}$ and $v_2 + e + \frac{m}{2}$ are distinct.

Case 3. $v_3 + v_4 = x$ or $v_{n-2} + v_{n-3} = x$ with $n \geq 6$.

We only consider $v_3 + v_4 = x$ and $n = 6$. when $n \geq 6$, the process of proof is similar to that in the Subcase 2 of Case 2. Let v_5 be the second rim vertex that adjacent to v_4. Thus we have the following.

(i) Either $v_4 = v_c + v_5$ or $[v_4, (v_c + v_5)]$ exists; and

(ii) Either $v_5 = v_c + v_4$ or $[v_5, (v_c + v_4)]$ exists.

Subcase 1. Either $v_4 \neq v_c + v_5$ or $v_5 \neq v_c + v_4$.

The proof is similar to that in the Subcase 1 of Case 2.

Subcase 2. $v_4 + v_c = v_5$ and $v_5 + v_c = v_4$.

Thus we have the center $v_c = \frac{m}{2}$, the rim vertices v_3, $v_5 + \frac{m}{2}(v_4)$, v_5 and the isolate $v_3 + v_5 + \frac{m}{2}$. There exists a rim vertex $v_3 + \frac{m}{2}$ that adjacent to v_5 by the isolate. Since $n = 6$, the rim edge $[v_4, v_5] = (v_5 + \frac{m}{2}) + v_5 = 2v_5 + \frac{m}{2}$ must be an isolate. Otherwise, either $v_1 = 2v_5 + \frac{m}{2}$, $v_2 = 2v_5$ or $v_2 = 2v_5 + \frac{m}{2}$, $v_1 = 2v_5$. If $v_1 = 2v_5 + \frac{m}{2}$ and $v_2 = 2v_5$, then the rim edge $[v_2, v_3] = 2v_5 + v_3$ implies $v_5 + \frac{m}{2}(v_4)$ is adjacent to $v_5 + v_3 + \frac{m}{2}(x)$, a contradiction. If $v_2 = 2v_5 + \frac{m}{2}$ and $v_1 = 2v_5$, we have the same result. But the isolates $v_3 + v_5 + \frac{m}{2}$ and $2v_5 + v_3$ are distinct.

Case 4. $v_i + v_{i+1} = x$ for some $i \in \{4, 5, \dots, n-4\}$, $n \geq 8$.

The verification is similar to Case 3. □

Theorem 2. $\rho(F_n) = 2$ *for* n *even,* $n \geq 6$.

Proof. Now consider the following mod sum labelling f of $F_n \cup 2K_1$. Let

$$v_c = 25(\frac{n}{2} - 1).$$

$$\{v_1, v_2, v_3, v_4, \dots, v_{n-1}, v_n,\} \cong \{3, v_c+(2\frac{n}{2}+1), 5, v_c+(2\frac{n}{2}-1), \dots, 2\frac{n}{2}+1, v_c+3\}.$$

Let $V(F_n) \cup 2K_1 - V(F_n) = \{u_1, u_2\}$ and

$$u_1 = 25(\frac{n}{2} - 1) + (n+4), \ u_2 = 25(\frac{n}{2} - 1) + (n+6).$$

Modulo is $50(\frac{n}{2} - 1)$. It is easy to verify that f is a mod sum labelling of $F_n \cup 2K_1$. By Lemma 4, thus we have $2 \leq \rho(F_n) \leq 2$ and the result follows. □

3. Symmetric complete bipartite graphs

Symmetric complete bipartite graph $K_{n,n}$ for $n \geq 3$ is not a mod sum graph in [5]. In this section we shall determine the mod sum number of $K_{n,n}$ for $n \geq 2$. Let $r = \rho(K_{n,n})$ and $S = V(K_{n,n}) \cup rK_1$. There exists a finite subset L of $Z_m - \{0\}$ (m is suitable modulo). In this labelling, we suppose that (X, Y) is a bipartite of $K_{n,n}$, where

$$X = \{x_1, x_2, \dots, x_n\}, \quad Y = \{y_1, y_2, \dots, y_n\}.$$

When $n = 2$, $K_{2,2}$ is a mod sum graph. Let $X = \{1, 4\}$, $Y = \{2, 3\}$, modulo is 5.

We thus need to consider only the case when $n \geq 3$.

Lemma 5. $\rho(K_{n,n}) \geq n$ *for* $n \geq 3$.

Proof. For $n \geq 3$, $K_{n,n}$ is not a mod sum graph. So $\rho(K_{n,n}) \geq 1$. It must exist a $x_i + y_j \in S - X \cup Y, i, j \in \{1, 2, \ldots, n\}$. Without loss of generality, we may suppose $x_1 + y_1 \in S - X \cup Y$, thus we have $x_1 + y_i \in S - X \cup Y, (i = 1, 2, \ldots, n)$.

Assume the contrary, $x_1 + y_1 \in S - X \cup Y$ and $x_1 + y_i \in X \cup Y, i \in \{2, 3, \ldots, n\}$.

Case 1. $x_1 + y_i \in X$. Thus we have $x_1 + y_i$ is adjacent to y_1. Hence $[x_1 + y_i, y_1] = x_1 + y_i + y_1 = (x_1 + y_1) + y_i$ implies that either $x_1 + y_1 = y_i$ or $x_1 + y_1$ is adjacent to y_i, a contradiction.

Case 2. $x_1 + y_i \in Y$. Thus we have $x_j (j = 1, 2, \ldots, n)$ is adjacent to $x_1 + y_i \in X$. Hence $x_1 + y_i + x_j = x_1 + (y_i + x_j)$. Because $y_i + x_j$ exists. Either $x_1 = y_i + x_j$ or x_1 is adjacent to $x_j + y_i$. If $x_1 = y_i + x_j$, when $i = 1$ we have $y_i = 0$, a contradiction. If x_1 is adjacent to $y_i + x_j$, then $y_i + x_j (j = 1, 2, \ldots, n) \in Y$. Thus $Y = \{y_i, y_i + x_1, y_i + x_2, \ldots, y_i + x_n\}$. Since $|Y| = n$, there exists some $y_i + x_j = y_i, j \in \{1, 2, \ldots, n\}$, we have $x_i = 0$, a contradiction.

Hence there are at least n distinct vertices $\{x_1 + y_1, x_1 + y_2, \ldots, x_1 + y_n\} \in S - X \cup Y$. □

Theorem 3. $\rho(K_{n,n}) = n$ *for* $n \geq 3$.

Proof. Now we give a mod sum labelling f of $K_{n,n} \cup nK_1$.

$$\{x_1, x_2, \ldots, x_n\} \cong \{3, 18, \ldots, 3 + (n-1)15\}$$

and

$$\{y_1, y_2, \ldots, y_n\} \cong \{4, 19, \ldots, 4 + (n-1)15\}$$

Let $S - V(K_{n,n}) = \{u_1, u_2, \ldots, u_n\}$ and

$$\{u_1, u_2, \ldots, u_n\} \cong \{7, 22, \ldots, 3 + (n-1)15\}.$$

Modulo is $15n$. It is easy to verify that f is a mod sum labelling of $K_{n,n} \cup nK_1$. By Lemma 5, thus we have $n \leq \rho(K_{n,n}) \leq n$ and the result follows. □

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